Relating Practice and Research in Mathematics Education

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PREFACE

The fourth Nordic Conference on Mathematics Education – Norma 05 – was held 2nd –6th September 2005 in Trondheim, Norway with 101 participants from 14 countries. The previous conferences were Norma 94 in Lahti (Finland), Norma 98 in Kristiansand (Norway), and Norma 01 in Kristianstad (Sweden). This volume contains all the plenary lectures at Norma 05, and a selection of the papers presented, including workshops and short communications.

The theme of the conference – *Relating practice and research in mathematics education* – was intended to focus on how different practices in preparation for learning are related to teachers' views on the learning of mathematics, and how these views are related to basic theoretical aspects of mathematics education. Research in this tension between theory and practice will develop more knowledge about practices that can foster improved learning in mathematics. It will therefore enhance knowledge about favourable approaches to education in school and to pre-service and in-service education of teachers. The conference was primarily designed for teachers and teacher educators, researchers and graduate students in mathematics education, in particular from the Nordic and Baltic countries, but participants from other countries were most welcome. The conference did attract many particpants and most noticably, a great number of doctoral students from all Nordic countries.

We experienced that the participants enjoyed the conference days in sunny September at Nidelven and were deeply engaged in serious and highly professional and supportive discussions in a friendly and relaxed atmosphere. We want to thank all who contributed to Norma 05 with plenary lectures, paper presentations, workshops and short communications and all those who shared their work and took part in the discussions. We would also like to thank Sør-Trøndelag University College for hosting, and financially supporting the conference. Thanks also go to Tapir Academic Press for being willing to publish the conference proceedings in this book, and to Radisson SAS Royal Garden Hotel for conducting the arrangement in a smooth way. The Nordic Graduate School in Mathematics Education contributed with resources for the workshop on classroom research and for two of the plenary lecturers, for which we express our gratitude. Finally we also want to thank the Norwegian Center for Mathematics Education for financial support. For the editors of the book it was a pleasure to work with all the contributions and we hope readers will enjoy the outcome. Now we are looking forward to the next Norma-conference that will be held in Denmark in 2008.

Linköping, Kristiansand and Trondheim, October 2006

Christer Bergsten, Barbro Grevholm, Heidi S. Måsøval, Frode Rønning

WHEN RESEARCHERS ARE RELATING PRACTICE AND RESEARCH IN MATHEMATICS EDUCATION

Barbro Grevholm
Agder University College, Norway

In this paper an overview of the contributions in this book is given. An account of the five plenary lectures is introducing the chapter, followed by the NoGSME workshop and the twenty four papers that were presented. Finally the six short communications follow and the activity workshop. The review process is described and a short comment on where to find other sources of research from the Nordic countries. Finally I try to discuss what kind of research we see here and how it could influence the development in society in the future. In this way I want, on behalf of the editing committee, to thank all the contributors for the serious and deep work they have all done in order to prepare their papers. Without the hard work and thinking done by all the contributors we would have had no conference or book. It is challenging to consider the many hours of work and endless efforts that lie behind an artefact like this. A warm thanks to all the writers.

WHAT IS IN THIS BOOK

The content of this volume represents what took place during the fourth Nordic Conference on Mathematics Education in Trondheim, 2nd – 6th September 2005. The broad variety of content in the mathematics education studies reflected by these contributions show how the field of study has grown and is now represented by many new researchers. At least 18 doctoral students have published in this volume from their studies and this indicates many promising ongoing studies. The five plenary speakers painted a coherent and challenging overarching picture for the work in the conference. The many impressive findings from the research studies offered learning from knowledge in many different areas to the participants. Several larger studies have also been reported here. The NoGSME workshop gave the valuable opportunity to learn from interpretation of mathematics classroom data. Participants from fourteen countries contributed through their varied cultural experiences to lively and fruitful discussions linked to the presentations.

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THE REVIEW PROCESS OF THE PAPERS

All contributions in the book have passed through a triple review process. First the abstracts were reviewed by the programme committee and accepted or rejected, and after that the papers were presented at the conference where they were subject to discussion and critique by the audience. After the conference a peer review process followed, where each paper was read and commented on by at least two colleagues. Papers were then rewritten according to suggestions from reviewers and sent to the editors, who took a decision of acceptance or rejection based on the reviews and the revision of the papers. Finally the editors reviewed the accepted papers once more and changes needed were communicated with authors, who were allowed to make final changes. This is the first time the Norma conference has used a proper peer review process and the reason why we introduced this now is that we consider the Nordic community of researchers in mathematics education ready and ripe for such a process. The field of mathematics education has grown substantially in size in the Nordic countries since the last Norma conference and a number of new graduate programmes in mathematics education have been created. The review process was intended to contribute to the raising of the quality of the papers in this volume. We hope that this will also have as a consequence that we in the future will find more papers by Nordic and Baltic researchers in international scientific journals. There has not been a strong tradition in the Nordic countries to publish in such journals. In these countries much interesting research is currently going on and it is worth sharing with an international audience.

OTHER SOURCES OF PUBLISHED RESEARCH IN MATHEMATICS EDUCATION IN THE NORDIC COUNTRIES

There is a strong tradition in the Nordic countries of publishing on research in conference proceedings. The first Norma conference took place in Lahti 1994 and was documented in a book edited by Erkki Pehkonen (1994). The second conference followed four years later (Breiteig & Brekke, 1998). The documentation from the third conference came quite recently (Bergsten & Grevholm, 2005). The book from the fourth conference is this one. The fifth conference is planned to take place in Denmark in 2008 and a group of researchers has just started to prepare for it. Besides these conference proceedings there are also other sources for an overview of the research in mathematics education in the Nordic countries. For example, Wedege and Antonius (2004) have published a bibliography about research in Denmark over the years 1965 to 2003. Bergsten (2002) has written an overview of Swedish research in mathematics education and Björkqvist (2003) has published a report that gives the state of the art in Sweden in 2002. Recently Ole Skovsmose (2006) gave a report and an evaluation of ten licentiate theses in the Swedish Graduate School. We are part of a growing field of research and hope to have still more to offer in the future.

THE RED THREAD OF THE FIVE PLENARY PRESENTATIONS AT NORMA 05

Five plenary addresses took place and here I will present them first. Then follow twenty four papers, the NoGSME workshop on classroom research, six smaller contributions (short communications) and an activity workshop.

Mogens Niss, as the first plenary speaker, presented on *The concept and role of theory in mathematics education*. He carefully investigates the general notion of theory and based on this discussion moves to theories in mathematics education. He poses a number of crucial questions about theories such as ‘What are they?’, ‘Where do they come from?’, ‘What foundations do they have?’, ‘What are their roles?’, ‘Are they good enough?’, and ‘What should we strive for in terms of theory development and use?’. He continues to engage in deep discussions of all these questions, which do not have any easy answers. In his conclusions he states that only the mathematics education research community possesses the multifaceted expertise necessary for constructing consistent theoretical frameworks that are rich enough to cover the whole field in its complexity. It is unlikely that we shall ever arrive at just one theoretical framework for mathematics education research, unifying all researchers in the field. Several comprehensive, respectable, but competing, theories of mathematics education are more likely to arise.

From this theoretical level we were taken into the practice of classroom research with an illustration of the role of theory in such studies. Thus the second lecture was on *Students’ goals in mathematics classroom activity* by Simon Goodchild. He inquires into the goals students hold in their regular mathematics classroom activity. His data comes from a year-long ethnographic style case study of a regular mathematics class comprising 14–15 year old students in England. He carefully deals with the issue of establishing a theoretical framework for learning in classrooms, where he takes three complementary perspectives. The research questions are: ‘What are the goals, rationales, and interpretations, towards which students work in their regular mathematical activity?’ and ‘What are the features of the classroom, arena and setting, which comprise that socio-cultural context in which students engage in mathematical activity?’ He concludes that to the casual outside observer this class appeared to function well. His engagement with the class suggests that it functioned socially but not as a place where teaching and learning mathematics was effective. The teacher did not agree. It is difficult to convince teachers that their well managed classes might benefit from closer examination. The key question to ask might be what do the students think is going on.

From the classroom we were taken to a specific artefact that is normally used in classrooms, the textbook. The third plenary lecture given by Birgit Pepin had the title *Making connections: An exploration of connections made in and ‘around’ mathematics textbooks in England, France and Germany*. The study explores the ways in which mathematics textbooks vary at lower secondary level in these three countries. Connections are explored in terms of interconnectedness of mathematical content knowledge. Tasks and exercises are investigated with respect to cognitive demand and contextualisation. Learners in the different countries are offered different mathematics and given different opportunities to learn that mathematics. It is argued that textbooks play an important part in the mathematics education process. Both their content and mediation need to be understood in terms of a wider context in order to establish shared understandings, principles and meanings.

From this study of the important artefact in the classrooms we were taken to a study of the pupils in mathematics classrooms. The fourth plenary was given by Markku Hannula. *Understanding affect towards mathematics in practice* was the title of his lecture. He stressed that affect in mathematics classes is a complex phenomenon to study. The complexity needs to be reflected in both the conceptual model and the representational system developed for the study. In his paper a framework for this kind of research is discussed. The present view sees student behaviour as self-regulated and suggests three main concepts for a theory: cognition, motivation, and emotion. Only emotions are directly observable in expressive behaviour. Observed emotions can be used for making inferences about motivation. Student behaviour and verbal expressions provide further information about student emotion, motivation and cognition. These should not be taken at face value, since they are subject to the student's conscious regulation. Hannula points out that triangulation of multiple kinds of data provide interpretations that are more reliable.

Finally Barbara Jaworski introduced us to the research and development project LCM, *Learning Communities in Mathematics*. The project aims to design and study mathematics teaching development for improved learning of mathematics through inquiry communities between teachers and didacticians. It involves twelve didacticians at the university college, forty teachers and the pupils from eight schools. They are inquiring in/into mathematics, in/into mathematics teaching, and in/into researching mathematics learning and teaching. An episode from research in the first phase of the project exemplifies this inquiry in three layers. In the theoretical sections the concept of inquiry community is questioned and explained as a basis for developmental research. After that the methodological and operational perspectives of the LCM-project are discussed. The presentation ends with a discussion of outcomes and issues emerging from the ongoing project.

Thus the five plenary presentations gave the conference participants a good overview of vital issues from the role of theory in mathematics education research to the practice of such research in one mathematics classroom, to focus on an important artefact in the classrooms, to focus on the individuals in the classrooms and finally to the issue of how to develop teaching and learning mathematics through inquiry in a wider community. With this as a background participants could enjoy to choose among almost forty different presentations at the conference. From them we are able to offer twenty four papers, six short communications and a workshop report in this book.

THE NOGSME WORKSHOP ON MATHEMATICS CLASSROOM RESEARCH

The Nordic Graduate School in Mathematics Education (NoGSME) organised a half day workshop on classroom research during the conference. For this NoGSME had invited Simon Goodchild to carry out the work. The aims of the workshop were to demonstrate that classroom episodes can be interpreted in a variety of ways, each with a validity that arises from whatever theoretical perspective held by the interpreter, to demonstrate that we learn about these multiple perspectives by listening to each other, and from questioning the beliefs, values, etc. upon which interpretations are articulated, to demonstrate the role of theory in making interpretations and the role of empirical evidence in the development of theory.

The workshop started with two brief presentations of classroom episodes by Heidi S. Måsøval and Barbara Jaworski and they exposed different approaches arising from different theoretical positions. Participants then continued with work in small groups on transcripts of episodes with two 14–15 year old students. Feedback was given to the work in groups by Simon, including analysis. Finally a plenary discussion took place, where all participants had the opportunity to question and respond to issues arising from both parts of the workshop. About eighty persons took part in the workshop and left it enjoyed and convinced that analysis of classroom episodes can contribute to teacher knowledge about teaching and learning of mathematics and inspire to initiatives of change and intervention in normal classroom work.

THE PAPER PRESENTATIONS AT THE CONFERENCE

Claire Berg writes about *Focus on teachers' reflections and the role of the researcher in collaborative problem solving in algebra: Building a learning community*. Her paper is related to ongoing research concerning the possibility to enhance the learning and teaching of elementary algebra through the creation of a learning community. Related to elementary algebra, it explores teachers' reflections when engaging in problem solving in collaboration with a researcher. It also explores the role played by the researcher. The data presented illustrate the way a community of teachers and a researcher engaged in a mathematical task through their reflections emerging from the process. Preliminary results suggest that engaging in such activity increases teachers' awareness of their own ways of working in mathematics and of the complexity of the teaching situation.

The concept of understanding in mathematics textbooks in Iceland is in focus in the contribution by Kristín Bjarnadóttir. She claims that the importance of understanding concepts and procedures is emphasised in most mathematics textbooks written in Iceland, from the earliest writings in Icelandic up to present day. Assertions are found at regular intervals that the emphasis on understanding was not promoted in earlier times. In order to analyse what lies behind these statements, the literary meaning of the Icelandic word "skilja", usually translated as "understand", is compared to the corresponding words in Danish, English and Latin, and also related to the modern sense of understanding in mathematics.

Collaborative problem solving in geometry without teacher intervention is written by Raymond Bjuland. His paper reports on research that focuses on student teachers working collaboratively in a problem-solving context without teacher involvement. One episode from the group discussion has been chosen to illustrate how elements of the students' reasoning process are revealed in dialogues. By taking a situational, socio-cognitive approach to studying communication and cognition the students' use of different strategies has been identified. In their attempts at coming up with a solution to one of the problems, the students discuss different ideas and geometrical concepts. The analysis puts emphasis on how they attribute meaning to the concept of similarity. One question is when it is suitable for the teacher to become involved in the group discussion.

Hans Jørgen Braathe talked about *Identity and agency in mathematics teacher education*. He discusses theoretical frameworks presented in the book "Identity and agency in cultural worlds". The framework is used to analyse texts about learning produced by one student teacher during the compulsory mathematics course in the pre-service education. The texts are seen as reports of her perception and understandings of the "figured worlds" of mathematics education in which she participates as a learner. This is meant to indicate her positions and reflections of her "authoring identities" as learner and performer as teacher of mathematics.

Martin Carlsen writes about *Geometric series and mathematical mastery and appropriation* in a paper that discusses upper secondary students' work. Analyses of student dialogues in a collaborative problem-solving small group are conducted to identify what the students are doing together mathematically when working on and discussing mathematical problems. The analytical approach puts emphasis on the words the students use, how they use concept-related definitions and formulae, and the students' argumentation when discussing and applying the concept. The analyses show both mastery and appropriation of the concept of geometric series. The students' use of ordinary and formula-related words and argumentation indicate some appropriation difficulties.

Ingvald Erfjord and Per Sigurd Hundeland write about *Teachers' reflections on the use of ICT tools in mathematics: Insights from a pilot study*. The paper is based on data from a case study at a secondary school. The aim is to present an analysis of the nature and content of two teachers' reflections on the use of ICT tools in mathematics at grade 9. A central method in the study is the use of focus group interviews to gain insights into teachers' perspectives. Two teachers are reflecting on the use of ICT in their mathematics teaching recalled by a video recording of three students' work in the lesson. A table is presented with questions addressing the reflections from the teachers. The table indicates teachers' ability to address a variety of questions, both linked to the concrete activity recalled by the video and on a more general level concerning ICT teaching in mathematics.

Students' attitudes, choice of tools and solutions of mathematical tasks in an ICT rich environment is written by Anne Berit Fuglestad. In a three year development and research project in lower secondary school the aim was to develop and study students' competence to use ICT-tools and to make reasonable choices of ICT-tools to solve mathematical problems. Her paper presents some of the students' solutions in computer files, some observations and answers to a questionnaire on their work. The students were asked some questions about their attitude and about what tools they chose for a specific task and why. The paper also outlines a subsequent project which aims to build learning communities with teachers and didacticians in order to develop ICT competence with mathematical tools and use of ICT in inquiry processes.

Anne Birgitte Fyhn's presentation had the title *Sami culture as basis for mathematics teaching.* A QRS-system is a meaningful system that for a given group of people gives meaning to Quantity, Relations and Space. Each cultural group has its own QRS-system. Her paper focuses on how the Sámi culture's elements 'lavvu', 'lasso', 'skis' and 'duodji' can function as basis for mathematics teaching. The elements are described and mathematised, and made accessible to mathematics treatment. They are further categorised into two groups, Relations and Space. The possibilities for how to use them in mathematics teaching was pointed out.

Artefact mediated classroom practice is the title of the paper by Sharada Gade. Her paper reports on a classroom study of meaning made by students working in small groups at an upper secondary school. As it is a naturalistic study, it observes student activities within the classroom from a socio-cultural-historical perspective. It addresses the artefact mediated classroom practice, in which student participation and meaning making is guided and apprenticed by the teachers. The mathematics classroom constitutes a culture, which is also the medium for meaning making, mediated by artefacts within intentional practices.

Örjan Hansson writes about *Preservice teachers' conceptions of the function concept and its significance in mathematics*. The purpose of his paper is to examine preservice teachers' conceptions of the function concept and also their conceptions of the significance of functions in mathematics. Still another purpose is to study the effects of an intervention regarding the function concept. Two groups of preservice teachers who are specializing in mathematics and science are participating in the study. He reports on findings indicating changes, firstly in the preservice teachers' view on the function concept related to the intervention, and secondly with respect to different themes of relevance in their reasoning on the significance and presence of functions in various contexts.

Rolf Hedrén reports on *Problem solving – what and how do pupils learn?* His study forms a part of a project, RIMA, where pupils aged 13 – 16 years and their teachers were working with ten rich problems during mathematics lessons. Hedrén discusses the work of two groups, which were video and audio tape recorded during their work. They were also interviewed individually after the problem solving sessions, and their written solutions were collected. He tried in this way to find out what opportunities for learning arose during the problem solving and how the pupils made use of these opportunities. Most of the pupils, though bewildered in the beginning, could find procedures and strategies to solve the problem. In that way they got a better understanding of the concepts involved. We honour the memory of Rolf, who died peacefully in April 2006.

Per Sigurd Hundeland, Ingvald Erfjord, Barbro Grevholm, and Trygve Breiteig write about *Teachers and researchers inquiring into mathematics teaching and learning: A case of linear functions*. The teachers in their study are participants in the LCM- (Learning Communities in Mathematics) project run at Agder University College. The project emphasises the development of communities of teachers and researchers focused on inquiry in mathematics and teaching. This paper deals with a well-planned lesson in an eleventh grade class in a Norwegian upper secondary school. Data are presented that illustrate how the teachers intervened to reach certain goals that had been identified through the planning process. The interventions identified discourses, which showed that there was a discrepancy between how the teachers interpreted and used a certain word for a mathematical concept and how the students interpreted the same word. The findings illuminate common challenges faced when trying to build a community of inquiry in the classroom.

Marit Johnsen Høines and Beate Lode presented *Teaching: An investigative approach*, a project where the preconditions for a subject based and reflective approach in the context of practice teaching in teacher education are investigated. Five second-year student teachers and three tutors were invited to participate in the investigation during a compulsory 30 ect credits study of mathematics education. Focus is on how the preliminary analysis provides a basis to inquire further into the didactical conditions for including a subject based discussion within the conversation in practice teaching. Contradictions are discussed implied by the findings that practice teaching communication is imprinted by an evaluative approach that can restrain the development of a subject based, and reflective approach.

Kjærand Iversen presented on *Students' response to one-object stochastic phenomena*, a study where 485 students in lower secondary school (age 13–16) were asked to solve probabilistic problems in a written test. Their responses to three of the tasks, which were all related to compound stochastic events, are considered. The aim of the investigation is to study logical inconsistencies in the students' answers. In each task a question is presented to the students in two different ways. A high degree of inconsistency was found in the responses for all age groups. Further some results from a follow up clinical interview are reported.

Limits and infinity – A study of university students' performance is the title of the paper by Kristina Juter and Barbro Grevholm. They describe how students work with limits of functions, including the concept of infinity. These concepts are complex but necessary for mathematics studies. Student solutions to limit tasks were studied to reveal how students handled limits of functions and infinity at a Swedish university. As a background to the students' and teachers' situations, textbooks and curricula were also analysed. One of the results is that many students were unable to handle limits correctly. Some of the students gave correct answers with incorrect explanations to tasks. Textbooks that are used at upper secondary schools do not provide much theory or tasks about limits and infinity, so most newcomers to university do not have a well developed image of these concepts.

The paper on *Students' beliefs and attitudes towards mathematics* is written by Kirsti Kislenko, Barbro Grevholm, and Madis Lepik . The focus of the study described is students' beliefs and attitudes towards mathematics teaching and learning. Some preliminary results from research carried out in Norway in 2005 are given, which focus on first year students in upper secondary school. In addition the answers from the ninth grade students in 2005 are briefly compared with students' responses from 1995, when corresponding data were collected within the KIM project in Norway. Both of these studies use a questionnaire elaborated in 1995. Some of the aspects related to a similar study amongst Estonian students, that will take place in spring 2006, are also discussed.

A study of teachers' views on the teaching and learning of mathematics, their intentions and their instructional practice was presented by Bodil Kleve. It is an ethnographic study of how teachers in lower secondary school are implementing the current mathematics curriculum, L97, in Norway. The methods in the study include focus groups, individual conversations, self estimation, and classroom observations. She found different degrees of coherence between what teachers say they do and what they actually do in the classroom. Kleve focuses on the methodology and the analysis of data from one teacher who wants to emphasise students' conceptual understanding in mathematics as L97 does, rather than exercising drill and procedures. Possible constraints preventing him from carrying out "ideal" teaching and some of L97's recommendations are considered.

Teacher research in mathematics teaching is the title of the paper by Jónína Vala Kristinsdóttir. The article accounts for an elementary school teacher's experience from working at developing her way of teaching mathematics. Sketches from her teaching are given to highlight the processes she went through. They are part of a three year long study of her mathematics teaching, where she worked on improving her way of communicating in the classroom and using her experience with the children to make decisions about her teaching.

Reidar Mosvold writes about *Teaching "Mathematics in everyday life".* His paper presents a case study of one teacher, with the aim to distinguish strategies of connecting mathematics with everyday life and the underlying beliefs. He claims that a case study provides the opportunity to portray, analyse, and interpret real individuals and situations in their uniqueness. The article presents practical examples of how a teacher connects mathematics with everyday life by arranging small projects and using other sources than the textbook.

Heidi S. Måsøval presented the paper *Efficiency or rigour? When students see the target as 'doing' more than 'knowing' mathematics.* Her paper has the objective to analyse an observation of three student teachers' collaborative work in a problem-solving context. Two episodes are described which reveal incidents of specializing, generalizing, conjecturing, and convincing. Her analysis begins by investigating the student teachers' difficulties in changing representational system from natural language to formal language (using a symbolic notation including fractions and letters). Further analysis describes how reasoning processes involved are constrained by the didactical contract.

Girls' beliefs about the learning of mathematics is written by Guðbjörg Pálsdóttir. She claims that there has been an increased attention to research on beliefs about mathematics and mathematics education. It has become one of the central elements of study in mathematics education. She reports from a qualitative research study on the beliefs of four Icelandic teenage girls about mathematics, the study of mathematics, and themselves as learners of mathematics. Their descriptions and thoughts are viewed in the light of theories and recent results from overseas quantitative investigations on girls' beliefs about mathematics and the study of mathematics. The main conclusions are that these girls view mathematics as a process, place emphasis on understanding and solving the problems at hand, are self-confident, well organized and study hard, and do not often use elaboration strategies.

The teacher as researcher: Teaching and learning algebra is written by Per-Eskil Persson. Together with a colleague he performed a study of which factors that facilitate or obstruct students' learning of algebra. This study is the context for the reflections in the paper and he briefly presents the aims of the study. Starting with three examples from the study, he describes and discusses how these, together with the theories connected with them, helped to interpret and better understand what happened in the classroom and how theories and research could be used to improve teaching practice. He also tries to show the relationship between the challenges of having the double role as a teacher and an observer, and the benefits of doing action research, both for the students and for the teacher/researcher. Finally, Persson discusses how the process of going through a research education has affected him as a person and a newcomer in the didactical field.

Wolfgang Schlöglmann talked about *Mathematics learning without understanding – cognitive and affective background and consequences for mathematics education*. Mathematics learning without understanding and learning by rote are widespread phenomena. Learning by rote contradicts the view that learning necessitates understanding. This phenomenon is illuminated from the cognitive as well as the affective viewpoints. Recent results from neuroscience are used to understand the process of such learning, suggesting that this kind of learning is based more on the form of the description of a problem than on its content. This "knowledge" is very inflexible and not transferable to other problems. If acquiring competencies that are useful also outside school is the goal of mathematics learning at school, then efforts in the classroom towards developing concepts and using demanding problems ought to be intensified.

Jeppe Skott and Tine Wedege presented on *Researching potentials for change: The case of the KappAbel competition*. They report on the design and methodology of a study in belief research, addressing the question of the potential and perceived influence of the KappAbel competition on the mathematical attitudes and practices of the participating teachers and students. They outline the main methodological considerations and also discuss, and to some extent challenge, what they consider dominant approaches to research on teachers' beliefs. Three interconnected methodological difficulties are addressed: (1) the problem of not using conceptual frameworks that are well grounded empirically; (2) the question of overemphasising teachers' views of mathematics for their educational decision making; (3) the problem that no terminology carries unequivocal meanings, and more than an indication of agreement or disagreement with the rhetoric of reform is needed to outline teachers' and students' school mathematical priorities.

SHORT COMMUNICATIONS

Kari Hag and Peter Hästö presented on *Peer-assessment in mathematics at the Norwegian University of Science and Technology*. Maugesten and Lauvås in 2003 reported on improvements in student performance after the introduction of peer-assessment methods in a class of about a 100 mathematics student teachers at Østfold University College. The authors wanted to investigate whether similar gains could be made at the Norwegian University of Science and Technology and also consider the effect of peer-assessment in a more controlled study. They report on a pilot study conducted during the Spring Term 2005, in which peer-assessment was tried out in one section (consisting of student teachers) of the mathematics course Multidimensional Analysis.

Ričardas Kudžma presented on *Algebra – geometry relation in teaching functions and calculus*. He points to the fact that algebraic calculations, formal applying of formulas, often dominate in mathematics classes. He claims that such teaching distorts the essence of mathematics and forms unfortunate attitudes to the subject in society. By including more geometry into algebra and calculus the situation can be improved. A few examples of showing harmony between algebra and geometry are presented. Emphasis is put on the direct-inverse functions relation, its geometric interpretation and its applications.

Jari Lakka talked about *Building addition and subtraction strategies in early primary school mathematics.* First and second grade pupils' thinking strategies in addition and subtraction tasks were researched in the multiple case study. Using a phenomenographical approach he interviewed six first graders and eleven second graders three times during the school year 2003–2004 and found four categories in counting.

Thomas Lingefjärd presented *On teaching problem solving and solving the problems of teaching*. George Polya's first principle in teaching is active learning: "let the students discover by themselves as much as feasible under the given circumstances". Lingefjärd found that by adopting this principle in a specific way, it is possible to get students to change their attitudes towards mathematics and become more involved in the subject. Besides the more immediate feedback and reactions in the classroom, it also resulted in more students passing exams. They even choose mathematics as a continuation course in their teacher program. So the benefits have proved to be substantial.

Alistair McIntosh talked about *A strategies approach to mental computation from years one to ten*. Many countries are moving from an over-emphasis on written computation in the early stages to a greater emphasis on developing mental computation through a strategies approach. For the past five years the author has been working with schools in Australia on a project funded by the federal government and two states to develop an organised approach to mental computation from the basic addition and subtraction facts to decimals, fractions and percents. His paper describes the background and gives a brief account of the research and curriculum development aspects of the project.

Gülçin Tan talked about *Teachers' and students' perceptions about the newly developed mathematics curriculum: A case from Turkey*. The purpose of the study is twofold: (1) to find out how the fifth grade classroom teachers and the students perceive the newly developed mathematics curriculum (NDMC), and (2) to investigate how the NDMC is put into practice. The study was conducted with a sample of 65 fifth grade students and two fifth grade classroom teachers from one of the pilot schools in Ankara. Both qualitative and quantitative data were collected. The results indicated that most of the students perceived learning mathematics as enjoyable and easy in the NDMC. For the teachers, the NDMC reflects a constructivistic approach and provides opportunities for the students to learn mathematics with understanding. Generally, both the teachers and the students are satisfied with the NDMC.

A WORKSHOP WITH ACTIVITIES

May Renate Settemsdal, Toril Sivertsen, and Ingvill Merete Stedøy-Johansen organised a workshop about *Math clubs – a door opener to mathematics*. At the Norwegian Center for Mathematics Education they have an ongoing project developing math clubs and are investigating several aspects of the effects of participation among the children and youngsters. Two master studies with different focuses on the math clubs were presented briefly in the workshop. The math clubs have been designed for pupils 5, 9 and 14 – 15 years old. In the workshop the focus was on the target groups for the master studies, 5 year olds and 14 – 15 year olds. The stage of the math clubs was set, and the theory behind the studies was presented, as well as the main results. The audience took part in the activities, and was invited to try to characterize the tasks and challenges the pupils are given.

CONCLUDING WORDS

In her closing plenary lecture at CERME 4 Margret Brown (2006) offered three categories for research:

- Basic research – aims to develop knowledge and understanding without any immediate practical outcome;
- Strategic research – aims to inform practice and policy but not necessarily at a detailed level of implementation;
- Applied/practice-based research – aims to develop products/artifacts/ processes which have an immediate use.

In what categories would the research presented at Norma 05 fall? The theme of the conference – *Relating practice and research in mathematics education* – was intended to focus on how different practices in preparation for learning are related to teachers' views on the learning of mathematics, and how these views are related to basic theoretical aspects of mathematics education. Thus it is no surprise that many of the papers deal with what must be called applied or practice-based research. There are at least seven papers that focus on teachers and the practice of teachers and five papers are studies of teacher education or student teachers. The theme obviously inspired many of the contributors to present work closely related to it. There is also a handful of papers on collaborative work in groups and another handful of papers dealing with specific concepts in mathematics such as probability, algebra, and limits. We also find papers on gender aspects, the use of textbooks, students' beliefs and on learning of mathematics. Most of the studies are investigating grown ups, a handful of studies focus on lower secondary pupils, a few on upper secondary and very few on younger pupils. Is this a consequence of the conference theme or is it a general trend to study older learners?

How will all these studies influence mathematics education in the future? Will it improve the teaching, learning and assessment of mathematics? Will it help to build positive attitudes to mathematic and to its application (Brown, 2005)? In her presentation Margret Brown discussed some of the problems of being involved in policy-related research and development and gave some recommendations for action. She points to the need for researchers to become involved in policy-related work but also to the difficulty to hang on to one's professional and personal intergrity and academic values when doing so. Are we in the Nordic community of mathematics education researchers too far from the policy-makers in society in order to influence the development strong enough? I suggest that one part of the Norma 08 conference will investigate the outcome of the research from our community, when it comes to influencing societal development. That would need some preparations to be done in the coming years.

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Plenary presentations

STUDENTS' GOALS IN MATHEMATICS CLASSROOM ACTIVITY

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This paper outlines an inquiry into the goals students hold in their 'regular' mathematics classroom activity. The reason for exploring students' goals as a means to understanding the classroom is provided and a theory of goals based on the works of Stieg Mellin-Olsen and Walter Doyle is developed. This theory is then worked into a framework of cognition in practice, just one of three theories of cognition that are used within the inquiry in which a principle that Hans-Georg Steiner has described as 'complementarity' is applied. There follows a brief introduction to the research methods and a discussion of some of the outcomes from a year-long ethnographic style case study of a regular mathematics class comprising 14–15 year old students.

INTRODUCTION

In this presentation I want to show how a study of students' goals in 'regular' classroom activity has the potential to reveal fresh insight and understanding of, what to the most of us is, a very familiar place. I will illustrate this with data that arose from my own ethnographic-style case study in which I joined a year ten mathematics class for every lesson for almost one complete year. However, before I can do this I need to spend some time developing part of the theoretical framework which was crucial in bringing fresh illumination. I want to offer, briefly, my principal arguments for the attention I give to theory, for the particular theory I use, and the construction of a theoretical framework that uses apparently inconsistent theories of cognition. This account of my study is also a story of a personal journey that led me into fresh and challenging insights into teaching and learning mathematics. Hence, I commence this presentation by outlining my point of departure.

THE START OF A PERSONAL JOURNEY OF RESEARCH INTO MATHEMATICS CLASSROOMS

My introduction to educational research followed many years of professional practice as both teacher and teacher educator. Like many in the mathematics education research community, I came as a practitioner looking to research for answers to questions that arose from my experience, rather than as a theoretician seeking to validate, test or refine the theory I was espousing. The basic question that I

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wanted to ask was: ‘*what on earth is going on in my classroom?’*, or, more generally ‘in mathematics classrooms?’ This might appear a rather naïve question, and surprising after 20 years of practice but I believe it will resonate with other experienced teachers. I observed students who did not learn from my well prepared and, in my opinion, imaginative lessons and I observed students learning things that I was sure I had never taught. Also, rather humbling, I observed trainee teachers that I was supervising engaging with classes with apparent effectiveness and ease using approaches that I would never even consider – or sometimes had tried but without success. Despite all my experience, the mathematics classroom remained a mysterious place, and yet, rather perversely I had learnt to experience it as one of the most ‘familiar’ places in which to be.

I found it reassuring that my experience was supported by research evidence. For example Brenda Denvir and Margaret Brown (1986) undertook a painstaking study in which they analysed subject content for internal hierarchies of mathematical skills, they carefully assessed each student’s current knowledge and prepared teaching programmes that would match the student’s learning needs to the subsequent points on the content hierarchy. After all their care and effort the results revealed, I quote, “the acquired skills, whilst consistent with the hierarchical framework were not the ones that had been taught” (Denvir & Brown, 1986, p. 153). Other work that coincided with my experience included the research into misconceptions and common errors (e.g. Hart, 1981) that reveals students learning things that were never intended.

‘What on earth is going on in mathematics classrooms?’ does not constitute a research question! A moment’s reflection leads to the slightly more focused question, ‘*what are the students doing in mathematics classrooms that leads to the difference between what the teacher expects to happen and the observed outcome?’*. Then, Richard Skemp helps to focus the question sharply when he asserts “If we want to know what someone is doing, we attach at least as much importance to their goals as to their outwardly observable actions” (1979, p. 2). Thus my attention was led to students’ goals. However, still before a research question could be properly framed it was necessary to engage with the theory and literature on students’ goals, especially in the context of classroom activity.

STUDENTS’ GOALS

There is a large body of literature on the subject of goals, motivation, engagement and other related issues. I reviewed some of this from perspectives of cognitive psychology, constructivism and socio-cultural theories and there seemed to me, as a teacher-researcher, three separate accounts that had the potential to help me frame a research question and unravel some of the mystery of the mathematics classroom. These three accounts came from the writings of Stieg Mellin-Olsen, Walter Doyle and some accounts of constructivism. They were attractive to me because it was evident that they emerged from their proponents’ engagement in classroom learning.

Rationale

Stieg Mellin-Olsen developed Activity Theory, in which goals take a central place, and he proposed a theory of goals based on students' *rationales* for learning (Mellin-Olsen, 1987). He argued that if students are going to learn then they must approach their activity with a rationale for learning. Mellin-Olsen identified two possible rationales for learning, an 'S-rationale' and an 'I-rationale' (1981, 1987). A student has an *S-rationale* for learning if she recognises that the subject matter is intrinsically interesting or of value, S- because the subject matter is, for the student, 'socially significant'. A student is in possession of an *I-rationale* for learning if she sees its value only for some instrumental purpose such as getting good grades or possibly seeking teacher approval.

Purpose

Walter Doyle writes of classes that he has studied in the United States of America in the following terms: "An overriding impression one gets from studying academic tasks in American classrooms is that the curriculum is represented to students as *work*" (Doyle, 1986, p. 371). This view is consistent with many of the classrooms with which I am most familiar and it raises the question: is this representation of the curriculum reflected in the *purpose* that students see in their classroom activity? Do students see the purpose of the tasks given to them by the teacher as about work, that is, the achievement of *production* targets set by their teacher? Alternatively, do students see their purpose to be *learning*, in which meaningful productivity is about their own gains in knowledge skills and understanding. This contrasts with Mellin-Olsen's account because whereas 'a rationale for learning' is about the reasons that students have for engaging in classroom activity, the distinction between learning and production purposes, I believe, belongs to characteristics of the student's activity when engaged in the task set by her teacher; here the student's purpose relates to what she expects to get from the task.

Interpretation

The constructivist accounts of learning focus attention at a different level, that is, at the individual, conceptual *interpretation* made by the student. In these accounts attention is drawn to a possible discrepancy between the 'observed' activity of a student and the impact that the activity might be having on the students' knowledge, understanding or level of competence. For example, Les Steffe and Heide Wiegel observe the possibility that students might have all the appearance of being actively engaged in the tasks set by their teacher but this might not result in learning. They write: "One should not mistake, however, children's activity in the use of their schemes for making modifications in those schemes that might constitute learning" (1992, p. 455). Bent Christiansen and Gerd Walther have also noted the possibility that students, consciously at least, may not be intent on making sense of what they are doing; they write: “blind activity on a task does not ensure learning as intended” (1986, p. 250). Even the intensity of the activity is not necessarily a measure of its effectiveness, as Grayson Wheatley observes when considering the role of reflection in mathematics learning, “It is possible that students may be so active that they fail to reflect and thus do not learn” (1992, p. 536). Thus, I believe, a crucial component in a student’s activity on a task is the student’s own metacognition, which Alan Schoenfeld teases out into three strands: knowledge about one’s own thought processes; control or self-regulation; and beliefs and intuitions (1987, p. 190). For convenience, although I admit the term could be misleading or suggest oversimplification, I refer to this interpretation as *reflection*, and if the student does not engage in reflection then I refer to what they are doing as *blind activity*.

The research question

In the foregoing ‘goals’ have been identified at three distinct levels within the compass of classroom activity. At the uppermost level, as the student enters the mathematics classroom, it is of interest to ask what *rationale* for learning the student has. Then as the student engages with the set tasks, interest focuses on what *purpose* the student perceives in her activity. Then, because it has been noted that activity by itself is insufficient to bring about learning, it is appropriate to inquire after the nature of the student’s *interpretation* of the task. This leads to a formulation of the first research question:

> *What are the goals, rationales, purposes, and interpretations, towards which students work in their regular mathematical activity?*

The question above is rooted in theory and answers should help to gain an improved understanding of what is happening in the classroom, especially, if the methods adopted manage to elicit the answers from the perspective of the students. However, it opens up another question: *what do I mean by regular mathematical activity*? If, ‘regular mathematical activity’ means *the activity that goes on in ‘ordinary’ mathematics classrooms*, that is, without any special provision made for research activity, it only leads to yet another question: *What is an ‘ordinary’ classroom?* No two classrooms are identical, each brings together a unique group of students and teacher each with their own histories of engaging in mathematics and each person, relating uniquely to other people in the room. Thus to address the first research question it is also necessary to formulate another crucial question that relates to the characteristics of the classroom in which the exploration of students’ goals takes place. For this a theory of activity or practice in classrooms is required.

DEVELOPING A THEORETICAL FRAMEWORK FOR CLASSROOM RESEARCH

Apart from the need to place the first research question into context and thereby make it meaningful there are other compelling reasons for classroom researchers to work from a well defined theoretical framework, here I will summarise three of these. But before going any further I want to make it clear that when I use the word classroom I am not merely referring to the physical space enclosed by walls, windows and doors but the whole cultural, historical, social entity including all the resources, participants, and the activity that goes on therein. It is within this social, cultural, temporal and material context that, at the micro level, the individual student is situated, and it is the student who is intended to learn from the activities set by the teacher.

Seeing beyond our preconceptions

The first and most compelling reason for having a well-defined theoretical framework, for me as a *teacher*-researcher, is that *theory offers a means of seeing beyond my own preconceptions*.

Developing a coherent science of classroom research

The second reason comes from a belief that one of our roles as researchers is to make a contribution to the 'science' of our discipline. Without careful attention to theory how can we hope to make a meaningful contribution to the development of a coherent understanding of those things which matter to us?

Respectable theory transcends common sense

A third reason for asserting the importance of theory is that a 'good' theory should help researchers understand what is going on in the classroom, in the words of Andy diSessa "respectable theory … transcends common sense" (1991, p. 226). Moreover, as Richard Pring argues, theory is inseparable from practice and "to think of practice apart from theory … is to create (a) … false dualism" (2000, p. 78).

Teachers have said to us in our work in the KUL-LCM[1] project at Agder University College, that they want us to 'give' them some ideas for their classes. I would rather share some theory that helps them to see what is happening in their classroom in a new light so that their practice develops, not on the basis of discrete ideas 'given to them' but, on their own imaginative responses to their students' learning needs. Good theory has the potential to empower teachers to develop their own practice. The teachers know their students, they know their resources, they know the culture

[1] LCM is supported by the Research Council of Norway within its KUL programme 'Kunnskap, Utdanning og Læring' (Knowledge, Education and Teaching: Norges Forskningsråd, project no. 157949/S20)

of teaching and learning in their school, they are in a much better position to know what will work. However experienced we didacticians[2] are in classroom work we cannot match the knowledge teachers have of their own situation that is crucial in the adaptation of any 'good' idea if it is to 'work' in their classrooms. As Alan Bishop notes:

> My own response to the challenge of the complexity of the classroom is not to seek salvation in the textbook full of ideal lessons, … but to seek better ways to understand the classroom. It is only complex because of our ignorance and if we could understand it better, if we could interpret it more richly, then perhaps we could learn how to handle it better. (1985, p. 25)

I assume in the above 'we' includes all of us who at sometime teach mathematics in classroom situations. So my third reason for claiming the importance of theory is that it is necessary to enlighten understanding about how students respond in classrooms. Thus, one goal of our research effort is to produce evidence that stands the chance of convincing teachers that the theory we didacticians espouse is valid, reliable and has practical implications for their practice.

A theoretical framework

Establishing a theoretical framework for learning in classrooms, into which a theory of goals can be woven, is far from easy. There are two major difficulties to address. The first concerns the complexity of the mathematics classroom as a context for learning. The second arises from the variety of theories, and sub-theories available. I will address these in order.

Alan Bishop is quoted above reflecting on the complexity of the classroom and I have recounted my own personal experience. I will only add to this a quotation from Paul Ernest that hints at the nature of this complexity.

> … the mathematics classroom is a fiendishly difficult object to study. For the mathematics classroom involves the actualised relationships between a group of students … and a teacher … with a variety of material and semiotic resources in play within a set of temporally and geographically delimited spaces. Furthermore, each student, teacher, classroom, school and country has a life history with antecedent and concurrent events and experiences which impinge on the thin strand chosen for study within all this complexity: periodic mathematics lessons. (in Goodchild, 2001, p. 7)

[2] Didacticians are people who have responsibility to theorise learning and teaching and consider relationships between theory and practice. In the KUL-LCM project we refer to the university educators as 'didacticians' in order to recognise that both teachers and didacticians are educators and both can engage in research. I find the term particularly appealing because it reminds me that I am, beneath the researcher exterior, a teacher who is concerned with the quality of teaching and learning.

Second, the array of theories used within Mathematics Education is quite bewildering. In a broader study into the development of the discipline of mathematics education Anna Tsatsaroni, Stephen Lerman & Guo-Rong Xu (2003) looked into the theories used by authors publishing papers in PME[3], ESM[4] and JRME[5] over a 12 year period 1990–2001. They identified over 8 broad areas of theory such as: traditional psychological and mathematics theories; psycho-social, including re-emerging theories; sociology, sociology of education, socio-cultural studies and historically oriented studies; recent broader theoretical currents, feminism, postructuralism and psychoanalysis; etc. It can be seen from these brief headings from their list that each of the areas in fact includes a number of discrete sub-theories within its compass. However, in the context of the complexity of the classroom the consideration of a multiplicity of theories available holds the potential for the development of a theoretical framework that matches the complexity of the classroom.

The principle of 'complementarity'

I believe I need theory that accounts for individual cognition, also I need theory that explains cognition within socio cultural practice and I need something in between these two extremes. That is, I also want a theory that enables me to analyse the engagement of students in activity mediated by the range of tools and resources at their disposal. The idea of using a multiplicity of theories in the context of mathematics classroom research is certainly not new, for example Hans-Georg Steiner argues for a theory of mathematics education based on a principle of 'complementarity' that he explains, in part using the words of H. H. Pattee "the principle of complementarity requires simultaneous use of descriptive modes that are formally incompatible. Instead of trying to resolve apparent contradictions, the strategy is to accept them as an irreducible aspect of reality" (H. H. Pattee in Steiner 1985, p. 15). I will outline how I construct a theoretical framework that enables one to take the three perspectives of the classroom I want.

However, before going further I need to put the record straight on one matter. In my abstract for this lecture I suggested that Stephen Lerman uses the metaphor of a 'zoom lens' to explain the application of different theories to examine classroom activity. In preparation for this lecture I went back to his writing where he introduces the metaphor of a 'zoom lens' (Lerman, 1998, 2001), and followed this by personal correspondence with him. I now have to admit that my abstract seriously misrepresents Lerman's position. Lerman is consistent in working within a theory of 'cultural, discursive psychology', he does not, as I have suggested apply a number of theories. The 'zoom lens' refers to the researcher's focus on different 'views' of the classroom: micro (individual student), macro (classroom culture) or meso (a

[3] PME: The International Group for the Psychology of Mathematics Education

[4] ESM: Educational Studies in Mathematics

[5] JRME: Journal for Research in Mathematics Education

variety of views in between). Lerman's approach is quite different from what I am suggesting here[6].

A theoretical framework taking three complementary perspectives

I do not think it is arguable to assert that the generally accepted function of mathematics classrooms is that individual students will learn facts, concepts, skills, strategies and personal qualities (D. E. S., 1985). Further, it is the explicit intention of mathematics education that the mathematics learnt can be used and applied in situations far removed from the classroom. For this I want a theory of learning that focuses on individual cognition, the perspective I choose is social-constructivism (e.g. Ernest, 1997), and I give special attention to metacognitive activity. At the macro level my focus is on the student participating in a community of practice, here meaning and learning are appropriated through engagement in the cultural and social practices of the classroom. At this level a theory of practice, or situated cognition (Lave, 1988; Lave & Wenger, 1991; Wenger, 1998) provides the perspective I need. Between these two extremes, at a meso level is the student engaged in the tasks and activities set by the teacher and here it is Activity Theory (e.g. Engeström, 1999) that provides a useful perspective.

I want to stress a point made above, it is not my intention to create a single theory from these perspectives and I also want to emphasise their basic inconsistency, which would make any attempt to weave them into a single theory incoherent. To make the point, consider, for example, what 'learning' means from each theoretical perspective. From the constructivist position learning is about the individual interpretation of experience that results in the construction and adaptation of mental schemata. From the position of activity theory learning is about the internalisation of socially mediated activity. From the perspective of a theory of practice "the primary focus … is on learning as social participation" (Wenger, 1998, p. 4). Teasing out all the inconsistencies between these theories is beyond the scope of this lecture. However, the question might be asked '*why not just take one of the theories and extend it to include aspects of the others*?' My response to this is in the form of another question: is cognition about construction, internalisation or participation? Different processes are implied by each of these and I question why it should be one or the other. I think we learn in a variety of ways, and although I do not believe that behaviourism should be considered as a productive theory for

[6] In one sense I do the same as Lerman and focus at different levels but at each level I ask the same questions but from different theoretical perspectives. What does a social constructivist theory tell me about a student's learning? What does activity theory tell me about a student's learning? What does a theory of practice (or situated cognition) tell me about a student's learning? These are different theories, I do not try to compound them to make a single complex theory that somehow matches the complexity of the classroom, rather I treat them separately to provide complementary views from different perspectives. This is not so much a zoom lens as (to continue the metaphor) setting up the camera in different positions.

mathematics education (see for example Ernst von Glasersfeld's condemnation of behaviourism in von Glasersfeld, 1987; also Steffe & Kieren, 1994) it seems from my observation that it is alive and well in the minds of our students!

There is only space here to consider the classroom from one of these theoretical perspectives and how the goals identified fit into it. Even in this partial account it will be necessary to leave significant gaps, I will attempt to draw attention to these gaps as I proceed. Because, I hope, it will be more novel and provocative I will consider the classroom from the perspective of a theory of practice.

A THEORY OF PRACTICE – THE MACRO PERSPECTIVE

Cognition in practice

Within this theoretical context Jean Lave explains "'cognition' in everyday practice is … distributed – stretched over, not divided among – mind, body, activity and culturally organized settings" (1988, p. 1). Later in the same volume Lave goes on to explain that "… 'cognition' is constituted in dialectical relations among people acting, the contexts of their activity, and the activity itself" (1988, p. 148). It is these 'dialectical relations' that become the unit of analysis from this theoretical perspective. Lave then offers a diagrammatic representation of these 'dialectical relations', which I have adapted in the context of the mathematics classroom (see fig. 1).

In the diagram the arrows represent the dialectical relations which I interpret as the interaction between the elements named at the end of each arrow as they exist in a continual process of mutual reformation and development. There are two fundamental parts to the diagram, the classroom as experienced by the students and teacher, this is represented in the lower two thirds of the diagram, and the constitutive order represented by the top third. There is, in my account, a further level that is not represented in this diagram, which is the level of the individual student observed from the perspective of social-constructivism. This omission in Lave's model should not be surprising given the inconsistency of the theories. However, as I remarked earlier, in this account I will consider only the macro level of description and analysis.

The Constitutive Order

Notice first that the part representing the constitutive order is enclosed within a rectangle to signify that this aspect of classroom practice is not directly influenced by the day to day conduct of the participants within the classroom but, never-the-less the vertical dialectical relation with the experienced classroom does exist. What happens in the classroom is to a large extent determined by the structure of the national education system, which is interpreted through the curriculum and assessment structure, and at a local level through funding decisions that take place both outside school, and budget and resource decisions within school. Additionally the design of the scheme of work, tests and so on take place in arenas, such as department meetings, which are removed from the classroom. Decisions taken outside the

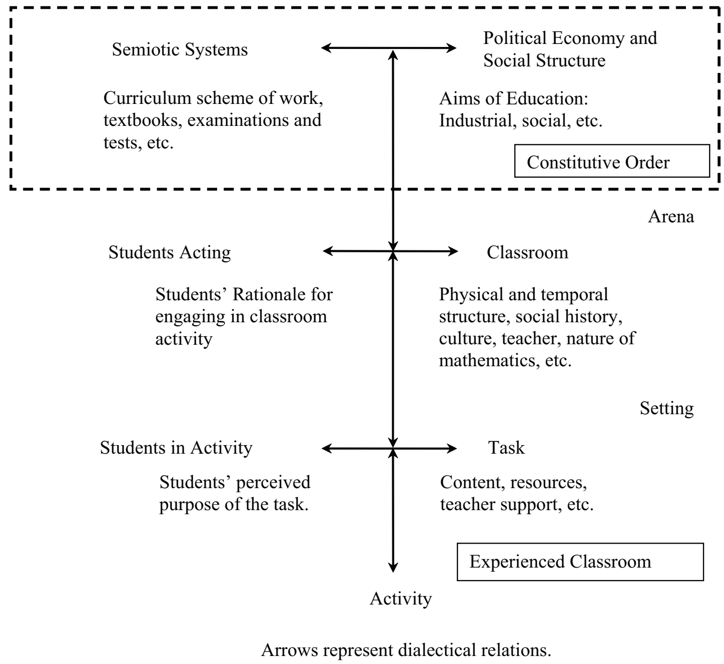

Figure 1. A theoretical model of practice in mathematics classrooms, based on Lave's model of cognition in practice. (1988, p. 179).

classroom can have a major impact on what is possible inside. For example, within the KUL-LCM project at Agder University College the participating teachers from the 'videregående skoler' (upper secondary schools) appear to be very conscious of the pressure of time on them, experienced through the curriculum demands and assessment requirements, and consequently they are rather more cautious than the teachers in the 'grunnskoler' (primary and lower secondary schools) about trying out new ideas.

Decisions regarding the national curriculum will be made in the light of political, social and economic needs. For example, in the UK, significant changes have taken place within the curriculum as a result of a political agenda to 'raise standards' which is interpreted narrowly as improving student performance in written tests and examinations. As a result there have been radical changes especially with regard to the prescription of 'approved' or strongly recommended teaching approaches (See, for example an evaluation of the impact of the 'National Strategy' in the primary curriculum: Ofsted, 2003). The impact of national policies is experienced within the daily routine of the classroom although it may only be when someone stops to question why things exist as they are that the pressure is realised. However the arrow in the diagram does go both ways, the relationship is dialectical. As, for example in Norway over the last few months there has been work on a new curriculum for schools which has included a period of consultation including teachers.

The point that I am making is: many of the features of the classroom are decided before the participants come together, and the influence they have on these features is rather weak and indirect. In their daily routine they do not choose the textbook, the time of the lesson, what lesson it follows, the room in which it is taught, the school or department policies, for example, on calculators or homework, possibly even the seating arrangements within the class.

The experienced classroom: 1. The Arena

Consider, now, students and teacher coming together for a lesson, what do they experience? That is, the attention turns to the level of the classroom as ‘arena’. As students, and teacher, enter the specialised social arena of the mathematics classroom their experience is shaped by a very wide range of cultural practices, social interactions, material resources and expected modes of behaviour. To provide a full picture of the classroom ‘arena’ it would be necessary to describe physical features of the room, the resources available and used, what events immediately precede the lessons, what time of day the lessons are scheduled. It would be necessary to describe the texts used by the students and teacher, both the main text and the supplementary texts used occasionally or just for a specific topic. It would be necessary to describe the way the teacher manages the curriculum content, approaches her task and the actions she takes to ensure effective learning, maintenance of good order within the classroom, and the way the students respond to her approach and management. As Sally Brown and Alistair McIntyre’s (1986) research exposed, teachers see …

> their *actions* as instrumental to, firstly maintaining some *normal desirable state of pupil activity* and, secondly, promoting some kind of *progress*. But the actions they took and the standards they expected of the activity and progress, were determined by the *conditions* they perceived to be impinging on their teaching. (Brown, 1989, p. 4, italics in original)

Thus it would be necessary to know what the teacher believes to be a ‘normal desirable state of pupil activity’ and what conditions she feels constrain their practice.

It would also be necessary to describe both the language the teacher uses when explaining mathematics and the way that she orchestrates her explanations with questions, both at whole class and small group or individual levels. It would also be necessary to give some detail about the range of mathematical content experienced by the students and how they are led to evaluate their own performance through tests and other feedback. All of these features and more are part of the students’ participation and, in Wenger’s terms the ‘reification’ (1998) of mathematics classroom practice; that is, the process through which students can realize and understand the mathematics classroom in some tangible or concrete sense.

In summary, every feature of the mathematics classroom, especially the activity that takes place therein, contributes to the students’ understanding of what it means to engage in *mathematics classroom practice*.

Students' goals at the level of the arena

Students come into the classroom arena with expectations about what the practice of the arena is about. In the description of goals that I have described earlier it is at this level that interest focuses on students' rationales. I maintain that students' expectations inform their rationale for engaging with the tasks set by their teacher. Further, as has been explained earlier, Mellin-Olsen asserts that a prerequisite for effective teaching and learning is that students enter the classroom with a rationale for learning. The assumption that I am making is that students' rationales will be in dialectical relationship with the other features of the arena. It seems fairly obvious that their rationales will be shaped by the characteristics of the arena, but their rationales will also contribute to the conditions that impinge on the teacher's actions and thus have a direct influence on the arena.

The experienced classroom: 2. The Setting

At the next level in Lave's model is the 'setting', that is the level at which the student engages in the tasks set by the teacher. It is at this level of participation that the practice is 'reified', which I will interpret as, becomes meaningful, in the students' activity. As with the description of the arena there are many issues that characterise and influence students' activity in their set tasks. These include the way in which the teacher intervenes in the work of individual students and the manner in which help and support is given. Also the interactions between students and the characteristics of their mutual engagement in tasks, these could be described in a number of ways, such as collaborative, peer tutoring, scaffolding, and so on. The way in which students use available resources to provide structure to their activity is also a feature of the setting, this is an important feature in learning mathematics as Celia Hoyles observes, albeit from a constructivist perspective: "Different tools lead to different representations of concepts; they evoke different cognitive structures and support different reasoning patterns" (Hoyles, 1990, pp. 124, 125). Also, the actual mathematics content experienced through their tasks and activities forms a very significant part of the students' setting.

Students' goals at the level of the setting

It is at the level of the setting that I find the notion of student's purposes, which has been developed from Doyles' work, of value. The manner in which tasks are presented to students will have an impact on what they perceive to be their purpose in those tasks. Conversely, the students' expectations, manifest through their perception of purpose will be yet another condition that constrains the teacher's actions. Sensitive teachers will try to match activities to students' expectations, and, the teacher will seek ways of modifying students' expectations if she believes they do not fit into a 'normal desirable state'.

THE RESEARCH QUESTIONS

I am now, in a position to state the second research question, recall the first is:

> *What are the goals, rationales, purposes, and interpretations, towards which students work in their regular mathematical activity?*

The second I express as:

> *What are the features of the classroom, arena and setting, which comprise the socio-cultural context in which students engage in mathematical activity?*

Research methods and data collected

Before discussing the data that arose from my attempts to address these questions it is necessary to describe the methods adopted. Because I was using theories that emphasise the cultural and social nature of learning, cognition and meaning it was necessary to explore students' activity as close to the 'natural' situation as possible. To achieve this I joined a 'regular' class for every mathematics lesson they had for very nearly one complete year. I only missed the lessons in the first few weeks while the teacher established her routine. Then during the course of each lesson I would observe whole class episodes and when students were engaged in their individual or small group activity I would observe, listen and eventually engage students in conversation about what they perceived to be the nature and purpose of their activity. I made audio recordings of both the whole class episodes and the time spent with individual students. At the end of the period of field work I conducted an extended interview with the teacher. The data did not emerge in any well structured form and the analysis required interpretations to be made on the basis of multiple pieces of evidence, that is not only what the students said to me in the totality of the conversation but also using their conversation with other students, their written work, and their appeals for help to the teacher, test scripts, self-evaluations completed in the normal course of their work and a variety of other contextual information. The following presentation of data will be necessarily brief, a full account is available in the monograph of the research (Goodchild, 2001).

In the remainder of this presentation I will use data that arose from my time with the class to present a picture of the classroom arena, that is, a partial answer to research question 2, and, as far as possible, this will be done through the teacher's own words. I will follow this with a discussion of the rationales that I was able to deduce from the conversations and to support this I will reproduce some very brief utterances from the students. Unfortunately space prevents me describing features of the setting that would offer a comprehensive answer to the second question from the perspective of a theory of practice. However, I will offer a discussion of the students' purposes at the level of their setting, and this again will be illustrated using the students' own words. I will then use these elements to try to unravel some of the mystery of one mathematics classroom from the perspective of a theory of practice.

Throughout my work if describing an individual person, either in the abstract or to protect identity I use female pronouns. This is a stylistic device and should not be taken to refer to either sex. When I refer to particular students I use pseudonyms that match the sex of the student, and then use whichever pronoun is appropriate.

The Classroom Arena

The overriding impression I had of the arena was that there was a very high proportion of individual and small group teaching performed by the teacher circulating the room. I asked the teacher why 'she' did not employ much class teaching, preferring to engage with small groups of students repeating the same things to each group, I suggested that it seemed as if she 'avoided' whole class teaching. She responded:

> I quite enjoy whole class teaching but it's very difficult in a class that has a broad spectrum of ability … I do, do quite a lot of class teaching because [when] their ability is very similar, it's very easy to do, but with that class it's very difficult. But I wasn't avoiding it [whole class teaching] because … I felt at some times it would really be counter productive in the way it worked, and a lot of them [i.e. the students] would have just sat there and not, either taken anything in, or not been prepared to ask, whereas on an individual level they're much more prepared to ask. [I] run around like anything in the class which can be a pain but at least then they're saying look can you help me, look am I doing this right? What do I do now?

From a careful analysis of this extract it would be possible to make inferences about the teacher's beliefs and attitudes regarding teaching and learning, about learners and communication. I include the passage to provide a surface picture of life in the classroom where the teacher spent most of her time moving from student to student or group to group providing almost individual instruction. When I first went into the classroom I was struck by a fairly high level of talk 'off task' and I reflected in my field notes

> Does a degree of 'idle' chatter and lack of attention to task actually enable the teacher to fulfil (her) role? If they all worked hard, all the time, they would demand (T's) attention a lot more, (T) would not manage it. There appears to be a tacit understanding between students and teacher. Closer attention to task would require a different approach to problems/difficulties. (Field notes, 11.10)

My conversations with students revealed that they perceived her as being the only authority in the classroom and the only resource that could help them when in difficulty. Consequently the demands on her for individual attention were very high.

On the nature of the mathematics experienced by the students the teacher remarked:

> …it was almost as if we were doing a lot of stuff in isolation and there wasn't any build up of skills and quite often, I don't know, a term and a half would pass 'oh yeah, you remember doing that, remember we did decimals nine months ago, well we're going to use it again today'. And I felt myself '… they can't remember this, having to go back over stuff' – and there's nothing wrong with that but having to go back a long, long way and then suddenly finding ' … I've only got three weeks to do this, this and this". Whilst the intention is good in practice it didn't quite work – and there's this conflict.

This reflection by the teacher is certainly consistent with my own account of the classroom, and with the mathematical content that the students experienced, to which I will come later. To the above I should add that the scheme of work divided the material to be covered during the year into blocks of work that each covered five or six weeks, each block concluding with a test. The dates for the tests were published at the start of the year, these seemed to result in the teacher feeling under continual pressure to get through the prescribed content, here in a very direct way we can see the impact of the constitutive order on the classroom arena.

I also want to add to this from my own observation that the teacher did achieve a sense of coherence through the way that she responded to what I believe were the students' expectations. Her management style was to be generally moderate, she rarely needed to assert her authority, and she was effective in maintaining attention and students' engagement with the tasks she set, within the manageable limits that I have described earlier. Thus, in my opinion she did achieve a '*normal desirable state*' within the '*conditions'* of her mathematics classroom. Peter Woods, in his extensive ethnographic study of schools (Woods, 1990), writes that students expect their teacher to maintain order, to set work, to make the work easy, and if possible, to be 'good humoured'. The teacher in my study achieved all of these and I believe the students both liked and respected her. Her approach to making the work 'easy' was through a process that Walter Doyle describes as the familiarisation of content (1986, p. 373). Added to the teacher's remarks about the fragmentation of the mathematics programme her approach to novel tasks was to break them down into small routine steps that were of fairly low demand, often routine calculations that would normally be carried out on a calculator, and possibly measuring tasks. The effect of this was noticeable, for example when I asked a student what he believed to be the purpose of an algebra exercise in multiplying out brackets, he responded 'practising multiplication tables'. On another occasion, when the topic was about making sense of different views of three-dimensional objects a student told me that he believed the task was about measuring and drawing.

Rationales for learning

In the conversations and other data I sought evidence of students' rationales. I will outline separately students' S-rationale, I-rationale, and a further significant rationale that became evident to me through the study.

S-Rationale

I could not identify a rationale for engagement in every encounter or conversation with a student but in about half of those occasions when a rationale was apparent it was characteristic of an S-rationale. I reproduce some of their statements, but it must be remembered my interpretation was made using all the evidence relating to the conversation so the statement alone may not be totally convincing.

> … so if you're an engineer … [I – what you're doing here might help you …?] Think so (Alan, tape 28)
>
> Well I want to be an engineer (Alan, tape 50)
>
> If you work in shops (Ben, tape 6)
>
> If you're working a till and that … (Ben, tape 17)
>
> Horses like, if you put on like horses … (Ben, tape 17)
>
> If you have stuff like D.I.Y. (*i.e. Do it Yourself – home improvement and maintenance*) … not in my job (Eddy, tape 29)
>
> [I- Is it important to you whether the maths is useful?] Yes, otherwise there'd be no point in doing it – I'd rather do another subject. (Eddy, tape 29)
>
> … like D.I.Y. (Eddy, tape 56)
>
> doing some work at home (Eddy, tape 73)
>
> It depends really if you got sort of job you might need … like my dad – electrician (Gary, tape 35)
>
> … you need maths … have to do equations in science (Gill, tape 11)
>
> It's good if you're working in a shop (Jacky, tape 51)
>
> Need to angle the weapon you're shooting (Neil, tape 30)
>
> [I – So have you used this before?] Yeah, but on my computer at home. (Oscar, tape 32)

Some observations on these brief extracts: first, note that the student's belief from which the rationale was expressed appears quite stable over time we see Alan relating the mathematics to the work of an engineer, Ben to working in a shop, Eddy to work in the home, Neil to a career in the army, and so on. Also note that the rationale was often related to employment, usually in the context of the students' own aspirations or their parents. Eddy said he wanted to join the Police Force and could not see any value in mathematics for that but could see some value in the context of home improvement. Students rarely gave a reason for engaging in the practice of the classroom because the mathematics was useful in the context of their concurrent activity outside school, in this respect Oscar's comment, made when the topic was Pythagoras' rule, is one of these rare occasions, I also suspect that Ben's comment relating work on ratio and proportion to gambling on horse races might be related to his concurrent interest outside school.

I-Rationale

In about one quarter of the encounters where a rationale was discernable it could be interpreted as an I – rationale. Some of the statements made by students were:

> In case we're going to have it in our exam (Chris, tape 12)
>
> Helps you in the exam (Pam, tape 48)
>
> Get a good grade in my exam (Pam, tape 68)
>
> Because that's the way they're going to do it in an examination … that's the way they teach you throughout your education (Harry, tape 75)
>
> Maths and English is what they ask for in most jobs (Irene, tape 19)
>
> To revise it for my GCSE's (Irene, tape 36)

Here we see the predominance of statements relating to examinations and tests, which Irene relates to future employment prospects. Harry's comment 'that's the way they teach you throughout your education' appears to be projecting an instrumental rational on the whole process, which given the political rhetoric that accompanies educational reform in the UK seems to be an understandable position to take.

Another rationale

In the conversations with students I often encountered students who were engaging in the tasks set by the teacher, but they did not reveal either of the rationales for learning. They expressed a rationale that related merely to the practice of the mathematics classroom – they engaged in the activity, apparently for no other reason than, because this is 'normal' and therefore expected in the arena of the mathematics classroom.

> We should work hard as the same as all things (Ann, tape 48)
>
> Because (the teacher) wants us to (Ben, tape 39)
>
> I just thought hurry up and finish it, so I didn't really bother … gives more time not to do anything (Tina, tape 52)
>
> [I- What is the point of copying?] so I don't get into trouble for not doing my work (Claire, tape 57)
>
> I want to get the work done (Gary, tape 6)
>
> [Why do you think you are doing this work?] don't know (Fiona, tape 16)
>
> 'cause 'she' (the teacher) said so' (Fiona, tape 37)

> [why do you think you are doing this work?] don't know really … [I – you always work hard, why?] don't know. (Gill, tape 26)
>
> We've done some work, it's all wrong but we've done some work (Jacky, tape 44)
>
> It's what I do (work) when I come to maths lessons (Leon, tape 67)
>
> [why do you think you do it?] Got to, takes up the lesson (Lisa, tape 9)
>
> Because we have to … I don't know, no reason, just got to (Roger, tape 33)

These statements were made by students who were actively engaged in the tasks set by the teacher but they seemed to approach their activity without any discernable rationale for learning. So what was their rationale for engaging? This question has led me to propose an additional rationale which I call a P-rationale: they engage in the activity of the classroom because, simply put, this is what is expected of them in this arena. In other words, their rationale is about adopting the behaviour of the practice, therefore I label the rationale P- for practice. It should be noted that it is different from the S- and I- rationales proposed by Mellin-Olsen because the P-rationale that I propose is *not* a rationale for learning.

The classroom setting

As I indicated earlier, there is insufficient space to offer a comprehensive account of the classroom setting here. The setting is the link between the publicly experienced arena and the private domain of each student's conception. Characteristics of the setting include the nature of the classroom discourse arising through interactions between students and teacher, and between students themselves. Additionally the setting is distinguished by other features of the classroom that the students (and teacher) choose to use as 'tools' (or mediating artefacts) to resource and structure their activity. A key element of the analysis of the setting is to explore how the mathematical content of the lesson is experienced by the students. As indicated earlier the detail of this part of the inquiry can be found in the monograph, Goodchild (2001). The discussion here moves on to students' goals at the level of the setting, which I have referred to above as their 'purposes'.

Students' Purposes

Consistent with Doyle's impression of classrooms in the US (Doyle, 1986) my observations of the classroom I am describing here revealed that the curriculum was presented as work. Activity was presented to the students as 'work' to be done and production targets to be met, very often with the additional rider inserted by the teacher 'for me'. Thus students were presented with tasks to be completed not with the intention to facilitate learning and that they would be the beneficiaries but rather in the way that work is presented to factory employees.

Learning purpose

As with identifying rationales the identification of purpose was based upon all the available evidence relating to a conversation, not just the brief statements that I reproduce here. In less than 40 % of the encounters in which a purpose could be discerned did the student identify learning as their reason for engaging in the task. Typical responses included:

> [I- so why are you doing it?] to learn (Ben, tape 6)
>
> To understand doing equations (Chris, tape 35)
>
> To learn … how to learn this … remind how to do it. (Claire, tape 65)
>
> To learn about this … I know how to do the sums … (Gary, tape 6)
>
> I like learning (Harry, tape 20)
>
> Learning how to draw plans and to use measurement and work things out (Kevin, tape 2)
>
> [I- Why do you think it's worth your while working?] you will learn something (Steve, tape 51)
>
> to make sure that we understand (Mary, tape 57)

As with all the interpretations made from such statements it is always important to be cautious about taking things at face value. There is an expectation that students come to school to learn and that one of the roles of a teacher is to help students to understand. I do not wish to minimise even further the milieu of learning within the class but it has to be recognised that students may make statements like those above as an automatic response. However, amongst all the statements it is Harry's that attracts my attention: 'I like learning'.

Production purpose

By contrast over 60% of the encounters with students exposed that they perceived 'work' to be the purpose of their activity. Typical of the statements made are:

> I copied him from there and I started … from D4, I was away (Ben, tape 36)
>
> [Are you learning anything from it?] No because I've already done these before. (Dave, tape 9]
>
> Get on with my other work and finish it … hurry up and finish it so I didn't really bother (with interpreting the result). (Tina, tape 52)
>
> [What's the point of copying?] So I don't get into trouble for not doing my work. (Claire, tape 57)
>
> Just doing them and get them right … I know how to do the sums … I just want to get the work done. (Gary, tape 6)

> [Why are you copying?] we help each other .. we work better … we do more. (Ian, tape 70)
>
> We could have done twice as much work [I-So do you think the important thing is to do as much work as possible] Yeah, as long as you know what you're doing. (Jacky, tape 11)
>
> Least we can show that we've done some work (Jacky, tape 44)
>
> [I- Are you learning anything from it] no I don't think so. (Mark, tape 29)
>
> I do the hard ones last … you've got all the others to do. If you leave that figuring out till last you've got all the others …'(Neil, tape 57)
>
> [I- Why are you drawing it?] I don't know, no reason, I've just got to …
>
> [Does it matter very much?] No. (Roger, tape 33)

It can be seen that copying from another student, or occasionally from the text is a fairly common activity and this is most often justified as to show that 'work' has been done. However, as Oscar remarks 'when I copy I don't think about what I'm copying' (tape 74) the value of the activity is questionable on any terms. Neil's comment is revealing, it might be argued that learning is most likely to arise by confronting appropriate challenges. Neil's approach to hit some unspecified production target is to leave the questions in which he perceives a challenge until last, probably that means they are never attempted.

Synthesis of students' goals viewed from the perspective of cognition in practice

It is difficult, here, to produce a convincing argument for the synthesis I offer because, given the complexity of the mathematics classroom, even viewed from one theoretical perspective, it is necessary to consider much more of the contributory evidence than I have been able to set out here. However, I will conclude this account on a fairly depressing note by extracting from my summary of the classroom that follows a much fuller description, analysis and interpretation of the evidence available (Goodchild, 2001, pp. 226, 227).

From the perspective of situated cognition it is the 'practice' that is recognised as being 'goal directed' rather than the activity of the students within the practice. It follows that students' statements of goals may only reveal characteristics of their expectations in the practice of the classroom. They 'expect' the mathematical activity to be relevant to employment; they 'expect' to be given 'work to do'; they 'expect' to be required to produce correct answers. Furthermore, students 'expect' to learn from their activity because they are informed that this is the function of school. In this sense the statements of rationales and purposes expose students' expectations regarding the 'practice' they have learnt through years of experience of mathematics classes. If the curriculum is to function then it has to adapt to meet the students' expectations – the consequences of a mismatch can lead to uncomfortable experiences for the teacher, as described by Robert Davis and Curtis McKnight

(1976). From this perspective students' accounts of their activity – and observations of students in activity – are manifestations of their expectations. To paraphrase Sylvia Scribner and Michael Cole (1981) the main outcome of experience in mathematics classrooms is to learn how to *do* mathematics classroom practice. In the classroom, mathematics is set within a specialised 'classroom' discourse, it allows students to locate and follow cues and signals, skip over peripheral text and apply a variety of resources to bring the highly stylised tasks to some form of resolution. Activity within the described classroom is not mathematics and for all its pretence it is not about the students' current or future experience of the world outside the classroom. Success in this classroom practice does not prepare a student for the practice of mathematics or any other activity outside the classroom. Success may, however, reveal a student's potential to participate in a particular type of practice.

CONCLUSION

The attention that I have given to the goals that students expressed, in particular their rationales and purposes has both corroborated evidence about the classroom that was available from other sources and, significantly provided a perspective that reveals, in part, what it was like for the students. In every respect, to a casual observer, it would appear to be a 'regular' happy classroom. The students were, I think without exception, content with their teacher, she met their expectations in her management of the class and the subject. The students entered the classroom with a rationale for engaging in the activity, sometimes this was focused on learning sometimes just on 'playing the game'. Whichever rationale they held it did not undermine the teacher in her role, in fact the rationales they held possibly enabled the teacher to function within the constraints and conditions that she experienced. The students perceived much of their activity as being 'work' and there appeared to be, from their perspective a clear, causal relationship between 'hard work' and effective learning. This could be expected: students are so often exhorted to 'work hard' at their studies. Less often are students informed about how work might be more effective in their learning. To the casual, outside observer this class appeared to function well. My engagement with the class suggests that it functioned socially but not as a place where teaching and learning mathematics were effective. The teacher of the class has read the full report and commented briefly in a telephone conversation that whereas she recognised the class and her own practice, and found the analysis of interest, she felt the students had understood more of the mathematics they encountered than I suggest. Researchers are sometimes led to make interpretations that differ from our informants; this is one such occasion. I note that the teacher engaged with many more students during the course of each lesson, on average each of her interventions lasted less than two minutes. My conversations would last between 15 and 40 minutes. The teacher also gave me access to all the students' records that she maintained. The teacher's contrary opinion has led me to reflect on the interpretation that I have presented, but not to change it.

We seek improvements in students' learning experiences. It is difficult to convince teachers that their well managed classes might benefit from closer examination. I argue that an exploration of students' goals can provide a productive instrument to be used in that examination. It is insufficient to ask what 'what on earth is going on in my classroom?' The key question, I believe, is to ask: 'what do the students think is going on?'

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UNDERSTANDING AFFECT TOWARDS MATHEMATICS IN PRACTICE

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Affect in mathematics classes is a complex phenomenon to study. This complexity needs to be reflected in both the conceptual model and the representational system developed for the study. The present view sees student's behaviour self-regulated and suggests three main concepts for a theory: cognition, motivation, and emotion. Of these concepts, only emotions are directly observable in expressive behaviour. Observed emotions can be used for making inferences about motivation. Student behaviour and verbal expressions provide us with further information about student emotion, motivation and cognition, but these should not be taken at face value, since they are subject to student's conscious regulation. Triangulation of multiple kinds of data provides interpretations that are more reliable.

INTRODUCTION

Human behaviour is a complex topic to study. Reid (1996) has distinguished between an effort to create theories *of* and theories *for* a topic of interest. It is not likely that we can ever develop a single theory of affect in mathematics education that would cover all relevant issues. Instead, we can build theories for understanding affect in mathematics education that can inform practice and future research. One of the choices with the present article is to focus on the level of the individual student. Despite the individualistic approach, one must be aware of the importance of classroom culture, social and sociomathematical norms (Cobb & Yackel, 1996) as well as the social context of the student and the school, but the present view to these issues is to look at how these are embedded and manifested in the individual students as personal beliefs.

This paper will begin with a look at how affect has been approached in research on mathematics education. Then it will discuss some characteristics of good conceptual frameworks for research on affect and elaborate a conceptual framework used by the author. Finally, the paper will elaborate how these concepts can be observed in practice.

There are two main traditions to look at affect in mathematical thinking and learning. The first tradition is to measure affective traits and see their relation to

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achievement. For example, Ma and Kishor (1997) synthesised 113 survey studies of the relationship between attitude towards mathematics and achievement in mathematics. The causal direction of the relationship was from attitude to the achievement. Although the correlations were weak in the overall sample, they were stronger throughout grades 7 to 12, and in studies that had done separate analysis of male and female subjects.

Also more generally, educators are interested in the possible causal relationship between affect and achievement. Based on a literature review of self-concept and achievement in learning, Chapman, Tunmer, and Prochnow (2000) proposed a hypothesis for the developmental trend for their causal relationship. During the first school years the causal relation would be from achievement to self-concept, for next years there would be a reciprocal relationship, and in the upper secondary school the causal relationship would be from self-concept to achievement.

Some evidence for this developmental trend in mathematics has been found (e.g., Hannula, Maijala, & Pehkonen, 2004; Linnanmäki, 2002), but further systematic research is needed to confirm it. Gender differences in self-confidence favouring males have been well confirmed (e.g., Leder, 1995), and some studies indicate that also the relationship between self-confidence and achievement is affected by gender (e.g., Hannula et al., 2004).

Another research tradition has been looking at the affect of an individual as an important aspect of mathematical problem solving. In *Mathematical Problem Solving,* Schoenfeld (1985) defined an individual's beliefs or "mathematical world view" as shaping how one engages in problem solving. Looking at more rapidly changing affective states, Goldin (2000) explained in detail how people may adopt either positive or negative affective pathways. A slightly different approach has been to look at how teaching approaches or characteristics of a problem influence the affective experience of the solver. For example, Liljedahl (2006) describes tasks that allow *Chain of discovery* and sustained engagement. One student teacher of his study describes her experience as follows:

> Of all the problems that we worked on my favourite was definitely the pentominoe problem. We worked so hard on it, and it took forever to get the final answer. But I never felt like giving up, I always had confidence that we would get through it. Every time we got stuck we would just keep at it and suddenly one of us would make a discovery and we would be off to the races again. That's how it was the whole time – get stuck, work hard, make a discovery – over and over again. It was great. I actually began to look forward to our group sessions working on the problem. I have never felt this way about mathematics before – NEVER! I now feel like this is ok, I'm ok, I'll BE ok. I can do mathematics, and I definitely want my students to feel this way when I teach mathematics … (Liljedahl, 2006, p. 232)

In previous research on affect, the typical approach has been to either classify or measure students' affect. Anxiety measures have been used extensively in mathematics education research in the 70's and somewhat later the trend was to identify students' attitudes on a positive-negative dimension (Zan, Brown, Evans, & Hannula, 2006). Educational psychology has been interested in motivation where a typical approach has been to identify between different categories of motivation, e.g., intrinsic vs. extrinsic motivation or classifying motivation into mastery, performance or ego-defensive orientation (Murphy & Alexander, 2000).

Such an approach to classify or measure affect has been shown to have predictive power, but it does not give an understanding of how affect is developing on a personal level – or how to change affect. For example, attitude towards mathematics is often defined as an inclination to evaluate mathematics favourably or non-favourably ('I like ...', 'It is important',..). Such attitude may be affected by situation variables (e.g., teacher behaviour), automatic emotional reactions of the student (based on some traumatising event(s) in the past), expectance of outcome (beliefs), goals of the student (e.g., career aspirations), or social variables (attitudes of the family). Different cause of negative attitude would call for different action, but a single attitude measure would not suffice (Hannula, 2002a).

Research on psychology of emotions in laboratory settings has revealed the dynamics of millisecond-to-millisecond processing of information in our brain. For example, a typical study of *attentional blink*-phenomenon would require subjects to monitor a screen where a series of words would appear one after another, task being to identify two words of distinctive colour and repeat these words afterwards. Attentional blink impairs the identification of the second target word. This impairment is affected by the interval between the words, but also by the emotional content of target words (Anderson, 2005).

The general research on the psychology of emotions has distinguished that there are differentiable basic emotions with distinctive physiology (e.g., Buck, 1999; Power & Dalgleish, 1997). There is still a debate going on about the actual number of basic emotions, but at least happiness, sadness, anger, fear, and disgust are on most lists. Other basic emotions might be, for example, interest and attachment. These different basic emotions are likely to also have distinctive effects on cognitive processes.

Laboratory studies have provided convincing evidence of affective bias of cognitive processing: Anxiety relating to attentional bias and depression to mnemonic bias (Williams, Watts, MacLeod, & Mathews, 1988). Emotions have also an effect on problem solving, but both negative and positive feelings can either facilitate or impede, and the dynamics are not easily understood (Schwarz & Skrunik, 2003).

DEVELOPING A FRAMEWORK FOR RESEARCH ON AFFECT IN PRACTICE

Some general principles

Schoenfeld (in press) has developed a schematic model (Figure 1) of the research process. Mathematics educators can easily identify its similarity to the modelling cycle. Schoenfeld argues that all empirical research involves the following processes:

- initial conceptualization, in which the situation to be analyzed is seen and understood in certain (often consequential) ways;
- the creation or use of a conceptual model, in which specific aspects of the situation are singled out for attention (and, typically, relationships among them are hypothesized);
- the creation or use of a representational/analytic system, in which the aspects of the situation singled out for attention are represented and analyzed;
- the interpretation of the analysis within the analytic model; and
- attributions and interpretations from the preceding analytic process to the situation of interest (and possibly beyond) (Schoenfeld, in press, p. 9).

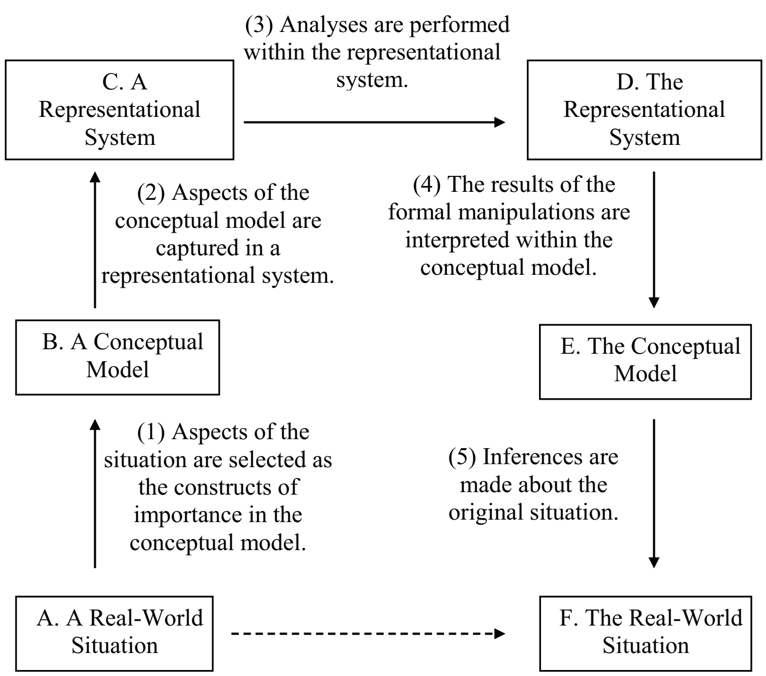

Figure 1. A schematic representation of the process of conducting empirical research (Schoenfeld, in press, p. 9, with permission)

When we try to move from laboratory settings to classrooms, we face a tremendous increase in the complexity of the situation. This requires development of a more comprehensive conceptual model. However, it is difficult if not impossible to create conceptual models that would take into account all relevant aspects of a classroom situation. Furthermore, the more sophisticated the conceptual model becomes the clumsier it becomes as a tool for thinking.

At the same time, the problem of representation (measurement) becomes more difficult as we move out from the laboratory. Only a fraction of affects at play is observable directly as facial expressions. Part of the affective experience is purely subjective, and hence not observable. Unlike in laboratories, we usually can not measure physiological measures like skin conductance and heartbeat frequency nor do repeated measures to minimise the effect of noise.

The basic elements to conceptualise affect

Let us first focus on the requirements for a conceptual model for research on affect in classrooms. The model should include concepts and relations that have earlier been proven relevant, the concepts in the model need to be operationalisable in classroom settings, and the model should be kept simple enough to be intelligible. From the point of view of a teacher, the model also needs to inform practice.

Let us begin with identifying relevant concepts. In research on affect in mathematics education, the research has typically used McLeod's (1992) classification of affect into emotions, beliefs, and attitudes. In my own research, I first made a distinction between cognitive and emotional aspect of affect. Literature on emotion indicated the importance of goals in relation to emotions and thus pointed to the concept of motivation (Hannula, 2004a). Schoenfeld's 'Theory of Teaching-in-context' (1998) includes elements of the same three domains: Teachers decision-making is based on their knowledge (cognitive aspect), goals (motivational aspect) and beliefs (emotional aspect). In addition, some researchers of mathematical beliefs have identified motivational beliefs as an important subcategory of mathematical beliefs (e.g., Kloosterman, 2002; Op 't Eynde, DeCorte, & Verschaffel, 2006).

Coming from a different research tradition, educational psychology, Meyer and Turner (2002) had conducted a study on motivation, conceptualised according to the prevalent tradition to include two components: cognition and motivation. However, as a conclusion of their work, they had identified a need to expand their conceptualisation. They summarise their concern for the limitations of the established frameworks as follows:

> Historically, psychologists have adopted three components to describe human learning: cognition, motivation, and emotion [...]. Yet, theorists and researchers have tended to study these processes separately, attempting to artificially untangle them rather than exploring their synergistic relations in the complexity of real life activities. (Meyer & Turner, 2002, p. 107)

They call for “new theoretical syntheses and research programs that integrate emotion, motivation, and cognition as equal components in the social process of learning” (Meyer & Turner, p. 107). Schutz and DeCuir (2002) also bring up these three basic elements and suggest that goals are “the transactive point for our understanding of constructs such as cognition, motivation, and emotion” (p. 127).

Self-regulation as the systemic frame

Choice of concepts is not enough; we need also to have an understanding of their relations. The approach taken here is to look at human behaviour from the perspective of self-regulation (e.g., Boekaerts, 1999). Zimmerman and Campillo (2003) have characterised self-regulation as “self-generated thoughts, feelings, and actions that are planned and cyclically adapted for the attainment of personal goals” (p. 238). This characterisation is otherwise suitable, but it does not fully apprehend the role of unconscious and automatic self-regulation. If these are accepted, planning can not be seen as necessary for self-regulation.

Boekaerts (1999) outlined the three roots of research on self-regulation: “(1) research on learning styles, (2) research on metacognition and regulation styles, and (3) theories of the self, including goal-directed behavior” (p. 451). Based on these schools of thought, she presented a three-layer model for self-regulation: the innermost layer pertains to regulation of the processing modes through choice of cognitive strategies, the middle layer represents regulation of the learning process through use of metacognitive knowledge and skills and the outermost layer concerns regulation of the self through choice of goals and resources.

Most research has focused on the two innermost layers and little effort has been made to integrate motivation control, action control or emotion control into theories of self-regulation (Boekaerts, 1999, p. 445). Boekaerts and Niemivirta (2000) have proposed a broader view for self-regulation that would accept a variety of different control systems, not only metacognition:

> [Self-regulation] has been presented as a generic term used for a number of phenomena, each of which is captured by a different control system. In our judgment, self-regulation is a system concept that refers to the overall management of one’s behavior through interactive processes between these different control systems (attention, metacognition, motivation, emotion, action, and volition control). … In the past decade, researchers involved in educational research have concentrated mainly on activity in one control system – the metacognitive control system – thus ignoring the interplay between the metacognitive control system and other control systems. (Boekaerts & Niemivirta, 2000, p. 445)

Although the scope of the present paper does not allow elaboration of all the abovementioned control systems, the present view sees self-regulation to be much more than mere metacognition. Most notably, the important role of emotion is acknowledged.

When we look at cognition, motivation and emotion in self-regulation, I would like to use a metaphor of orienteering with a map and compass (Table 1) (at least the Nordic audience knows this sport).

Orienteering is based on the map that is used to identify one's location. A good map also informs about roads, path, ravines and thickets that may ease or impede moving in such a terrain. On this map, there is marked a course, controls that one should find. To help this task, a compass is used to identify the direction of the next control. In self-regulation, cognition provides information of environment and oneself in relation to it (map). A well-informed person also is aware of how easy or difficult certain actions are likely to be for him/her. Motivation gives direction for behaviour; there are individual needs one wishes to satisfy and goals one wants to reach (the course). The role of emotions in self-regulation is to regulate self towards satisfaction of needs. Emotions may influence physiology to adapt to situation (flight or fight –response) or bias cognition according to most urgent needs (anxiety biases attention towards threats). Emotion can be seen to regulate the psychophysiological self towards the direction of the goal (compass).

Domain Concept	**Cognition**	**Motivation**	**Emotion**
Self-regulation	Information about self and environment	Direction for behaviour	Goal-directed self-regulation
Orienteering metaphor	The map	The course	The compass (giving direction)

Table 1. Cognition, motivation, and emotion in self-regulation and in orienteering metaphor.

When focusing on cognition, motivation, and emotion, it is important to understand their rapidly fluctuating state and the more stable 'trait' aspect (Table 2). In the cognitive domain of mind, there is the continuously changing 'landscape' on thoughts, which relates to equally rapidly evolving neural activation pattern of the brain. However, there are also rather stable neural structures in the brain, which enable some neural patterns to activate easily, others with some difficulty, and yet others they disable. These more stable structures are reflected in the concepts, facts, scripts and other schemata that are stored in memory. Likewise, there is the continuously evolving emotional state, which is partially embedded in the neural activation patterns and partially in the biochemical system of hormones and neuropeptides. There is the biologically founded structure of basic emotions in the background, but also an 'emotional memory' that is based on previous experiences. These prime activation of certain emotions in certain situations (e.g., in a mathematics class). For motivation, there are rather stable needs, values, and desires, but also more frequently changing 'at-present-active' goals. These issues will be elaborated more thoroughly later on in this paper.

	Cognition	Motivation	Emotion
State	Thoughts in mind	Active goals	Emotional state
Trait (memory)	Concepts, facts, scripts etc.	Needs, values, desires	Emotional dispositions (attitude)

Table 2. State and trait aspects of cognition, motivation, and emotion

Emotions

Nowadays, there is general agreement that emotions consist of three processes: Physiological processes that regulate the body, subjective experience that regulates behaviour, and expressive processes that regulate social coordination (e.g., Buck, 1999; Power & Dalgleish, 1997; Schwarz & Skrunik, 2003).

Although researchers have not agreed upon what they mean by emotions, there is agreement on certain aspects. Primarily, emotions are seen in connection with personal goals: They code information about progress towards goals and possible blockages, as well as suggest strategies for overcoming obstacles. Secondly, emotions are seen also to involve a physiological reaction, which distinguishes them from non-emotional cognition. Thirdly, emotions are seen also to be functional, that is they have an important role in human coping and adaptation. (e.g., Buck 1999; Damasio, 1995; Lazarus, 1991; LeDoux, 1998; Mandler, 1989; Pekrun, Goetz, Titz, & Perry, 2002; Power & Dalgleish, 1997).

The three main issues that researchers have not agreed upon are the borderline between emotion and cognition, the number of different emotions, and whether emotions are always conscious. According to Buck (1999), emotions have three mutually independent readouts: Adaptive-homeostatic arousal responses (e.g., releasing adrenaline in the blood), expressive displays (e.g., smiling), and subjective experience (e.g., feeling excited) (Table 3). Here, all these readouts are regarded as part of the emotional state. Unlike generally in mathematics education (e.g., McLeod, 1992; Goldin, 2002), the term emotion is not restricted to intensive, 'hot' emotions. Hence, for example, a mildly sad mood is considered as an emotional state.

Readout target	Readout function	Accessibility	Learning
I Autonomic/ endocrine/ immune system responding	Adaptation/ homeostasis	Not accessible	Physiological adaptation
II Expressive behaviour	Communication/ social coordination	Accessible to others (and self)	Social development
III Subjective experience	Self-regulation	Accessible to self	Cognitive development

Table 3. Three readouts of emotion (Buck, 1999)

Resent advances in our understanding of the neuropsychological basis of affect (e.g., Damasio, 1995; LeDoux, 1998) have radically changed the prevalent view of the relationship between affect and cognition. Emotions are no longer seen as peripheral to cognitive processes or as ‘noise’ to impede rationality. Emotions have been accepted as necessary for rational behaviour.

There are two main routes for emotions to arise (LeDoux, 1998; Power & Dalgleish, 1997). The first route is an automatic, preconscious emotional reaction from a relatively simple stimulus (e.g., sound, object or a concept). Such automatic emotional reactions are based on earlier experiences that have left an association (a memory trace) between the emotion experienced in a situation and a specific element of the situation. In the mathematics class, such a stimulus might be, for example, the tone of voice of the teacher, peer’s laughter, or the concept ‘fraction’. Such emotional reactions are fast and have evolutionarily provided shorter reaction times to possible threats. On the hindsight, they lack flexibility and are difficult to change once formed. The other route to an emotional reaction is based on (possibly unconscious) analyses of personal goals and elements in the situation. This latter reaction is more flexible and possible to affect through conscious deliberations.

Whichever the route of emotional reaction to arise, the emotional state has an influence on cognitive processes. Emotions direct attention and bias cognitive processing. For example, fear (anxiety) directs attention towards threatening information and sadness (depression) biases memory towards less optimistic view of the past. Emotions also activate action tendencies (e.g., fight or flight –response).

Metalevel of affect

DeBellis and Goldin (1997, in press) have introduced meta-affect as the affective counterpart of metacognition. This metalevel is essential for understanding affect in real-life settings. The concept was further elaborated in Hannula (2001) where the ‘metalevel’ of mind was divided into four aspects (Table 4).

Metacognition (cognitions about cognitions)	Emotional cognition (cognitions about emotions)
Cognitive emotions (emotions about cognitions)	Meta-emotions (emotions about emotions)

Table 4. The four aspects of the meta-level of mind.

Within each of these four aspects, we can separate the aspects of monitoring and control. The elaborated approach slightly sharpens the definition of metacognition by restricting steering to cognitive steering. This definition does not include direction of attention and bias of cognitive processing that is caused by emotions.

Emotional cognition is the sister of metacognition. It includes the subjective knowledge of one's own emotional state and emotional processes. Students are aware of the different emotions they have in different situations and they even know of their typical emotional reactions in mathematics class. The subjective knowledge of one's own emotions is the basis for emotional expectations in different situations, and thereby it directs the approaches one has towards mathematical situations. Emotional cognition also includes the conscious regulation of own emotions, which has been shown to be important in effective problem solving (e.g., Carlson, 2000; Zimmerman & Campillo, 2003). Emotional cognition and metacognition are the two kinds of cognitions one can have about one's own mind. These aspects of mind subjects in interviews can tell about. Therefore, these concepts are of interest with respect to methodology of mathematics education.

Emotions exist in relationship with goals and sometimes (e.g., during problem solving) goals may be cognitive. In this paper, emotions that are related to cognitive goals are called cognitive emotions. Frustration and curiosity are examples of typical cognitive emotions that partake in regulation of problem solving. Cognitive goals may be explicit, like when one wants to remember a fact or a procedure, or when one tries to solve a mathematical problem. Sometimes the goal may be vague, like 'to understand' a topic.

Meta-emotions are emotional reactions to one's own emotions. These meta-emotions code important information about the appropriateness of the emotion in question and they control that emotion. Presumably, all humans share the goal to experience pleasure and avoid unpleasant emotions. Humans have also the capacity to tolerate unpleasant emotions if reward of pleasure is to be expected later. For example, successful problem solvers are prepared to tolerate frustration on their way towards solution. There are, however, different norms and individual coping strategies concerning emotions. Therefore, the same emotion may be more stressful for one individual than the other.

Motivation – Why students bother?

Teachers tend to attribute poor performance at least partially to lack of motivation. Students who are not trying to learn are less likely to do so. Researchers try to measure students in order to identify those without motivation – and possible causes to the lack of it. In the present view, the approach has been to look from the other direction: Why most students are putting an effort to learn mathematics. In his ICME9 presentation, Shlomo Vinner pointed to the core of all human behaviour: "…human behavior, as well as human thought, is determined by human needs" (Vinner, 2000, p. 120). This is the view of this paper as well.

Motivation is seen as the inclination to do certain things and avoid doing some others. In the literature (e.g., Ryan & Deci, 2000), one important approach has been to distinguish between intrinsic and extrinsic motivation. Another approach to motivation has been to distinguish (usually three) motivational orientations in educational settings: Learning (or mastery) orientation, performance (or self-enhancing) orientation, and ego-defensive (avoidance) orientation (e.g., Lemos, 1999; Linnenbrink & Pintrich, 2000). Murphy and Alexander (2000) also see interest (situational vs. individual) and self-schema (agency, attribution, self-competence, and self-efficacy) as important conceptualisations of motivation.

All the above approaches fail to describe the quality of the motivation in sufficient detail. This is perhaps inevitable, given that the authors' approaches aim to measure motivation, not to describe it. Two further issues relevant to a critique of mainstream motivation research need consideration: Acceptance of the importance of the unconscious aspect of motivation (Murphy & Alexander, 2000) and focusing on motivational states and processes rather than traits (Dweck, 2002). When motivation is conceptualised as a structure of needs, goals and means (Shah & Kruglanski, 2000), we can see that these vary a lot from person to person (Hannula, 2002b). The theoretical foundation of motivation as a structure of needs and goals was further elaborated in Hannula (2004b), where the following definition was introduced for motivation:

> Motivation is a potential to direct behaviour that is built into the system that controls emotion. This potential may be manifested in cognition, emotion and/or behaviour. (Hannula, 2004b, p. 3)

For example, the motivation to solve a mathematics task might be manifested in beliefs about the importance of the task (cognition), but also in persistence (behaviour) or in sadness or anger if failing (emotion). In cognition, the most pure manifestation of motivation is the conscious desire for something, but the manifestation may also take more subtle forms, such as a view of oneself as a good problem solver. Emotions are the most direct link to motivation, being manifested either in positive (joy, relief, interest) or negative (anger, sadness, frustration) emotions depending on whether the situation is in line with motivation or not. A part of emotions is observable as facial expressions and body language, but part of their nature is unobservable subjective experience (Buck, 1999; Hannula, 2004a). Although emotion and cognition can be observed only partially and are partially inaccessible even to the person him/herself, behaviour is always a trustworthy manifestation of motivation. Even when the person is unable to explain motives for own behaviour, inferences of the unconscious and subconscious can be made from behaviour.

Needs are specified instances of the general 'potential to direct behaviour'. In the existing literature, psychological needs that are often emphasised in educational settings are autonomy, competency, and social belonging (e.g., Boekaerts, 1999; Covington & Dray, 2002). The difference between needs and goals is in their different levels of specificity (Nuttin, 1984). For example, in the context of mathematics education, a student might realize a need for competency as a goal to solve tasks fluently or, alternatively, as a goal to understand the topic taught. A social need might be realised as a goal to contribute significantly to collaborative project work and a need for autonomy as a goal to challenge the teacher's authority.

This realization of needs into goals in the mathematics classroom is greatly influenced by school context, the social and sociomathematical norms in the class as well as by the students' views of themselves, mathematics, and learning. For example, in a teacher-centred mathematics classroom that emphasises rules and routines and individual drilling, there is little room to meet the students' needs for autonomy or social belonging within the context of mathematics learning. A classroom that reflects a socio-constructivist view of learning, on the other hand, provides plenty of opportunities to meet different needs and actually relies on students exhibiting their autonomy and social interactions.

Self-regulation of motivation

Based on a three-year longitudinal qualitative study (see e.g., Hannula, 2004a) instances of motivation regulation in actual students' lives have been identified. Three aspects of regulation were observed in empirical data: Deriving goals from needs, the influence of goal accessibility beliefs, and automated regulation of motivation.

In the comparative case study of two Finnish seventh-graders Eva and Anna (Hannula, 1998b, 2005), social needs were dominating Eva's goal choices, while competence was a more important need for Anna. In the specific social situation, these led Anna to give priority to learning goals while Eva's behaviour was determined by interpersonal relationship goals. Of course, this needs-goals relationship is mediated by personal beliefs. One may perceive a single goal to satisfy multiple needs and a need to be satisfied through multiple goals. Goals may also be seen as contradictory in a sense that reaching one goal might prevent achieving another goal. For example, mastery and performance are usually seen as competing motivational orientations (e.g., Linnenbring & Pintrich, 2000; Lemos, 1999). However, in an analysis of Maria and Laura (studied through grades 7 to 9) (Hannula, 2002b), mastery and performance were goals that supported each other. Maria was driven by her need for competence and mastery of mathematics was her primary goal. However, performance in mathematics tests was an important subgoal for her evaluation of reaching that goal. Laura, on the other hand, was primarily driven by her desire to gain a high status in class 'hierarchy'. Performance (outsmarting other students) was her main goal, while mastery of mathematics was an important subgoal. Furthermore, since Laura tried to hide possible gaps in her understanding, it was hard for her to participate in collaborative activities. Rather, she tried to learn mathematics through individual effort or with help from her father.

The second aspect of regulation of motivation is students' beliefs about accessibility of different goals. This is usually discussed under the term 'self-efficacy beliefs' (e.g., Philippou & Christou, 2002). In order for change in motivation to take place there must be a desired goal, and one's beliefs must support the change. Earlier (Hannula, 1998a, 2002a), the author has reported a case study of Rita, where a radical change in beliefs and behaviour included these two aspects. Using the terminology of goals, we may say that Rita had self-defensive goals dominating her behaviour in the beginning ('You don't need math in life'). However, this was later replaced by performance goals ('I will raise my math mark'). Behind this change, there was a new awareness of the importance of school success in general (change in goal values) together with more positive self-efficacy beliefs (success is possible). In the case of Anna and Eva (Hannula, 1998b, 2005), we can also see these conditions for successful goal regulation. Although both students saw mastery of mathematics as a desirable goal that was not accessible by simply listening to the teacher, only Anna managed to act according to this goal. One important difference between Anna and Eva was that Anna had higher self-confidence in mathematics and thus believed that she could learn mathematics through independent studying.

The third aspect to be discussed here, are automated emotional reactions as an inertia force to students' goal changes. There are two fundamentally different ways how emotional state may be changed (Power & Dalgleish, 1997). One way is the (possibly unconscious) cognitive analysis of the situation with respect to one's goals. Another route to change emotional state is through association to one element of the situation. Emotional associations are learned via classical conditioning and they form the core of attitude as an emotional disposition (Hannula, 2002a). Although they allow shorter reaction times to possible threats, they lack flexibility and are an inertia force of behavioural changes. Once formed, these associations are difficult to change. During school years, students usually develop some emotional disposition to different mathematical actions and goals. Therefore, emotional associations may prohibit change even when change would be 'rational'. In the case of Anna and Eva, one possible obstacle for Eva was her automatic emotional reaction, shame, when she needed to ask for help.

FROM MODEL TO MEASURES

A conceptual model is a theoretical construct that has no immediate connection with reality. Only when the relevant aspects of the model are captured in a representational system, it is possible to use it for a systematic analysis. The most 'natural' way to do this in a classroom is to use a human observer, who then would focus on observing the concepts and their relations defined in the model. The other extreme would be to audio- and video record the events in the classroom and then define exact 'rules' for interpreting the recorded data. For example DeBellis and Goldin (in press), have used the 'Maximally Discriminative Facial Movement Coding System' (Izard, 1983) that includes a score for each of three areas of the face: An eyebrow/forehead movement code, an eyes/nose/cheeks movement code, and a mouth/lips movement code for every hundredth of a second of time on tape.

Enactivism (see Hannula, 1998c, for details) and the interpretative research paradigms (Lincoln & Guba, 2000) have inspired a view of a researcher who is positioned in life history (preconceptions and situation), in a research community, and in relation with the research field and the research data analyzed (Figure 2). Research can not be seen as a simple linear process (research questions – collection of data – analysis of data – writing up report); one has to acknowledge the researcher's influence on the reality of the research field and data, as well as the influence of the writing process on analyses (Hannula, 2003, 2004a).

There are different sources for research data that the researcher may focus on (Figure 3). Some of the data can be collected in natural situations (e.g., collecting copies of student work) where the researcher's influence on the data is low. Some other data (e.g., questionnaires or interviews) are more influenced by the research situation to an extent that they can be said to be created by the researcher (together with the student). Some of these data are 'pure' in the sense that they do not include interpretation at the time of recording (e.g., video recordings), while some other data include already at the time of recording a large amount of interpretation (e.g., observer's journals). In the latter case one needs to be aware of the interpreters' preconceptions and life situation that may bias the interpretation. This classification is not propagating that research should be based on natural data with little interpretation. On the contrary, each of these pieces of data is likely to provide a different view to student thoughts and feelings. Combining multiple views is likely to produce the richest picture of the topic, but we need to be aware of the strengths and weaknesses of each different data.

Elaboration of the representational systems that can be used in research of affect will be divided into three approaches: Observation, interviews, and questionnaires. There are research methods that combine several of these approaches (e.g., video based stimulated recall interview). Such approaches are highly recommended for the triangulation they make possible. Such combined methods do not require a

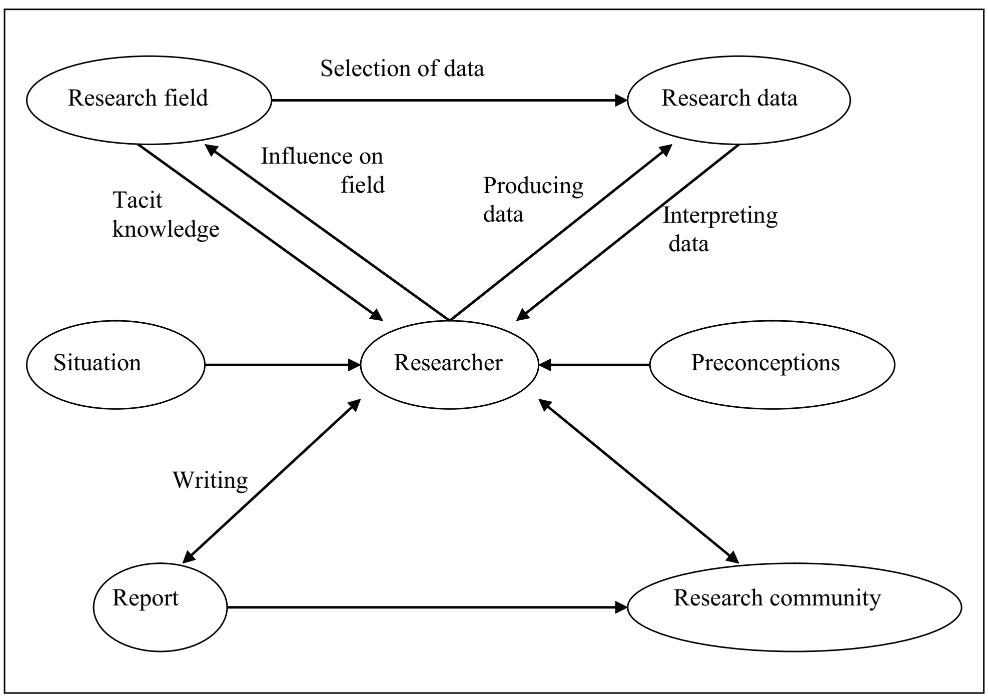

Figure 2. Aspects of methodology in a qualitative paradigm (Hannula, 2003)

specific analysis here, as strengths and weaknesses of each method will be discussed separately.

An observer in a classroom can observe things that reveal emotions. Facial expressions, posture and tone of voice tell about emotions in ways that humans are able to interpret more or less naturally. For example, facial expressions of basic emotions have been identified by respondents of unrelated cultures around the world although the accuracy of interpretation is compromised in less familiar (sub)cultures (Elfenbein & Ambady, 2002). Accuracy of interpretations can be increased through training.

Furthermore, it may be possible to identify a repeated emotional reaction pattern in a series of similar situations. For example, some students might have distinctive anxiety reaction whenever they need to go to the board, regardless of their confidence with the mathematical task. From such patterns, we can assume possible automated emotional reaction.

Observing motivation is more difficult. To be more accurate, we can not observe motivation, but we can make inferences about it. We can observe the student making choices and what makes the student happy, sad, angry and so forth. These observations give information about the student's motivation. We can assume students choosing alternatives that they perceive to lead towards their goals. Interpreting emotions is less straightforward. The situation where the emotional reaction takes place gives the first clue for the possible cause of the emotion (a goal). The nature of the emotional reaction is the second clue. Emotions are functional and they code significant information about goal directed behaviour (e.g., Power & Dalgleish, 1997). Happiness signals that the student is approaching a goal or has reached it. Anger signals that something (target of the anger) is perceived to block approaching that goal. Fear signals that something is seen to threaten an important

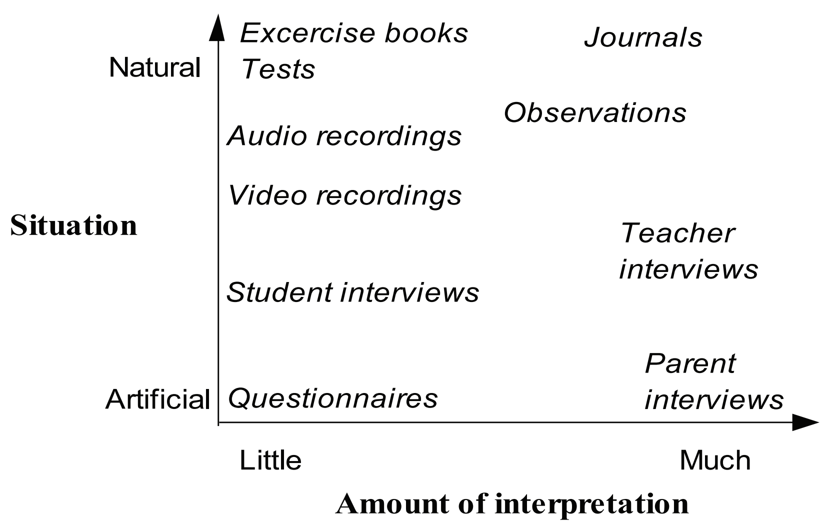

Figure 3. Possible data about students

goal. Sadness is an emotion related to a situation, where a goal is seen no longer to be accessible.

Regarding cognition, the observer can notice progress and moments of success and failure. Such information is important because it often relates to (or even causes) emotional reactions. Students use language to express their thoughts in communication. We see language here in a general sense as any signals (gestures, mimicking etc.) whose meaning can be interpreted by others. Whatever students say about themselves, math, and so forth, is observable. The student may also say things about own emotions and motivation, and interpretation of such utterances is similar to interpreting interviews (see below).

In an interview situation, we can focus on the content of the talk. The student may tell about emotions, beliefs, and motivations. However, this is restricted to what the student is aware of and is willing to tell. What the students choose to tell about reflects also the kind of identity they wish to express to the interviewer. This, on the other hand tells about the desired and actual identities of the student. However, the interviewer needs to be careful about not too eagerly accepting what the interviewee is portraying. In an interview, we can also observe the interviewees facial expressions, posture, tone of voice, which can tell us about either their emotion in the interview situation or their emotions associated with the content.

It is also possible to make a narrative analysis of the interview (e.g., Polkinghorne, 1995). The genre, plot, style etc. chosen by the student tell about the identity of the narrator or about an identity they wish to express in the situation. In a narrative analysis attention should be paid to emphasis, repetition, and telling through negation, which all signal higher personal relevance. It is also important to pay attention to spontaneous talk and silence. Whenever the student brings up a topic spontaneously, it signals relevance or other meaning attributed to the topic. Unwillingness to respond, on the other hand, hints that the interviewee might avoid the topic for some reason.

Much of the research on affect is based on questionnaires, and they are efficient tools for collecting information from a large group of respondents. However, over- or misinterpreting data collected through a questionnaire is easy. Typically, these tools provide us only the surface beliefs of the respondents that reflect their expressed identity. However, with a well-designed instrument it is possible to reach the hidden dimensions of affect – at least on a general level of a large sample. Another problem with questionnaire studies is that they typically reach only the relatively stable affective traits, not more rapidly changing affective states. Yet, it is possible to collect data with a questionnaire during any process, for example problem solving (e.g., Vermeer, 1997). With such ’on-line’-questionnaires, it is possible to collect data of the fleeting emotions and changing goals in the process.

SUMMARY

Empirical study of affect can not be separated for the development of a conceptual framework. All empirical studies are based on a system of theoretical concept and their hypothesised relations that guide (implicitly or explicitly) the development (choice) of tools for representing (observing, measuring) these concepts. An argument is made in this paper, that to study affect in mathematics education in contexts of actual classrooms there are three main elements to pay attention to: Cognition, emotion, and motivation. All of these three domains have a more rapidly changing state-aspect and more stable trait-aspect.

In the field of mathematics education, motivation has been a somewhat neglected topic. The main argument for the importance of motivation is that what the student wants, influences strongly the student's way of perceiving the mathematics classroom, his/her way of engaging with mathematics content and thus his/her experiencing of mathematics classroom.

Motivation (in this approach) is seen as a potential that is not directly observable. Therefore, emotional expressive behaviour becomes a crucial element in the representational system of both emotion and motivation. Emotions can be observed while the student is engaged with mathematics, but also in interviews and other situations where students talk about their relationship with mathematics.

Although emotions are relatively easy to observe, their interpretation into motivation is far more complex. It is important to combine the observation of emotions with the social and mathematical context where the emotion was observed and with the interpretation given by the people themselves. A narrative analysis is a useful tool to get beyond the surface interpretation of what people actually think and feel about issues they talk about.

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LEARNING COMMUNITIES IN MATHEMATICS: RESEARCH AND DEVELOPMENT IN MATHEMATICS TEACHING AND LEARNING

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THE LCM PROJECT

The Learning Communities in Mathematics (LCM)[1] project aims to design and study *mathematics teaching development* for the improved learning of mathematics through *inquiry communities* between teachers and didacticians. The project, which involves 12 didacticians at a university college in Norway and 40 teachers in 8 schools (spanning ages 6–17), focuses on inquiry at three layers:

– *Inquiry* in/into mathematics
– *Inquiry* in/into mathematics teaching
– *Inquiry* in/into researching mathematics learning and teaching

For all participants in the project, *being* an inquirer and *becoming* an inquirer are central aims of participation (Jaworski, 2004a; Wenger, 1998). These ideas will permeate this paper as I present the project and discuss issues it raises for the development of mathematics teaching and learning. The paper addresses the research question

> In what ways has this research involved each of its participants in reflection and inquiry and how has the inquiry nature of the research contributed to growth and development of learners within the project? (CLI1 – See Appendix 1)

I first set the scene by presenting an episode from research in Phase 1 of the project to exemplify inquiry in the three layers. This is followed by a section in which I look briefly at the socio-political context in Norway to situate the research in the

[1] The LCM Project is supported within the KUL Programme (Kunnskap, Utdanning og Laering – Knowledge, Education and Learning) of the Norwegian Research Council (Norges Forskningsråd, NFR). Project number 157949/S20.

2007. In C. Bergsten, B. Grevholm, H. S. Måsøval, & F. Rønning (Eds.),
Relating Practice and Research in Mathematics Education. Proceedings of Norma 05, Fourth Nordic Conference on Mathematics Education (pp. 71–96). Trondheim: Tapir Academic Press

current educational scene. A theoretical section follows, in which I question and explain the concept of *inquiry community* as a basis for developmental research. I then discuss the LCM project from methodological and operational perspectives and end with a discussion of outcomes and issues emerging from the project so far.

INQUIRY IN THREE LAYERS – AN EPISODE FROM ONGOING RESEARCH

Stages in a developmental research process

In order to enable understanding of the episode, I draw attention to a cycle of key stages of activity as designated in the project, denoted as *the design cycle*.

- Design
- Action
- Observation
- Reflection
- Feedback

The project proposal suggests that teachers, with didactician support, will *design* activity for the classroom. This design includes a didactical consideration of what they want to achieve: what mathematics is the focus of the lesson; how they will approach this with pupils; what tasks they will give to pupils; what resources they will use; and so on. Design, at this stage is a theoretical object, albeit using much practical knowledge. Such theoretical consideration will be addressed in the later section on theory in the project. The design stage is followed by classroom activity, involving innovation in the classroom, with elements of *action* and *observation*. Here the teacher transforms elements of design into classroom practice – *acting* in the practical setting and *observing* this action. Inevitably action requires the teacher to face unexpected situations and make decisions in the moment which will be noticed (observed) for subsequent reflection. After the lesson, the teacher reflects on what occurred in the classroom and particularly on the decisions s/he had to make. *Reflection* adds a new dimension to the design process. Now, real elements of practice can be brought into the design. It is no longer a theoretical object (albeit based on previous experience) but starts to include directly related classroom experience and the teacher's associated thinking. Issues can be articulated and shared with others. This is the *feedback* stage. Didacticians can get involved in talking with teachers about these issues. Together, teachers and didacticians can modify the design in accordance with what has been learned. What I describe here is essentially a developmental research process leading to deeper knowledge about teaching and learning processes. Epistemologically speaking, knowledge is seen to grow through the whole design cycle both for individuals and for the communities in which they participate.

The example from which the episode is taken[2]

One of the project schools is an upper secondary school (in Norway, videregående skole) with three mathematics teachers committed to the project. They each teach a class in the first year of the upper secondary mathematics programme. These classes have a parallel study plan and the teachers develop this together. For inquiry activity in the LCM project, they decided to focus on a unit of work on linear functions. In the project, there is a didactician team of three people associated with the school. Two of these didacticians visited the school in the autumn to work with teachers. Together, teachers and didacticians looked at the textbook chapter on linear functions, talked together about the mathematics of linear functions and addressed issues of how inquiry might become a part of the teaching and learning of linear functions.

At the end of a two hour meeting, the teachers indicated that they would like a full day to work further with didacticians in this process of thinking about the topic and how to teach it. Thus a meeting was organised at the college so that teachers and didacticians could go further with this process. This meeting was video-recorded for future analysis. Subsequently, in their school, teachers together designed tasks for their students. These tasks were written on four cards which would be given to the students in a lesson and from which students would work. Lessons were planned for early January. Three lessons took place at different times in the same day and these were video-recorded by didacticians.

After the filming, didacticians looked at the video and selected six extracts to view with the teachers in order to get teachers' reflections on their teaching and on the inquiry process in design and teaching. The meeting took place in an evening at the house of one of the didacticians. This was done to provide a comfortable environment outside school hours in which to relax and feel free to discuss frankly what had been experienced in the classroom. For each teacher two extracts from their lesson were viewed and the content discussed. Such viewing together, sometimes called *stimulated response,* allows issues related to classroom activity to be recognised, discussed and deeply considered by the research team – in this case three teachers and two didacticians.

One of the issues discussed had to do with the language of the tasks. Teachers had used the word "tegn" (draw), asking students to draw a set of number pairs. This was in the first card (Kort 1), and the wording of the card was as follows:

[2] The data that is discussed here constitutes a "Case" in our analysis in the project. It is a case of incidence of the design cycle operational in one school. This case has been deeply analysed and analysis is also discussed in two other papers: Hundeland, Erfjord, Grevholm, & Breiteig (this volume) and Fuglestad, Goodchild, & Jaworski, (2007).

Kort 1
X + Y = 7
Hver elev **skriver ned** to hele tall som til sammen blir 7.
Fins det flere slike tallpar?
Hvor mange slike finnes det?
Tegn disse tallparene på millimeterpapiret.

Translation into English:

Card 1
X + Y = 7
Each student **should write down** two integers which add together to make 7.
Are there other such number pairs?
How many such pairs are there?
Draw these number pairs on millimetre-paper

Students in all three lessons had queried the word "tegn". What did it mean? What should they do? Did they have to draw numbers? How should they do this?

In one of the video extracts discussed, a group of four students were sitting together round a table working on Card 1. They chatted as they worked, sometimes about their activity related to the card, sometimes about other matters. They seemed to have no difficulty in generating number pairs. However, when it came to "Tegn" they seemed unsure what they had to do. One girl raised her hand and the teacher came over to the group. The student asked what they were supposed to draw and the teacher reflected the question back to the group – how might they draw something to represent the pairs of numbers? Further discussion in the group left the students still unclear: should they draw a number of people, for example? Other groups were having similar problems. After some minutes the teacher dropped a strong hint: suppose the two numbers were represented by x and y … . There was an immediate response in the group with facial expressions changing from doubt to smiles. Someone reached for the squared paper. After a few minutes all had drawn linear graphs[3].

The episode represented a situation in which the teachers had wanted students to explore, inquire into, ideas related to linear functions. The wording of the card had been designed to promote such exploration. The students had appeared mystified as to what they should draw, so that the teacher, not wishing to give them the answer, had nevertheless provided a strong hint. The dawning realisation of what was required resulted in smiles and energetic activity. We might ask here, as

[3] What is written just above this note is a description derived as factually as possible from the data. What follows is more interpretative. Evidence for these interpretations can be found in data from the classroom and from meetings with the teachers. Providing the details of such evidence would take more space than this paper allows, but such evidence has been an important consideration in the analytical process.

researchers and teachers, to what extent the questions raised had achieved a higher degree of thinking and conceptualisation where relationships between numbers pairs and graphs are concerned. Also, how did such alternative activity modify students' expectations of classroom interaction?

It seemed clear, in reflection later, that the teachers had expected some graphical form of representation of the number pairs, but they could see that the form of words they had used was unhelpful in this case. They had wanted to generate inquiry on the part of the pupils, and not just tell them what to do. They could see here that they needed to be more clear about what was required. If they would use the cards another time, it would be important to change the wording of this part of Card 1. Such reflection might lead to redesign of the cards, and subsequent further trial in the classroom. Here we exemplify the inquiry process in the design of teaching, and teachers' developing knowledge about the nature and design of effective tasks.

The teacher portrayed in the video extract had been very sceptical about the proposed way of working with the cards, and whether the amount of time taken was really justified, given the weight of the syllabus and the topics to cover. He reported, however, that, in the following lesson, students had appeared much better aware of the concepts in linear functions and graphs than students in previous years, so he felt the cards task had been worthwhile. Jointly, the teachers thought that perhaps some activity of this sort, as part of their more traditional teaching, could be valuable in the future. A few weeks later, at an LCM workshop, two of these teachers reported on this activity and shared these reflections with teachers from other schools. We saw teachers' sharing of reflections, on learning from the inquiry process, as contributing actively to our building of an inquiry community in the project.

What we learn from this example

In the case reported above we see

- Pupils exploring relationships in linear functions
- Teachers designing and exploring classroom tasks to encourage pupils' mathematical exploration
- Researchers studying design and teaching to gain insight into learning and teaching development

The video extract captured a small part of the whole example I have portrayed here. The example shows an important part of the inquiry/design process in which teachers and didacticians have engaged in the project so far. It indicates that a lot was learned from this one case: a case of one cycle of design-action-observation-reflection-feedback in our inquiry process as a community within LCM. The pupils know more about pairs of numbers and their graphs in relation to linear functions. The teachers know more about an inquiry process in teaching linear functions, and students' responses to their teaching. Didacticians know more about the potential of the design-innovation-reflection-feedback process and how it can look in practice. Thus knowledge has grown for all of these participants, and in our community more widely. We get small insights into how such a process might contribute to teaching development in more general terms. The rest of this paper is about the wider principles and practices this embraces in the LCM project within the climate of mathematics learning and teaching in Norway.

A NEED FOR RESEARCH AND DEVELOPMENT IN MATHEMATICS EDUCATION IN NORWAY

Research on mathematical achievement in Norway

The need for research and development in mathematics education in Norway can be seen from a number of sources. First, mathematical achievement in Norway is not as strong as educators or society in general would like to see. This is reflected in both international and national studies such as TIMSS, PISA, KIM and the L97 evaluation (Alseth, Breiteig, & Brekke, 2003; Brekke, 1995; Lie, Kjærnsli, & Brekke, 1997). Students in schools are achieving below international averages. National studies show many areas of mathematics to be problematic for pupils (Brekke, 1995). An evaluation of the new curriculum introduced in 1997 (L97: The Royal Ministry of Education, Research and Church Affairs, 1999) suggests, for some areas of mathematics, that pupils are doing less well than in a review of the earlier curriculum (Alseth et al., 2003).

The social setting in Norwegian schools

Norwegian schools are organised according to year or grade groups 1 to 10 (grunnskole: ages 6–15). Within a year or grade group there may be several mixed-ability classes taught by members of the year group teacher team. Planning within these groups is designed overtly for social inclusion and diversity of experience. Thus teachers need to differentiate across a wide range of experience and achievement. Teachers are expected to teach all subjects, which means that it is hard for any teacher to know every subject in depth. It also means, largely, that subject specialisation is left until the upper secondary school.

Mathematics in teacher education in Norway

To start teacher education study in Norway, students need a general study competence from upper secondary school (grades 11–13). The first year in upper secondary school includes several compulsory subjects; one is mathematics with 5 lessons a week from a total of 30. To obtain general study competence, students need only this basic course in mathematics, although it is possible to do more. Compulsory mathematical study in teacher education programmes for teachers in grades 1 to 10 has been increased in recent years from 0.25 of a year of study to 0.5, with the option of a further 0.5 or 1 year for teachers who will specialise in mathematics. Thus there is a wide range of mathematical experience among teachers currently in practice in grades 1 to 10. Some colleges are starting to offer a master's course in mathematics didactics to students who will become teachers at this level. For upper secondary school, mathematics teachers usually have 1–3 years of education in mathematics from a university, some having a degree in mathematics.

It can be seen from this short account that although the situation is much improved currently, there are many teachers still teaching in grades 1 to 10 who have little formal competence within the subject and possibly little understanding of mathematics didactics.

Focus on teaching

As can be seen in the above sections, there are many factors which can contribute to pupils' underachievement in mathematics. It seems clear that teachers' confidence and competence in mathematics and in ways to approach the teaching of mathematics is one of the most important. We ask in the LCM project: what is needed for teaching to develop so that

- Teachers are *better* prepared to teach mathematics at a particular level
- Students at all levels achieve *better* in mathematics*?*

THEORETICAL CONSIDERATIONS

We work in a developmental research paradigm in which knowledge is seen as deeply rooted in the practices and activity in which groups and individuals engage (Jaworski, 2006; Jaworski & Goodchild, 2006; Wenger, 1998). Individual identity is seen to develop commensurately with educational practice in associated communities where knowledge is made explicit and available for scrutiny and improvement. We take Wenger's notion of *alignment* with social practice and add a critical element in which practices are a source of reflection and critique with a view to learning to act in ways that result in desired improvements. Research is both central to such critique, promoting development, and a means of analysing and documenting development (Chaiklin, 1993). A principle construct in our theoretical perspective is that of *inquiry community* which has theoretical, didactical, pedagogic and practice-related implications for our activity and study.

Why inquiry communities?

To inquire, according the Chambers' dictionary, means to ask a question; to make an investigation; to acquire information; to search for knowledge. Gordon Wells (1999, p. 122) sees dialogic inquiry as

> a willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them.

Thus we might see inquiry to be about questioning, exploring, investigating, and researching. In these ways it is central to approaches and heuristics in mathematical problem solving (Mason, Burton, & Stacey; 1982; Polya, 1945; Schoenfeld, 1985). It is also central to practitioner research designed to develop teaching (Cochran Smith & Lytle, 1999; Elliot, 1991). It leads to metacognitive awareness and ability to take responsibility for learning (Mason, 2002; Wells, 1999). For the LCM project the essence of inquiry as a basis for effective learning is what Cochran Smith and Lytle (1999) call an *inquiry stance*: a position of looking critically, in a positive way, at whatever we are engaged in, asking critical questions that take us deeper into the substance of our activity. An aim of such critical questioning is that we become more knowledgeable about our practice and so more able to function wisely and effectively. This applies to doing and learning mathematics, to developing approaches to teaching, and to conducting research.

Within the project, we recognise that we are humans working together to achieve mutual goals, but that we have different focuses and histories. Social groupings such as school, college, family and friends; social structures such as society and schooling; political systems; economic factors; cultural factors; all influence our working lives. We all engage in certain forms of practice and these practices are what we seek to understand more clearly in order to develop and improve what we do. An anthropological term, *community of practice,* can be used to capture the essential elements of such practices with respect to those who participate in them. A community of practice is a social grouping affording interactive engagement and an environment for learning in an area of mutual action, interest or concern. It has well defined norms of practice. New members can be seen as legitimate peripheral participants – they begin on the periphery of the practice and work towards the centre through their legitimate participation over time (Lave & Wenger, 1991). Examples of such a community might be the teachers in any school, or the group of didacticians in mathematics education at the college. A new teacher joining a school has to participate in school activities, experience the ways in which people in the school inter-relate and do their jobs, contribute to the day to day functioning, interpret their own activity and roles in relation to others around them and adapt to the norms and values of the practice as they experience them. If the new teacher has alternative ways of working, or different values, it is often hard to challenge the status quo in order to have the alternative ways accepted. It is hard for a newcomer to change the way things are. Sometimes the way things are could benefit from critical scrutiny but the established norms and values get in the way of critical questions or disturbing challenges.

Within the community of the project, we recognise that we (each member, and members collectively of the community) are a part of existing communities of practice, such as school, college, department, research group. We fulfil our roles (e.g., as teacher, didactician, researcher) through our participation in these communities.

We acknowledge the importance of these communities as collegial, supportive environments in which we need established relationships, ways of working, familiar routines, in order to function, to fulfil the expectations of our employment. Yet, the nature of our project is to challenge some of the practices familiar to us in order to see how learning and teaching can be improved – this is essentially what we mean by creating an inquiry community. We recognise here that the aims we outline can be seen as contradictory. On the one hand we seek established communities with norms of practice that define what is to be achieved and by what means and which provide appropriate support and guidance. On the other we propose a way of working (or being) in which we question practices and avoid taking for granted aspects of practice which can perpetuate unhelpful regimes. We want, first, to identify such elements of practice, and then to seek, actively, to find alternatives. The 'we' here indicates those who consider themselves members of the community. Thus, we aim to build a community of practice in which members have a clear sense of belonging (Jaworski, 2006; Wenger, 1998) and in which the belonging presupposes an inquiry stance. It is important to recognise that 'inquiry' is *not* the practice of the community. The practice is what the community does and seeks to achieve (such as teaching or researching). The role of inquiry is to look critically at how members of the community, through their practice, achieve community goals. Inquiry can be seen in operation at two inter-related levels: one, as a tool to challenge existing norms of practice and explore alternatives; and two, as way of being in practice that takes inquiry itself as a norm of practice. The second has developed the first as an established norm of practice. We can see here that inquiry, although not itself the practice, becomes deeply integrated with ways of operating within the practice.

This brings me to a definition of community of inquiry for the LCM project.

- A community of inquiry is a community (of practice) in which one of the norms is an attitude of inquiry (e.g., involving critical questioning).
- Inquiry is *not* the practice, but a way of approaching practice from within.
- Inquiry is used as a *tool* to develop inquiry as *a way of being*.

Inquiry is used fundamentally to enable development of teaching to enhance learning of mathematics. A *community of inquiry* in mathematics teaching and learning and their development affords:

- an inquiry approach to developing deep levels of understanding (e.g., of mathematics, or of teaching mathematics).
- A critically constituted community of practice to provide mutual support and a stimulating environment in which questions can be asked and hypotheses tested in order to work for more effective practice.

Central to our inquiry approach is the design cycle discussed above. This cycle offers a practical approach to critical development. It has didactical implications in terms of the design of material for the classroom, of tasks for workshops, of approaches to teaching and of research methods. It has both pedagogical and didactical implications in terms of its reflective element and feedback to new cycles of activity. As teachers or didacticians reflect on classroom or workshop activity, they are challenged to look critically at their involvement with learners in the activity and to rethink didactic and pedagogic processes in their overall design and decision-making.

A paradigm of Developmental Research

We see ourselves as part of an international movement in the development of a new research paradigm which we are referring to as *Developmental Research*. This, fundamentally means research that both studies and promotes development. It includes much of what is currently referred to as "Design Research", but goes further than design research.

Concerning design research, Kelly (2003) speaks of an emerging research dialect contrasting with dialects of *confirmation* or *description* which

> attempts to support arguments constructed around the results of active innovation and intervention in classrooms. The operative grammar, which draws upon models from design and engineering, is generative and transformative. It is directed primarily at understanding learning and teaching processes when the researcher is active as an educator. (Kelly, p. 3)

Design research focuses on the development of new 'products', such as classroom materials or software tools. These are refined through an iterative process of field research in which the proposed users (pupils or teachers) are studied in interaction with the developing product. The notion of product can be extended to include models of or approaches to teacher education (Wood & Berry, 2003).

Cobb, Confrey, diSessa, Lehrer, & Schauble (2003) suggest that design experiments afford a greater understanding of complexity in educational settings, describing it in terms of an 'ecology':

> Design experiments ideally result in greater understanding of a *learning ecology* – a complex interacting system involving multiple elements of different types and levels – by designing its elements and by anticipating how these elements function together to support learning. Design experiments therefore, constitute a means of addressing the complexity that is the hallmark of educational settings.
>
> Elements of a learning ecology typically include the tasks or problems that students are asked to solve, the kinds of discourse that are encouraged, the norms of participation that are established, the tools and related material means provided, and the

> practical means by which classroom teachers can orchestrate relations among these elements. We use the metaphor of an ecology to emphasize that designed contexts are conceptualized as interacting systems rather than as either a collection of activities or a list of separate factors influencing learning (Cobb et al., p. 9).

We see the development of such an ecology to be an essential part of the developmental process that promotes new forms of thinking and action and hence developments in practice. The role of the teacher concerning participation in and orchestration of activity seems a central factor in a learning ecology. Thus, the new forms of mathematical learning which the LCM project seeks, should develop through teachers' and didacticians' collaborative inquiry into approaches to teaching mathematics that use inquiry-based classroom tasks and activity. The ecology can be seen as the growth of a culture of *inquiry as a way of being* (Jaworski, 2004a).

Developmental research goes further than design research in seeking to include teachers as more than users of the products of design. Teachers should as far as is possible be partners in design, resulting in ownership of classroom approaches and deep knowledge of the principles behind classroom activity (Jaworski, 2004b).

What I have given in this theoretical section has been an account of the basis of the LCM Project. What I move to now is a methodological account of how the project has set out to employ this theoretical grounding to create the possibility to research the development of mathematics teaching and learning. We might ask here, how communities of inquiry can be *created* to improve teaching and facilitate learning of mathematics.

LCM PROJECT DESIGN

Key ideas as a basis for research and development

As set out theoretically above, the project seeks to create communities of inquiry in mathematics teaching and learning and their development. Key elements of project design, relating to notions of inquiry, are:

Inquiry in learning mathematics

- Teachers and didacticians exploring mathematically in tasks and problems in workshops;
- Pupils in schools learning mathematics through engagement in tasks and problems in classrooms

Inquiry in teaching mathematics

- Teachers using inquiry in the design and implementation of tasks, problems and mathematical activity in classrooms;

Inquiry in developing the teaching of mathematics

– Teachers and didacticians researching the processes of using inquiry in mathematics and in the teaching of mathematics.

In our creation of community, it is essential that teachers and didacticians can work together to share knowledge and develop practice. We see this as developing co-learning partnerships according to the following premise:

> In a co-learning agreement, researchers and practitioners are both participants in processes of education and systems of schooling. Both are engaged in action and reflection. By working together, each might learn something about the world of the other. Of equal importance, however, each may learn something more about his or her own world and its connections to institutions and schooling. (Wagner, 1997, p. 16)

In such a partnership between teachers and didacticians it is essential that both are seen as practitioners and both have potential to be researchers. The reality was that, in the early days, researchers in the project were the didacticians. It took time for teachers to see themselves as researchers and to start to take on a research role. However, it has been throughout a serious aim of the project that over time teachers would also become researchers in the project.

In line with these objectives, key elements of LCM design are as follows:

– *Creating Partnerships*: Didacticians and teachers work together for mutual benefit and support – both should be involved in design and implementation at conceptual levels for the success of innovation.
– *Designing Materials and Approaches*: Design of tasks for workshops and classrooms; design of approaches to learning and teaching; design of research/inquiry to learn about developmental processes and learning outcomes.
– *Reflective Action* in the use of designed materials and approaches and (critically) reflective questioning of outcomes.
– *Research* into all of the above in relation to research questions about the realization of inquiry communities and their contribution to improved learning.

The research scene here is a complex one. On the one hand we wish to study developmental processes related to design in the areas indicated. Thus research involves a study of the design process. On the other hand, the design process is itself a research process: as tools, models and approaches are designed and tested, questions are asked, people interact and knowledge grows. The objectives of research are not 'simply' to study the process externally, but to engage deeply and developmentally with the processes involved.

The LCM mission

In line with our conceptualisation of developmental research, our project *mission* may be expressed in the following terms:

- To study the processes involved in creating communication and collaboration between mathematics teachers and didacticians in which theoretical ideas and visions can be interpreted in practice.
- To produce insights into key issues in developing inquiry communities to enhance mathematics teaching and learning.
- To provide indications for sustainable practices in mathematics teaching development and learning improvement.

Our research questions, a subset of which can be seen in Appendix 1, derive from these aims. Our activity is developing in a number of phases. After an introductory phase, in which the basis of the project was established and schools recruited as partners, we moved into Phase 1 which is seen in retrospect as mainly a phase of *community building*. Phase 2, in which we are currently engaged, as this paper is written, is seen as a phase of *realising partnership*.

The LCM project works in close relationship with a parallel KUL project ICTML[4] – ICT in mathematics learning. Thus, activity at all levels uses and considers the use of ICT as a developmental tool.

Associated with both projects is a longitudinal study of mathematical achievement and attitudes to mathematics of pupils within our project schools. Test and surveys (linking to local and international studies mentioned above) have been conducted at the beginning of the project and once again during its life so far. Results offer both key details of achievement at different stages in the life of the projects, and important information for teachers about their pupils' mathematical performance. The latter is proving an important developmental tool in the projects. (See for reference, Andreassen, 2005; Andreassen, Breiteig, & Grevholm, 2005; Breiteig & Grevholm, 2006; Kislenko, Breiteig, & Grevholm, 2005).

Activity in the LCM Project

Choice of schools

Originally, we planned to work with six schools chosen for diversity of age range and sociosystemic factors. Seven schools volunteered and so we accepted all. There was some money in the project for support of schools. For any school we asked that:

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[4] ICTML, *ICT in Mathematics Learning*, is supported within the KUL Programme (Kunskap, Utdanning og Laering – Knowledge, Education and Learning) of the Norwegian Research Council (Norges Forskningsraad, NFR). Project number: 161955.

- The Principal or Headteacher must be supportive of the project – willing to support teachers, contribute to time and other resources and disseminate findings in and beyond the school.
- A minimum of three mathematics teachers must be participants in the project, attending workshops and engaging in task design and classroom innovation.

Three of these schools agreed to work also with the ICTML project, and a fourth school was recruited specifically to this project. Now, it too is developing practice according to LCM aims. Thus, in the combined projects we have eight schools.

In agreement with the schools, the available funds were distributed equally among schools, with schools free to determine their use. Contracts were agreed and signed between schools and college, and schools agreed to seek necessary permissions from pupils and their parents for research in classrooms. We collected data from all our discussions with principals and teachers at the stage of setting up these agreements.

Phase 1

Main elements, stages and cycles of developmental activity include

1. **Workshops at HiA**
   Teachers and didacticians working together on mathematical problems and tasks (using ICT). Creating a community of inquirers – we are all learning at a variety of levels.
2. **Teacher teams working in schools**
   Teachers building on experiences in (1), working together within a school to design tasks (including ICT) for the classroom. Drawing on support from didacticians according to needs.
3. **Innovative teaching in classrooms**
   Teachers teaching classes using inquiry tasks and ICT according to experiences from (1) and design at (2) to engage students in inquiry and learning in mathematics.

Typically, activity in 2 follows activity in 1, and activity in 3 follows activity in 2, with a cycling round from 2 or 3 to 1 and from 3 to 2. Activity and thinking in the three stages are deeply inter-related and a part of the design cycle.

Design of workshops

The workshops were conceived of as a basis for community building and for generating common understandings of inquiry processes. This has been largely interpreted in terms of producing problems or tasks to

- get didacticians and teachers working together, in inquiry-based activity, on mathematics;
- develop community, in considerations of mathematics, learning and teaching, with understanding, respect and confidence;

- lead to dialogue on mathematical tasks for classrooms, ways of working in classrooms and ways of achieving curriculum objectives.

Six workshops were held in Phase 1. Planning was done mainly by didacticians, with input or suggestions from teachers at various stages. Activity included plenary presentations by both didacticians and teachers, small group work (either in same grade groups or across grades) and plenary feedback sessions where insights and issues from group work are shared. Workshops in Phase 1 were designed around mathematical problems or tasks, mathematical topics, and inquiry-based activity. Data was collected from all planning meetings and from the workshops themselves and includes notes; audio-recordings of discussion; video recordings of workshops sessions, written material about tasks, problems and workshop schedules.

Design of school activity

Project design indicated that, in parallel with workshops, there should be activity in schools in which the workshop activity would stimulate ideas and planning for classroom work. No structure was given for school activity, except in saying that it was intended to plan or design inquiry-based activity for classrooms: perhaps, to plan a set of lessons involving inquiry approaches which teachers would teach and together explore the outcomes. Certain didacticians were associated with particular schools (3 to each school) and visited schools either on the invitation of teachers, or at their own request. Didacticians decided that they would respond to teachers' thinking and planning in schools, rather than attempting to direct such thinking. A question to be addressed was what support should/could be provided by didacticians? What needs would be recognised? In tackling this question we drew on various models for this kind of activity, for example:

- Japanese Lesson Study (e.g., Stigler & Hiebert, 1999);
- Learning Study (Marton et al., 2003);
- Formative Assessment Project (Black, Harrison, Lee, Marshall, & Wiliam, 2002).

Didacticians were concerned to keep open a range of possibilities in offering help and support and responding to perceived needs. Data were collected in the form of notes from meetings and decisions made, audio recording where possible, and materials produced for classrooms. The outcomes from this activity and analysis of data constitute a considerable learning experience for all involved as will be reported further below.

Classroom innovation

As a result of planning/design work in schools, it was proposed that teachers would teach the planned lessons and that didacticians would audio or video record the lessons wherever possible. Project design suggested that teachers would take some opportunity to observe each other in the classroom, and would meet at agreed intervals to reflect on the lessons, recognise issues, re-plan where necessary, and keep a record of thinking and outcomes. Didacticians would be available to provide support as needed and possible, collect data in classrooms and conduct focus group interviews. Data was expected to include teacher notes from lessons; recordings of lessons; observation notes from classrooms; and recordings of focus group discussions. The reality and practicalities were somewhat different from what had been proposed as indicated below.

A summary of approaches used in data and analysis is presented in Appendix 2.

WHAT WE ARE LEARNING FROM RESEARCH AND HOW THIS IS AFFECTING ONGOING PRACTICE

The account above has been concerned mainly with Phase 1 of the project, and what was planned for Phase 2. I shall say more about Phase 2 below. In Phase 1, the main elements of design (workshops, school activity and classroom innovation) were implemented. For example, 6 workshops took place, and parallel activity was undertaken in the 8 schools. Data was collected as far as circumstances allowed according to what was planned, and analytical work began. The large quantity of data, and preponderance of research questions, means that this analysis is ongoing in parallel with Phase 2 work. Learning from activity, experience, reflection and analysis is deep and significant at a range of levels.

Teachers, didacticians and design

At an operational level, the following took place:

- Didacticians designed workshops (with some input from teachers)
- Teachers designed classroom activity (with some input from didacticians)
- Didacticians have been responsible for research data and analyses (with some input from teachers)
- Teachers have been responsible for school development (with some input from didacticians)

But, in terms of the main principles of the project, community and inquiry, how are the various communities to be characterised and what inquiry-based thinking or activity has emerged? What does an inquiry community look like (or feel like) for those involved? For example, for

- The community of didacticians at HiA;
- The community of LCM teachers in one school;
- The classroom community for one teacher and pupils.

How can we recognise a community, as distinct from a group of individuals?

Realisation of community in the LCM project

The project was designed initially by didacticians, and teachers were invited to join. Thus, although agreements were made and contracts signed, knowledge and power in the project rested initially with didacticians. This made didacticians responsible for creating opportunity for teachers to develop knowledge and power within the aims and principles of the project. While the theoretical bases of inquiry and community were non-negotiable, there was nevertheless very considerable scope for interpretation of these ideals. It was essential for all to work together to learn what was possible and to examine critically the outcomes of joint activity and enterprise. I will mention just two areas of learning: the planning and interpretation of tasks for workshops; and the negotiation of activity within schools. Both of these raise issues for the nature and development of inquiry communities. Again, what I present is a descriptive account, but all statements can be backed up with explicit references to data.

Didacticians decided that we should work together with teachers on mathematics to encourage a mathematical dialogue, and to start to reach joint understandings about inquiry in mathematics. Thus mathematical problems were selected to generate joint activity in workshops and open up dialogue. There is considerable evidence of participation and enjoyment in work on the problems; some teachers took problems into their classrooms and reported, in subsequent workshops, on their activity with pupils. In the beginning, these problems were not related to particular areas of the curriculum; however, some teachers asked for clearer curriculum links. So problems were chosen in areas of *probability* (suggested by teachers in one school) and of *numbers (*relevant to classrooms across the age spectrum), to enable curriculum-related work. In all cases, choice of problems encouraged inquiry approaches in questioning, exploration and development of mathematical thinking. However, some teachers suggested that using such inquiry tasks demanded more classroom time than was available; they suggested there was not time for inquiry work in addition to what was required by the curriculum. It seemed that they perceived 'normal' work to be a basic necessity, a norm of school work, whereas inquiry tasks were a luxury for which time had to be found if possible. If not possible, then such activity could not be a normal part of school activity. Such thinking seemed more prevalent at higher levels of schooling (e.g., in higher secondary school) where the mathematics curriculum was seen as especially demanding.

These issues reflect the theoretical dichotomy between community of practice and community of inquiry, and show that in practice, creation of a community of inquiry is far from straightforward. Thus didacticians have been faced with an issue of how to work with teachers to foster a vision of inquiry-based work as a part of the normal teaching of the mathematics curriculum. This is a central issue in the project. We continue to collect data through which we can examine developing attitudes to and understanding of this kind of activity. Nevertheless it is an issue that, potentially, separates didacticians and teachers.

Activity within schools was left to teachers themselves to decide, although didacticians expressed their willingness to join in and support school activity. In the beginning teachers found it difficult to find time for meeting in schools. There seemed to be no 'natural' time that could be used. Some principals helped to organise suitable time, but in other cases, teachers had to find it themselves if they wanted a meeting. The visit of didacticians to a school was sometime a catalyst for a meeting to take place.

Didacticians tried to respond to teachers, encouraging teachers to decide on what they wished to focus planning for development. Ideas varied across the schools and between the teachers. One higher secondary school indicated a chapter in the mathematics textbook on which they would focus development. We have seen some of the details of this development earlier in this article. In other schools, teachers were less clear about a joint focus, and didacticians tried hard to help them to formulate their thinking. It became clear that there was room for the didacticians' role here to be more than a responsive one, and that some schools looked more explicitly for guidance from didacticians. Analyses of data help us to examine such issues and to characterise the associated processes of development. See Jaworski (2005) for a more detailed discussion of some of these issues.

Into Phase 2

I suggested earlier that Phase 1 was chiefly a phase of *community building*. During this phase – one year of activity as described above – the project community developed relationships, understandings and perceptions of goals. Phase 2 began in September 2005. I anticipated then that Phase 2 would be a phase of *realising partnership*. By this I meant that, having established our learning community, with understandings of inquiry processes growing in that community, we might share responsibility more equally between schools and college in making decisions about development in the project. Our first workshop of Phase 2 was then imminent, and had been planned after consultation with teachers in an open meeting at which all schools were represented. Teachers had expressed what they found valuable in workshops and what they would like coming workshops to offer them. Two main suggestions were that we should have an agreed mathematical focus, curriculum related, and that there should be opportunity in the workshop to plan for the classroom ("å planlegge et opplegg" was the Norwegian phrase which entered our project vocabulary).

So, for the first three workshops in Phase 2 we worked successively on probability and geometry, at the teachers' suggestion, and each workshop constructed small grade-related groups in which teachers planned together for the classroom. Two workshops included presentations from teachers to inform us all about their classroom activity related to this planning. Classrooms were video-recorded in some schools to capture interpretation of planning activity. There was enthusiastic feedback from teachers on the positive nature of these workshops. Didacticians too felt a sense of achievement related to the perceived quality of mathematical discussion and planning of classroom tasks. This paper is written after three workshops in Phase 2 and another 3 are planned for the rest of this phase. We are beginning a more detailed analysis of our Phase 2 data.

At this stage, nevertheless, despite some positive indications from schools, activity in schools does not yet come close to the aims of the project in terms of systematic planning and design, and associated classroom activity, by a teacher team in school. Operation of the teacher team in schools is problematic still, partly because teachers meet each other normally in grade-related teams, not subject teams. So getting together a team to talk about planning for mathematics has to be specially coordinated and this requires time and energy beyond the normal demanding school day. Also, the "å planlegge et opplegg" activity in workshops, contrarily, obviates to some extent the need to plan in school. Teachers can take the planned activity from the workshop directly into their classrooms. So we see individual teachers developing classroom work with pupils, with evidence of fruitful results for pupils' learning of mathematics; however, such development seems piecemeal and sporadic rather than systematic in building a strong developmental process within a school.

IN CONCLUSION

Although some of the remarks above may seem negative in terms of project achievement, they are important for the overall learning of the project about developmental growth and its practical implementation. We are working fundamentally at the practice level, trying to interpret theoretical ideas in ways that work in practice. It is perhaps not surprising that the path is not smooth. Didacticians discuss currently how best to collaborate with schools to achieve a developmental style which can work practically in interpreting the aims of the project. Development has to be sustainable for long term improvement of learning and teaching. Together teachers and didacticians have to find ways of tackling the practical issues to make it possible to realise theoretical and developmental aims. What we learn from engaging in this struggle helps us to see our aims in a deeper and richer context and to search for ways in which joint conceptualisations of community, inquiry, research and development can lead to new ways of working that are sustainable in the longer term.

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APPENDIX 1

Table of research questions

In our original proposal for funding we had 6 areas of research questions. We had to trim these to 4 areas to fit the funding we were given for the project. The table shows these four areas (keeping the original numbering for consistency), together with a selection of the research questions that are being asked and refined in these areas.

1 Mathematics Classrooms

a) Student/Mathematics/Curriculum/Assessment (SMCA)
Drawing on and extending recent research in this area in the Curriculum 97 study of mathematical achievement:

SMCA 1: What are students' mathematical conceptions in agreed curriculum areas? [Longitudinal + in-depth]

SMCA 2: What are the factors contributing to students' mathematical progress with respect to the current Norwegian curriculum in mathematics? [Longitudinal + in-depth]

What is the role of assessment in this relation between teachers' teaching and pupils' learning?
What is the relation between teachers' teaching and student learning?
How does the use of ICT software support the development of students' mathematical concepts?

b) Mathematics Teaching: Knowledge and Approaches (MTKA)

MTKA 1: What knowledge do teachers bring to mathematics teaching at agreed levels and what approaches are used? [Longitudinal + in-depth]

MTKA 2 What is needed for teaching to develop to address SMCA 1 and 2? [Longitudinal + in-depth]

How is our work with teachers helping them on their way of becoming inquirers?
What is the relation between teachers' teaching and student learning?
What is it that guides teachers when they decide about their teaching?
What is the role of assessment in this relation between teachers' teaching and pupils' learning?
Within the population of teachers and didacticians included in the KUL-LCM project is it possible to identify a diversity of forms of knowledge?

c) Inquiry in mathematical problem solving for students and teachers (IMP)

IMP 1: What are the issues that arise when teachers and didacticians prepare for and incorporate inquiry-based tasks related to the mathematics curriculum in whole class and small group learning?

IMP 2: In what ways does students' participation in mathematical inquiry relate to their achievement in mathematics?

IMP 3: What elements of the problem solving process can be identified in students' communication during enquiry based learning in small groups and whole classes at different levels of schooling?

In what ways can we identify relationships involving inquiry and students' mathematical thinking and understanding?
Is it possible to identify developments in classroom activity (teachers' activity, students' activity, or other features of the activity systems that can be identified)? It might be difficult to establish a cause-effect relationship with the KUL-LCM project but we can listen to teachers' 'stories' about any developments that have occurred in their practice.
What is the nature of inquiry in our project?

- How do we see inquiry in mathematical tasks and mathematical learning in classrooms?
- How do we (didacticians) see inquiry in the activities of teachers, thinking about their teaching and designing activity for teaching?

2 Language and Discourse in mathematics learning and teaching (L&D)

L&D 1: How do mathematical concepts emerge in the instructional discourse of the classroom and what mathematical understandings can be discerned? (SMCA 1and MTKA 1; IMP1 and 2)

L&D 2: In what ways can cross-cultural perspectives and international studies provide insights into discursive practices in Norwegian classrooms?

L&D 3: How can/should language and discourse become central to the planning and delivery of classroom mathematical activity? (SMCA 2 and MTKA 2)

What kinds of discourse are evident in the process of creating communities of inquiry in teaching and learning mathematics?

- What kinds of discourse promote inquiry?
- What artefacts and tools are created through the discourse?
- What are the developmental issues?

  How does inquiry as a tool emerge in discourses from the workshops: planning meetings, the workshop it self with teachers and evaluative meetings?

3 Information and Communications Technology in mathematics learning and teaching (ICT)

ICT 1: What roles does/can technology play in students' developing conceptions, competence and self-reliance in doing mathematics and solving problems? (SMCA 1 and 2; IMP 1)

ICT 2: What is needed in terms of teachers' conceptions of technology use in developing teaching approaches for effective student learning of mathematics? (MTKA 1 and 2; IMP 2)

ICT 3 What hardware and software provision is needed to support development of concepts of teachers and students and the wider growth and application of mathematical knowledge and understanding?

What is the current situation in schools concerning the use of ICT in mathematics?
How are computers used, and with what aims? What is the connection between curriculum plans, examinations and use of computers? What kind of software is used? How does the use of ICT software support the development of students' mathematical concepts? Can we find differences in ways of understanding concepts with and without ICT tools? Can we find impact from use of computers from students' explanations of their solution to problems?
What attitudes have students and teachers to the use of ICT in mathematics classrooms?
How independent and self-regulated are the students when working with computer software in mathematics, how can this be linked to their teacher's organizing, and if they are working alone or in peers?
How do teachers plan for and utilise computers in the mathematics classroom?
What is needed in teachers' own development concerning use of computers, and pedagogical insight into the use of ICT tools in mathematics? How do teachers develop their competence?

6 Communities of Learning and Inquiry within mathematics learning and teaching. (CLI)

CLI 1: In what ways has this research involved each of its participants in reflection and inquiry and how has the inquiry nature of the research contributed to growth and development of learners within the project?

CLI 2: In what ways has research capacity increased through the project and how might this be sustained in future learning and teaching improvement in mathematics?

In what ways can we identify relationships involving inquiry and students' mathematical thinking and understanding?

What are the features of KUL-LCM that mark it as an effective approach to teacher development?
- What is the nature of inquiry in our project?
- How do we see inquiry in mathematical tasks and mathematical learning in classrooms?
- How do we (didacticians) see inquiry in the activities of teachers, thinking about their teaching and designing activity for teaching?
- How do we see inquiry in our own activity: in ways we work with teachers and in ways we construct and conduct our research?
- What is the nature of community in our project?
- How do we see communities to develop?
- What are the roles of didacticians in developing partnerships with teachers?
- What kinds of communities are evident for teachers, and how do these develop?

APPENDIX 2

Summary of data and analysis of data within the project

Data are collected as follows

1. Data from application of longitudinal instruments in our schools
2. Notes from introductory conversations with principals and teachers
3. Notes from workshop design meetings; audio-recordings of discussion; tasks, problems and workshop schedules produced.
4. Audio or video recordings in workshops of plenary dialogue and some of small group dialogue. Written notes from activity of all groups.
5. Notes from teacher groups with careful record of discussion and decisions – possibly audio recording. Materials produced for classrooms.
6. Teacher notes from lessons; Audio or video recordings of lessons; observation notes from classrooms; recordings of focus group discussions.

Analysis is involving some or all the following stages and approaches:

1. Data Reduction: Factual summaries of recordings from all events (meetings, workshops, lessons, interviews);
2. Coding of summaries to elicit main activity and thinking in events. Extraction of commonly occurring codes;
3. Transcription where appropriate. Identification of sections relating to the commonly occurring codes and other criteria;
4. Scrutiny of key episodes chosen from (3) along with other significant episodes determined from the written reflections. Analytical accounts from episodes. Further analyses where appropriate and as determined (e.g., micro-analyses of discourse from certain episodes);
5. Synthesis of findings: basis of written papers for dissemination.

THE CONCEPT AND ROLE OF THEORY IN MATHEMATICS EDUCATION

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In theory there is no difference between theory and practice, but in practice there is
(attributed to US baseball player Yogi Berra)

INTRODUCTION

The notion and term of "theory" are essential in any discipline that perceives itself as scholarly or scientific. Hence "theory" is also essential in mathematics education as a research domain, where the term is frequently used in papers, books, and – not least – in Ph.D. dissertations.

On closer inspection, however, it is not clear at all what "theory" actually means in mathematics education. Nor is it clear where the entities referred to as theories invoked in mathematics education come from, how they are developed, what foundations they have, or what roles they play in the field. It seems problematic that a key entity for the furthering of research in a field is ill-defined and has an unclear status and function in the field. Therefore, we have to pay much more attention to non-trivial issues such as the ones just mentioned. As far as I can tell, in the international debate not much attention has been paid to these issues so far, even though the word "theory" comes up more and more often in research publications. The present paper is an attempt to contribute to placing these issues on the agenda of articulate and reflective discussions in our field. It is not the intention with this paper to survey or review different specific theories put to use in mathematics education research but to offer some general considerations of an overarching nature.

THE GENERAL NOTION OF THEORY

For a start let us consult a few dictionaries and encyclopedias to obtain a first approximation of the general notion of theory. As regards etymology, the root of the word is Greek, *theoria,* which means consideration or speculation and is derived from the verb *theorein*: to consider. In Aristotle, "theory" refers, more specifically, to deliberations on ideas, non-empirical matters or abstract contexts without regard to action or practice (Den Store Danske Encyklopædi, 2000).

2007. In C. Bergsten, B. Grevholm, H. S. Måsøval, & F. Rønning (Eds.),
Relating Practice and Research in Mathematics Education. Proceedings of Norma 05, Fourth Nordic Conference on Mathematics Education (pp. 97–110). Trondheim: Tapir Academic Press

The Collins Cobuild (1999(95)) *English Dictionary* lists four different meanings of the term theory.

1. "A *theory* is a formal idea or set of ideas that is intended to **explain something** [my emphasis].",
   in other words a more or less elaborate, connected and substantiated *edifice of formal concepts*, including *mechanisms for explanation*.
2. "If you have a *theory* about something, you have your **own opinion** [my emphasis] about it which you cannot prove but which you think is true.",
   in other words an unsettled *hypothesis* adhered to by someone.
3. "The theory of a practical subject or skill is the set of **rules and principles** [my emphasis] that form the basis of it.",
   in other words, the foundational *underpinning of a certain practice*.
4. "You use *in theory* to say that although something is supposed to be true or to happen in the way stated, **it may not in fact be true or happen that way** [my emphasis].",
   which points to the *difference* between more or less established *assumptions or predictions* and hard *reality*.

Princeton's *Wordreference.com* is not too different. It presents three different meanings of the word:

1. "a *tentative* theory about the natural world; a concept that is **not yet verified** [my emphasis] but that if true would explain certain facts or phenomena.",
   in other words an unsettled *hypothesis / a system of hypotheses* that may or may not hold; this more or less merges (parts of) the two first meanings in the Collins Cobuild dictionary.
2. "a *belief* that can **guide behaviour** [my emphasis]."
   i.e. a conviction forming the *basis for action*, close to the third meaning in the Collins Cobuild dictionary.
   Finally:
3. "a well-substantiated *explanation* of some aspect of the natural world; an organized system of accepted knowledge that applies in a variety of circumstances to **explain a specific set of phenomena** [my emphasis]."

i. e. an established and connected *edifice of general claims* covering several special cases.

On closer examination it turns out that most of these different meanings of the term "theory" are actually represented in publications within the discipline of mathematics education. We shall return to this point later in this paper.

In order for the subsequent discussion to have a clear focus it seems warranted to present a further approximation to the concept of "theory". Allow me to present my own definition of the concept:

A *theory* is a system of concepts and claims with certain properties, namely

- The theory consists of an *organised network of concepts* (including ideas, notions, distinctions, terms etc.) and *claims* about some extensive domain, or a class of domains, of objects, situations and phenomena.
- In the theory, the *concepts are linked in a connected hierarchy* (oftentimes of a logical or proto-logical nature), in which a certain set of concepts, taken to be basic, are used as building blocks in the formation of the other concepts in the hierarchy.
- In the theory, the *claims are either* basic hypotheses, assumptions, or axioms, taken as *fundamental* (i.e. not subject to discussion within the boundaries of the theory itself), or statements obtained from the fundamental claims by means of *formal or material* (by "material" we mean experiential or experimental) *derivation* (including reasoning).

In principle, for a system of concepts and claims to be called a theory, the system has to be *stable*, i.e. unchanged over a longer span of time, *coherent*, i.e. the components of the system have to be linked in a clear and non-contradictory way, and *consistent* in the sense that it is not possible to arrive at contradictory claims by means of the types of derivation permitted in the theory. In practice, however, many systems named theories do not possess all these features. This is particularly true of theories in development. Some theorists would add the requirement that all the non-fundamental claims in a theory have to be testable (or at least falsifiable) by logical or empirical confrontation with the domain(s) covered by the system. In other words, such theorists would discard what is sometimes called transcendental theories, i.e. theories in which the concepts and claims are so general and overarching that they do not apply in a straightforward way to a specific, empirically well-defined world. As numerous theories belonging to the humanities or the social sciences and employed in mathematics education are transcendental in this sense, I do not find it reasonable to exclude them from consideration here.

Theories *differ* in a multitude of respects. More specifically they differ with respect to the origin, nature and state of

- the *domain* (or class of domains) – and the kind of objects, situations and phenomena that populate it – which the theory refers to or is meant to cover;
- the *concepts* involved in the theory, and the conceptual *hierarchy* within which the concepts are organised;
- the *claims* actually or potentially made within the theory, including the fundamental claims underlying the entire theory;
- the way in which the *network* of concepts and claims of the theory is organised;
- the way in which claims are *derived and justified* within the theory;
- the degrees of *stability*, *coherence*, and *consistence* of the theory.

The variability within each of these features of a theory is sufficient to suggest that the notion of theory is not exactly a monolithic one. This observation gains even more momentum if we go on to consider the different purposes that theories may have in research. I have identified six such purposes. Some of these purposes are represented in, or are similar to, the dictionary entries quoted above. Of course, a theory can have several of these purposes at the same time.

One purpose of a theory is to provide *explanation* of some observed phenomenon occurring within a domain covered by the theory. Explaining a phenomenon by way of a theory means that the occurrence of the phenomenon can be obtained as a claim (or a consequence of a claim) in the theory. Explaining an observed phenomenon also includes specifying the conditions under which it occurred, and substantiating the fulfilment of these conditions within the theory.

Another purpose is to provide *predictions* of the (possible) occurrence of certain phenomena. Again, this means to establish the occurrence of the phenomena at issue as (possible) claims in the theory as a result of the (possible) fulfilment of the preconditions for their occurrence. It appears that there may well be a close link between explanation and prediction, since prediction, including a choice between possible scenarios, will often (but not necessarily always) rely on explanation of causes and mechanisms, and since explanation will often (but not necessarily always) give rise to predictions.

A third purpose is to provide *guidance for action or behaviour* by employing knowledge of claims, and the conditions of their validity in the theory, in order to plan and implement action or behavior so as to achieve desirable – or to avoid undesirable – outcomes.

A fourth purpose is to provide *a structured set of lenses* through which aspects or parts of the world can be approached, observed, studied, analysed or interpreted. This takes place by selecting the elements that are important for consideration in the context, while omitting others; by focusing on certain features or issues; by adopting and utilising particular perspectives; and by providing a *methodology* for the whole enterprise.

Yet another purpose is to provide *a safeguard against unscientific approaches* to a problem, an issue or a theme, including, for example, haphazard and inconsistent choices with regard to terminology, research methodology, and interpretation of results. This purpose is pursued by articulating underlying assumptions and choices and by making them explicit and subject to discussion; by situating one's research within some framework; and by declaring and describing its characteristics vis-à-vis possible alternatives.

The sixth and final purpose is to provide *protection against attacks* from sceptical or hostile colleagues in other disciplines. For instance, any mathematics education researcher has experienced criticism of our field from outside colleagues (especially in mathematics, psychology, or general education) concerning the foundation of our research and its results. And researchers in the humanities and social sciences at large have often encountered similar criticism. For quite a few of us / them the invocation of one or more theories, claimed to underpin research, may serve the purpose of counteracting such criticism.

It is now time to leave the general treatment of the concept and role of theory aside and move on to the part it plays in the field of mathematics education research.

THEORIES IN MATHEMATICS EDUCATION

The questions concerning theories put to use in mathematics education research that will preoccupy us in this section are the following:

- What are they?
- Where do they come from?
- What foundations do they have?
- What are their roles?
- Are they "good enough" (in a sense that has to be specified)?
- What should we strive at in terms of theory development and use?

WHAT ARE THE THEORIES PUT TO USE IN MATHEMATICS EDUCATION?

To begin with, four general observations pertaining to this question should be kept in mind.

Firstly, there is *no such thing* as a well-established unified "theory of mathematics education" which is supported by the majority of mathematics education researchers. On the contrary, different groups of researchers represent different schools of thought, some of which appear to be mutually incompatible if not outright contradictory. Moreover, for reasons that will be discussed in a later section of this paper it is not likely that we shall get a unified theory of mathematics education in a foreseeable future, if ever.

The second observation is that many mathematics education researchers relate their work to explicitly invoked theories *borrowed from other fields* (or at least from authors who belong to other fields), and often do so in rather eclectic or vague ways. Only rarely are theories "home grown" within mathematics education itself.

Thirdly, much discussion and debate in mathematics education research takes the shape of *"fights"* with and between theories. This may be potentially fruitful to the extent competing theories offer different perspectives on the same thing, whereas it is potentially futile, if not destructive, if the theories deal with different things and therefore are only competing in the superficial sense that "my topic object of study is more important than yours".

Finally, the fourth observation is that quite a few mathematics education *researchers do not explicitly invoke or employ any theory* at all in their work. Furthermore, many researchers who actually do invoke a theory in their publications do not seem to go beyond the mere invocation. In other words, some theoretical framework may be referred to in the beginning or in the end of a paper without having any presence or bearing on what happens between the beginning and the end.

Against this background, what theories are in fact being put to use by researchers in mathematics education? Let us begin by looking at theories that are, in principle, extraneous to mathematics education but which have been "imported" into the field.

First of all we find theories about the epistemology and sociology of *mathematics as a discipline* (e.g. Davis & Hersh, 1980; Ernest, 1991; von Glasersfeld, 1995; Hersh, 1997; Kitcher, 1984; Lakatos, 1976; Tymoczko, 1985), This is hardly surprising, as problems encountered in the teaching and learning of mathematics are often closely linked to the nature of mathematics as a discipline and to its subjective and objective relevance to life in society. Typically, theories about mathematics as a discipline, such as formalism, structuralism, empiricism, radical constructivism, social constructivism, semiotics, etc., relate to more general theories of knowledge or of social practice. In the early days of mathematics education research, theories of mathematics as a discipline were only sporadically found on its agenda, but since the 1980's it is customary for many a publication to declare its own position in this respect.

Traditionally, *statistics and exploratory data* analysis have occupied a prominent position as *the* tools for quantitative research. In the early days of mathematics education research, quantitative studies were predominant in empirical research, and even though different sorts of empirical studies entered the field in the 1980's and rapidly grew in significance and popularity amongst researchers, quantitative – and above all statistical – methods remain important tools. Evidently, the theories in question are the theories of (applied) statistics (hypothesis testing, analysis of variance, regression analysis), including exploratory data analysis, and psychometrics.

In addition to specific theories about the epistemology of mathematics, general *philosophical theories*, not specifically modified or tailored to dealing with mathematics, primarily theories of knowledge, are often invoked in mathematics education research (e.g. Bachelard, Kant, Peirce, Piaget (as a structuralist), Popper, Wittgenstein).

From the earliest days of mathematics education research, *general psychological theories* have been put to use in the field. This is easy to understand as early researchers took it for granted that the only extra-mathematical discipline of relevance to the study of mathematics education beside statistics was psychology. This is also reflected in the fact that the first international study group in mathematical education affiliated to ICMI was the study group on the psychology of mathematics education, PME, which at times even considered itself as *the* international group for mathematics education research. The psychological theories which have been invoked are behaviourism (e.g. Thorndike, Skinner) and neo-behaviourism, cognitive structuralism (Piaget as a psychologist, and followers), cognitive science in general, activity theory (e.g. Vygotsky, 1978; and partly Krutetskii, 1976), psychoanalysis. We can also find psychological theories specifically tailored or developed to focus on mathematics education (e.g Skemp, 1987; Fischbein, 1987; Vergnaud, 1991; APOS (the piagetian Action-Process-Object-Scheme theory developed by Dubinsky and others (e.g. Czarnocha, Dubinsky, Prabhu, & Vidakovic, 1999)).

Also *general pedagogical theories* about teaching and learning considered in psycho-social environments (e.g. concerning gender) are put to use in mathematics education research.

The same is true of theories from *linguistics* (structuralism, Chomsky, linguistic registers, semiotics, socio-linguistics (e.g. Bernstein)).

Today, theories from the *socio-cultural sciences* have gained considerable momentum in mathematics education. This has happened along with the extension of our fields of attention and vision to include classrooms and their subgroups, institutional contexts, social classes and groups, including cultural or ethnic minorities, socio-economic and political issues, etc. Thus the theories employed are imported from sociology (e.g. Beck, Bourdieu, Giddens, Habermas, Luhmann), anthropology (e.g. Lave, Saxe, Wenger), education (e.g. Dewey), political science and history. One of the main methodologies employed in current mathematics education research, *qualitative studies*, is rooted in these sciences, in particular anthropology and ethnology.

Finally, within this group of theories, recent trends focus on new developments in *neuroscience* which, by means of new brain scanning techniques, allow for studies of the working brain. This has given rise to physiological theories about the organisation of the brain and the mind that seem to have some bearing on mathematical cognition, especially arithmetical concept formation, and problem solving (e.g. Dehaene, 1997; Lakoff & Nuñez, 1997).

Beside theories such as the ones mentioned, mathematics education research has also developed what is sometimes called “home grown” theories designed to deal with a few or several aspects of mathematics education. Usually such theories are informed by extraneous theories like the ones listed above, so by using the word “home grown” we are not implying that the seeds themselves cannot be imported into the field from outside. The important thing is that the seeds are sown in the soil of mathematics education proper and that the resulting plants are cultivated in the discipline’s own gardens.

Apart from attempts – by Steiner and others (Steiner, 1985) – going back to the early 1980’s to provide a foundation for a comprehensive *“Theory of Mathematics Education”*, most home grown theories focus on a limited set of aspects. Above we have already mentioned the so-called *APOS theory* (Action-Process-Object-Scheme) which Dubinsky (Czarnocha et al., 1999) and colleagues have developed as an extension of Piaget’s work. This theory focuses on the learning of mathematics, in particular the formation of mathematical concepts, and is seen by some of those who advocate it as sufficient to cover most of the relevant features of mathematical learning. Tall and Vinner (1981), as well as Sfard (1991), and others, have laid the ground for a theory of mathematical concepts based on a distinction between *concept definition* and *concept image* and a distinction between *process* aspects and *object* aspects of concepts. The French school of *didactique des math-*

ématiques is based on three theories. Brousseau's theory of *didactical situations* in which the point of attention is teaching and teacher orchestrated student activity in the mathematics classroom (Brousseau, 1997). Chevallard's theory of the *didactical transposition* of scientific mathematical content into curricular mathematics (Chevallard, 1991), and Vergnaud's theory of *conceptual fields*, which focuses on the formation of mathematical concepts – especially in arithmetic – and sense-making thereof (Vergnaud, 1991). Yackel and Cobb (1996) have developed aspects of a theory of what has been called *socio-mathematical norms*, which focuses on the socially defined boundary conditions for what it means to be involved in mathematical activity. Schoenfeld and colleagues (Schoenfeld, 1999) have outlined what is meant to be a theory of *mathematics teaching*, Ball and colleagues (Ball, Lubiensky, & Mewborn, 2001, Hill & Ball, 2004) have identified elements of a theory of *teachers' mathematical knowledge for teaching,* and Skovsmose (1994) has worked towards a theory of *critical mathematical education*, in which he focuses on the social and societal roles of mathematics education and the possibilities of empowering students with a critical attitude to the use and misuse of mathematics in society.

Unfortunately, there is no room in this paper to carry through what is actually badly needed, a thorough and critical examination of the sense and extent to which each of the theories named as such in this section is a theory according to the definition I proposed in the previous section. It would be a significant and worthy task, which would further the scientific and scholarly health of our field, if such analyses were undertaken. Should the result of such analyses be that it is not quite justified to speak about theories, it would still be warranted to speak about theoretical perspectives.

WHERE DO THE THEORIES COME FROM?

As appears from the previous section, the far majority of the theories invoked or employed in mathematics education research come from *disciplines outside mathematics education* itself. Historically, statistics and psychology were the first "exporters" of theories into mathematics education research, statistics as the main tool in quantitative research paradigms, and they continue to play significant roles in the field, even though their relative importance has decreased as a consequence of the introduction of other theoretical perspectives.

Theories beyond psychology and statistics were introduced in mathematics education research more or less one by one, along with the adoption of new research issues and foci on the agenda of the field. These issues and foci include curriculum reform; the application of mathematics in extra-mathematical domains; philosophical characteristics of mathematics; mathematics classrooms; gender issues; linguistic issues; socio-cultural issues, including minority issues; student beliefs, affections, career perspectives; teacher preparation, in-service training, and teacher attitudes and beliefs; etc. (Niss, 2004)

When such new horizons are opened, we look to other disciplines that have dealt with similar issues and topics and seek their assistance.. However, when it turns out that the assistance offered by other disciplines is somewhat limited, because they are unable to account for the essential role of *mathematics* in the teaching and learning of mathematics, it becomes necessary to modify, reconstruct, or combine extraneous theories in order to tailor them to fit the needs of mathematics education, and, ultimately, to create specific theories having aspects of mathematics education in focus.

WHAT FOUNDATIONS DO THE THEORIES HAVE?

On closer inspection, most of the theories which play a role in mathematics education have foundations which are primarily *conceptual* in that they are based on the proposal and introduction of notions, distinctions and concepts of a rather general nature. Sometimes, but not always, the theories have a degree of partial empirical corroboration in the sense that they are mostly inspired by reflection on experience. Only in rare cases do they enjoy the sort of substantiation that may arise from systematic empirical or experimental testing. Only seldom do theories aspire to provide definite claims about the state of affairs in mathematics education. Instead, they are often *interpretive*.

All this implies that the theories we have looked at *never* remain *unchallenged* in the mathematics education research community.

WHAT ARE THEIR ROLES AND FUNCTIONS?

We saw in an earlier section that theories have different purposes, where the term "purpose" suggests an ultimate end of having a theory at all. The "role" of a theory, which will preoccupy us in this section, is somewhat different in that it points to the actual place of a theory in a larger picture, to the ways in which it is related to other components of the picture, and to the function(s) it serves in that context. With that understanding in mind, different theories have different roles in research in mathematics education.

Some theories serve as an *overarching framework* from which (parts or aspects of) the teaching and learning of mathematics can be viewed and approached. This constitutes a "top-down" approach, where the theoretical framework is given before and outside the specific piece of research in which it is being put to use. In principle – albeit not so much in practice – this implies that the only objects, situations, phenomena, and processes considered are ones that are permitted by and visible from the theoretical framework. Such theories tend to be based on one or a few fundamental concepts, like "reflective abstraction" in Piaget's theory of cognition or in the APOS theory, or like "the zone of proximal development" in Vygotsky's activity theory, "didactical" and "adidactical situations" and "the didactical contract" in

Brousseau's theory, like "social construction" in social constructivism, and so forth and so on.

Some theories focus on *organising a set of specific observations and interpretations* of singular but related phenomena into a coherent whole. This constitutes a "bottom-up approach", where no specific theory is imposed on the data – which, then, have not been collected according to a theory-based design – but is supposed to arise from them by concrete analysis. A typical example of this is "grounded theory", which is a way of establishing a specific theory grounded on the data given on the basis of a general methodology. In other words "grounded theory" can actually best be perceived as a meta-theory of how to obtain a specific theory.

Some theories have the role of providing the *terminology* – including the concepts and distinctions that come with it – involved in a particular piece of research. Examples of this are the "process-object duality" of mathematical concepts and "reification" (Sfard, 1991), "procept" (Tall, 1991), "concept image", "epistemological obstacle" (Bachelard, 1938; Sierpinska, 1994), the "epistemological triad" (Steinbring, 1989), "S and I rationales for learning" (Mellin-Olsen, 1981) etc.

Some theories offer a research *methodology*, primarily for empirical studies. Currently, qualitative methods are prevalent in mathematics education research, and the methodologies involved, which are borrowed from other disciplines in the humanities and the social sciences, help design and analyse observation protocols, interviews, questionnaires, student tasks, or to produce transcripts or video-clips, or to create fictitious individuals representing typical segments of a population, and so on. Meta-theoretical considerations propose method triangulation in empirical research so as to avoid biased interpretations of data caused by a research instrument in itself. When it comes to quantitative studies, applied statistics and its offsprings psychometrics and quantitative sociology continue to prevail as the way to obtain answers to the questions posed.

So far we have concentrated on the roles of theories that are not only mentioned in actual research but in fact utilised. However, when looking at lots of examples of actual research there are numerous cases where a theory is in fact being invoked, but where the relation between the theory and the specific piece of research seems to be missing, i.e. the research is carried out without really involving the theory which is being invoked. This means that references to theory tend to be rhetorical. In such cases, why, then, is a theory being invoked at all? What is its role in the research being conducted?

The only reasonable answer to this question seems to be that since, in such cases, the theory has not informed the research design, process, and inferences, its role must be limited to the publication part of the research. This suggests, then, that a theory is either being invoked in order to legitimise the piece of research at hand, so as to increase its scholarly solidity and credibility, or it serves as a means of announcing the author's adherence to or membership of a particular sub-community of researchers to which it may be seen as desirable to belong. In my view either possibility is unfortunate as it counteracts what ought to be a key feature of everything we do in research: uncompromising honesty about the nature of what we are doing.

ARE THEORIES PUT TO USE IN MATHEMATICS EDUCATION RESEARCH "GOOD ENOUGH"?

The crude and quick answer is "in general, no!" Here are a few details to elaborate on this answer.

Theories imported from other fields are largely *insufficient*, for the following reasons. Firstly, no single imported theory encompasses neither all of mathematics, mathematics learning and teaching, the relations between all kinds of individuals, groups, classrooms, institutions, communities, and societies with mathematics, nor all significant contexts and dimensions therein and thereof. Secondly, separate theories that deal with different domains pertinent to mathematics education cannot just be glued together so as to form a comprehensive "patchwork" theory of mathematics education. At least, so far nobody has been able to propose such a patchwork and demonstrated its universality. Thirdly, and perhaps most importantly, most imported theories are of a general nature, which does not allow them to offer a sufficient pool of specific results and concrete methodologies so as to provide complete guidelines for conducting a piece of research work.

Theories that are home grown in mathematics education (and their number is not large) do not suffer from the deficiency hinted at in the first of the abovementioned reasons, lack of specificity to *mathematics* education. However, each of them suffers from lack of comprehensiveness in its endeavour. Similarly, even if forces were combined they too have not been (cannot be?) glued together to form a comprehensive patchwork theory of mathematics education. With regard to the third respect, some home grown theories seem to have a potential for offering at least partial guidelines for conducting specific pieces of research. For instance, the French school of mathematics education research, combining the theories of Brousseau, Chevallard, Vergnaud, and complemented with Duval's theory of linguistic registers, does seem to outline a paradigm for research in mathematics education, at least in the context of the French educational system.

WHAT SHOULD WE STRIVE AT IN THEORY BUILDING IN MATHEMATICS EDUCATION?

Imagine that there existed a full-fledged theory of mathematics education. What would it look like, and what *minimum requirements* would it have to fulfil? In order for it to be comprehensive enough to be worth its name, it would have to contain at least the following of sub-theories, each accounting for essential traits of mathematics education:

- a sub-theory of *mathematics as a discipline and a subject* in all its dimensions, including is nature and role in society and culture;
- a sub-theory of *individuals' affective notions*, experiences, emotions, attitudes, and perspectives with regard to their actual and potential encounters with mathematics;
- a sub-theory of *individuals' cognitive notions*, beliefs, experiences, and perceptions with regard to their actual and potential encounters with mathematics, and the outcomes thereof;
- a sub-theory of the *teaching of mathematics* seen within all its institutional, societal, national, international, cultural and historical contexts;
- a sub-theory of *teachers of mathematics*, individually and as communities, including their personal and educational backgrounds and professional identities and development.

All sub-theories – and I do not claim that the list just given is exhaustive – have to account for *situating* their objects, situations, phenomena and processes *in all the contexts and environments* that influence them, be they scientific, biological, anthropological, linguistic, philosophical, economic, sociological, political, or ideological. Similarly, they must be geared to deal with both descriptive ("what *is* the case, and why?") and normative ("what *ought to be* the case, and why?") issues.

Moreover, each sub-theory has to live up to the *general requirements of a scientific / scholarly theory* (e.g. as outlined in my definition in the second section of this paper), including accounting for the ways in which its claims are obtained and justified.

The sub-theories *cannot just be juxtaposed*, they have to be *integrated* into a coherent and consistent whole, simply because they deal with issues, problems, and topics of which *mathematics* is a constituent component that lies across them all. In addition, as the sub-theories have non-empty intersections they have to be consistent in the ways they speak about entities and issues on which they overlap.

CONCLUSION

If we were to engage in the construction of theoretical foundations of mathematics education, it follows from the considerations given above that we – i.e. the didacticians of mathematics – have to be in charge ourselves. Of course, in this endeavour we need prudent import from any other field that has something important to offer. However, even if we need "a little help from our friends", only the mathematics education research community possesses the multifaceted expertise necessary for constructing consistent theoretical frameworks that are rich enough to cover the whole field in its complexity.

This being said, it is unlikely that we shall ever arrive at just one theoretical framework for mathematics education research, unifying all researchers in the field. This is a simple consequence of the fact that it seems possible to create several meaningful sub-theories for each of the five domains we have considered. Already for combinatorial reasons, this suggests that several comprehensive, respectable, but competing, theories of mathematics education are likely to arise from our endeavours to establish a theoretical foundation of mathematics education research.

Although these prospects may discourage some from engaging in attempts to establish a theory of mathematics education, I do not think it should. First of all we need to do much more serious work in order to come to grips with the ways in which our field can justifiably be perceived as the scholarly or scientific discipline we all think it is, or ought to be. Secondly, a most important outcome, even in the short term, of such endeavours would be a much richer and better box of sharp tools for critical analysis of research contributions of whichever kind than are at our disposal for the time being. This, in turn, would help raise the level of reflection and consciousness in our field, and not the least so with respect to novice researchers. One rather immediate result of that would be better research, and that wouldn't be too bad, would it?

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ABOUT MATHEMATICAL TASKS AND MAKING CONNECTIONS: AN EXPLORATION OF CONNECTIONS MADE IN AND 'AROUND' MATHEMATICS TEXTBOOKS IN ENGLAND, FRANCE AND GERMANY

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In the presentation we explore the ways in which mathematics textbooks vary at lower secondary level in England, France and Germany. We examine popular selling textbooks in each country and their treatment of 'Negative Numbers'. Connections are explored in terms of interconnectedness of mathematical content knowledge, and tasks and exercises are investigated with respect to cognitive demand and contextualisation. An analysis of the data suggests that learners in the different countries are offered different mathematics in textbook tasks. We contend that different countries provide different (mis)representations of mathematics for their students in school textbooks, in particular with respect to connectivity. Whereas in some countries students are inundated with skills, procedures and disconnected mathematical knowledge, in others students are allowed to develop an appreciation of its interconnectedness and generalisable nature. It is argued that textbooks, and tasks in textbooks, play an important part in the mathematics education process, and that both their nature and content need to be understood in terms of a wider context, in order for shared understandings, principles and meanings to be established, whether for promotion of classroom reform or simply for developing a better understanding of this vital component of the mathematics education process.

INTRODUCTION

Students spend much of their time in classrooms exposed to and working with prepared materials, such as textbooks, worksheets and information and communications technology (ICT) materials. Teachers often rely heavily on textbooks in their day-to-day teaching, and they decide what to teach, how to teach it, and the kinds of tasks and exercises to assign to their students. It is reasonable to argue, therefore, that such materials are an important part of the context in which pupils and teachers work. In recognition of the central importance of such documents, the

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framework for the Third International Mathematics and Science Study (TIMSS) included large-scale cross-national analyses of mathematics curricula and textbooks as part of its examination of mathematics education and attainment in almost 50 nations. Concerns have been expressed about the quality of textbooks, about the way they are written, and about their persuasive influence, and it appears that the textbook content, and how it is used, is a significant influence on students' opportunity to learn and their subsequent achievement (Robitaille & Travers, 1992). It is also commonly assumed that textbooks (with accompanying teacher guides) are one of the main sources for the content covered and the pedagogical styles used in classrooms.

Of central importance in this article is the concept of 'academic' or 'mathematical' task which is used here as an analytical tool for examining subject matter as a classroom process rather than simply as a context variable in the study of learning. The purpose of this presentation is to give a general overview of research in the areas of textbooks internationally in connection with mathematical tasks, and describe some of the themes that are emerging from the present study in relation to the literature.

TEXTBOOKS AND THE CURRICULUM

In order to explore possible influencing factors in cross-national differences in student mathematical achievement, it has been suggested that the curriculum is one of the key factors (Schmidt, McKnight, Valverde, Houang, & Wiley, 1997), and amongst those curricular factors the textbook has been identified as having potentially a large effect (Mayer, Sims, & Tajika, 1995).

Traditionally, there was an emphasis on improving material investments and guaranteeing universal access to public education, that is every child has access to his/her textbook. It was believed that universal access would provide more opportunities for learning, and understanding. More recently, the emphasis has shifted to the development of conceptual understanding, amongst other objectives, and there is thus a renewed interest in the intended curriculum, and in particular its quality, as one of the main factors of educational policy. Valverde, Bianchi, Wolfe, Scmidt, and Houng (2002) claim that

> textbooks are commonly charged precisely with the role of translating policy into pedagogy. They represent an interpretation of policy in terms of concrete actions of teaching and learning. Textbooks are the print resources most consistently used by teachers and their students in the course of their joint work. (p. viii)

Romberg and Carpenter (1986) noted that the textbook was consistently seen as "the authority on knowledge and the guide to learning [and that] … many teachers see their job as 'converting the text'" (p. 25).

Cross-national comparisons of mathematics textbooks appear to be important for US agencies, in the light of evidence that US textbooks constitute a 'sort of de facto national curriculum' (Mayer et al., 1995). In the European context several studies have supported the importance of textbooks in the mathematics classroom (see e.g. Bierhoff, 1996; Harris & Sutherland, 1998; Pepin & Haggarty, 2001).

Having established that textbooks are important artefacts in the classroom and that they strongly influence what happens in the classroom, in fact that they are the mediators between "the intent of curricular policy and the instruction that occurs in the classroom" (Valverde et al., 2002, p. 2) – they are part of the potentially implemented curriculum, we want to turn to textbooks and tasks.

TEXTBOOKS AND TASKS

It appears that textbooks, for better or worse, define and represent the subject for students, and they influence how students experience mathematics. Textbooks provide children with opportunities to learn, and to master skills, which are regarded as important by their government. Thus, it is important how texts are organised.

Teachers mediate textbooks by choosing and affecting tasks, and in that sense student learning, by devising and structuring student work from textbooks. In his research on mathematics instruction Schoenfeld (1988) argues that students, despite gaining proficiency at certain mathematical procedures, gained a fragmented sense of the subject matter and understood few connections that tie together the procedures they had studied. Moreover, he claims that the students "developed perspectives regarding the nature of mathematics that were not only inaccurate, but were likely to impede their acquisition and use of mathematical knowledge" (p. 145).

Doyle (1988) argues that the tasks teachers assign to students, and it has been shown that teachers use textbooks heavily for their selection of tasks (Kuhs & Freeman, 1979; Luke, de Castell, & Luke, 1989; Pepin & Haggarty, 2001), influence to a large extent how students come to understand the curriculum domain. Moreover, in his opinion, they serve as a context for student thinking not only during, but also after instruction. This premises that tasks, most likely chosen from textbooks, influence to a large extent how students think about mathematics and come to understand its meaning.

An investigation of tasks in textbooks is likely to bridge the gap, in some ways, between classroom practice and its relation to student understanding. For example, Doyle (1988) suggests that the way the subject matter is presented (i.e. as familiar work – routinised exercises that can be worked out of context and without significant understanding of the subject matter) has an influence on student perception of the subject matter (i.e. trivialise that subject matter) and in turn deprives students of the opportunity to understand and use what they have learnt.

Doyle (1988) argues that "to understand and improve the opportunities students have to learn the curriculum, it is necessary to examine how the curriculum is represented in the tasks a teacher requires them to accomplish with content." (p. 177). He uses *academic task* as an analytic tool for examining subject matter as a classroom process (rather than simply as a context variable). He suggests that an academic task approach to classroom research can be used as a "treatment theory to account for how students learn from teaching" (p. 167). Whilst recognising that there are other factors that influence student learning, he supports Shavelson, Webb, and Burstein (1986) proposing that academic tasks can serve as proximal causes of student learning from teaching. "Tasks influence learners by directing their attention to particular aspects of content and by specifying ways of processing information" (Doyle, 1983, p. 161).

In this sense the mathematical tasks that students are engaged with may influence not only what mathematics they are likely to learn but also how they come to think, develop, use, and make sense of the mathematics. And indeed, the differences between tasks that engage students appear important, either at surface level, or at deeper level, by demanding interpretation and the construction of meaning (Stein, Grover, & Henningsen, 1996).

Stein et al. (1996) highlight the importance of mathematical tasks as "important vehicles for building student capacity for mathematical thinking and reasoning" (p. 455). The underpinning assumption is that students must be provided with opportunities, encouragement and assistance to engage in thinking and sense-making in the mathematics classroom.

Other textbook studies have focussed on the analysis of problems presented in textbooks (Li, 2000). Using textbook problems as a window through which to view students' mathematical experiences has also been advised by Nicely (Nicely, 1985; Nicely, Fiber, & Bobango, 1986). Textbook problem analysis has also been carried out by cross-national comparativists (Stigler, Fuson, Ham, & Kim, 1986).

CONSIDERATIONS ON THE ANALYSIS OF TEXTBOOK TASKS

General considerations

Kilpatrick, Swafford, and Findell (2001) give a comprehensive view of what they regard as successful mathematics learning. They coin the term "mathematical proficiency" to capture what they think it means for anyone to learn mathematics successfully. In their view, mathematical proficiency has five "interwoven" and "interdependent" strands:

conceptual understanding – comprehension of mathematical concepts, operations, and relations

procedural fluency – skill in carrying out procedures flexibly, accurately, efficiently, and appropriately

strategic competence – ability to formulate, represent, and solve mathematical problems

adaptive reasoning – capacity for logical thought, reflection, explanation, and justification

productive disposition – habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy. (Kilpatrick et al., p. 5)

In terms of instruction they argue that the quality of instruction depends, for example, on the tasks selected for instruction (which includes their cognitive demand) and the meanings students assign to the mathematics, amongst other factors. It is also claimed that the teacher's expectations about students and the mathematics they are able to learn can "powerfully influence the tasks the teacher poses for the students, the questions they are asked, … in other words, their opportunities and motivation for learning" (Kilpatrick et al., 2001, p. 9).

The Programme for International Student Assessment (PISA) uses the following skills, in terms of classes of competencies, in their framework for mathematical competencies (OECD, 1999):

Class 1: Reproductions, definitions, and computations;
Class 2: Connections and integrations for problem solving;
Class 3: Mathematical thinking, generalisation and insight.

The OECD uses the term 'mathematical literacy' to assess the competencies across the three different classes, without hierarchical implications.

Both views of mathematical proficiency seem to have common characteristics and can be regarded as complementing each other.

Characterising/analysing tasks more specifically

Authors, such as Hart (2000), emphasise the connections made at classroom level, and she outlines different categories in terms of classroom connections. At the heart of it is the focus on the nature of the tasks or activities provided, the "task appropriateness" (p. 25). In terms of context and meaning of classroom task it appears sensible to establish the cognitive demands of a task in relation to pupils' existing competencies.

Smith and Stein's work (1998) is interesting and useful in terms of positioning the nature of tasks. They argue that the nature of the task determines what students learn. They also used four categories of cognitive demands as a way of classifying good tasks:

- Memorisation
- Procedures without connections to concept or meaning
- Procedures with connections to concept or meaning
- Doing mathematics

This is helpful in some ways, but does not distinguish between 'rote-doing' and 'doing-by-understanding', highlighted by Schoenfeld (1988).

Nicely (1985) asserts that mathematics textbooks have "a poor track record" in terms of higher order thinking skills (p. 26). He and his colleagues developed an analytical tool to evaluate textbooks. This consists of four lists which enable analyses and classification printed materials according to 'Type of content'; 'Level of cognitive activity'; 'Stage of mastery'; and 'Mode of response'.

In terms of cognitive level of student tasks he identified 27 verbs grouped into nine categories and arranged in an ordinal scale. Interestingly, their levels of cognitive demand (reflected in cognitive verbs and codes – according to which textbooks problems were analysed) ranged from lower levels (no task, observe, read; recall, recognise, repeat) to higher levels (prove, solve, test, design; evaluate).

This reminds the reader of Bloom's taxonomy (1956) as "a framework for classifying statements of what we expect or intend students to learn as a result of instruction" (Krathwohl, 2001). Kilpatrick (1993) asserts that the taxonomy attempts to reflect the differences of student's actions.

Without revisiting the whole theory, it is sufficient for the purpose of this presentation that Bloom's cognitive domain is organised into six hierarchical categories: knowledge; comprehension; application; analysis; synthesis; and evaluation.

CONNECTIONS, CONNECTEDNESS AND MATHEMATICAL KNOWLEDGE

There is a large body of literature emphasising the importance of relating procedural and conceptual knowledge. Hiebert and Carpenter (1992) believe that it is essential to make connections in mathematics if one intends to develop mathematical understanding. These connections should not only be those between procedural and conceptual knowledge, but also between the pieces of information that students already know. They question however whether students should be told about connections, whether teachers should stress those connections, or whether students should be given the opportunity to become aware of them themselves.

It is interesting to note that many official and important teaching documents mention the notion of 'connectedness' as a vehicle for better understanding. For example, the Initial Teacher Training National Curriculum (ITTNC) (TTA, 1999) says that newly qualified teachers should know

> how to use formative, diagnostic and summative methods of assessing pupils' progress in mathematics, including (ii) undertaking day-to-day and more formal assessment activities so that specific assessment of mathematical understanding can be carried out … (and) (iii) preparing oral and written questions and setting up activities and tests which check for misconceptions and errors in mathematical knowledge and understanding … and understanding of mathematical ideas and the *connections* [italics added] between mathematical ideas. (p.14, 9aii and iii)

The American National Council of Teachers of Mathematics (NCTM) claims that

> a central theme of Principles and Standards for School Mathematics is '<u>connections</u>'. Students develop a much richer understanding of mathematics and its applications when they can view the same phenomena from multiple mathematical perspectives. One way to have students see mathematics in this way is to use instructional materials that are intentionally designed to weave together different content strands. Another means of achieving content integration is to make sure that courses oriented toward any particular content area (such as algebra or geometry) contain many integrative problems – problems that draw on a variety of aspects of mathematics, that are solvable using a variety of methods, and that students can assess in different ways. (National Council of Teachers of Mathematics, 2000)

Earlier documents published by the National Council of Teachers of Mathematics (1989) and the National Research Council (1989) all point to the importance of students' developing interconnected understandings of mathematical concepts, procedures and principles, and not simply to memorise and apply procedures (Stein et al., 1996). According to the *Professional standards for the Teaching of Mathematics* (National Council of Teachers of Mathematics, 1991) one finds consistent recommendations for the exposure of students to meaningful and worthwhile mathematical tasks, tasks that make students think rather than simply repeating and using an already demonstrated algorithm.

Thus, in terms of both teacher and pupil understanding of mathematical concepts, it appears that an important question relates to how mathematical knowledge is structured and connected, and how this is done in tasks.

In terms of international and comparative research, and in terms of teacher knowledge, Ma (1999) compared Chinese and US elementary teachers' mathematical knowledge. She found that Chinese elementary teachers perceived mathematical concepts as interconnected, in contrast to US colleagues who perceived these concepts as arbitrary collections of facts and rules. She developed a notion of 'profound understanding of fundamental mathematics', and an argument for structured, connected and coherent knowledge (Ball, Lubienski, & Mewborn, 2001) and this was seen as one of the factors for student enhanced mathematical performance.

Research conducted at King's College in the United Kingdom (Askew, Brown, Rhodes, Wiliam, & Johnson, 1997) also pointed to the importance of connected knowledge, again of teachers. They revealed that highly effective primary teachers of numeracy paid attention to

> *Connections between different aspects of mathematics:* for example, addition and subtraction or fractions, decimals and percentages;
>
> *Connections between different representations of mathematics:* moving between symbols, words, diagrams and objects;
>
> *Connections with children's methods:* valuing these and being interested in children's thinking but also sharing their methods. (Askew, 2001, p. 114)

This orientation was called *connectionist* orientation, in contrast to the *discovery* and *transmission* orientations towards teaching numeracy which showed less effectivity in teaching.

In England the DfEE (2001), in their Key Stage Three National Strategy, suggest that

> Mathematics is not a group of isolated topics or learning objectives but an interconnected web of ideas, and the connections may not be at all obvious to pupils. Providing different examples and activities and expecting pupils to make links is not enough; pupils need to be shown them and reminded about work in earlier lessons. (1.46)

They also offer advice to teachers on how these connections may be made.

> As far as possible, present each topic as a whole, rather than fragmented progression of small steps …
>
> Bring together related ideas across strands …
>
> Help pupils to appreciate that important mathematical ideas permeate different aspects of the subject …
>
> Use opportunities for generalisation, proof and problem solving to help pupils to appreciate mathematics as a unified subject … (1.46)

Thus, the literature seems unambiguous about the fact that making connections is important for understanding. However, how this can be achieved is not clear.

THE STUDY

In this small-scale study we used our knowledge of textbooks and the analysis of textbooks to develop a deeper understanding of connections made in textbook tasks. See also (Pepin & Haggarty, 2001) and (Haggarty & Pepin, 2002). Textbooks which were identified as amongst the ones most frequently purchased for years 7 (6è*me*, *Jahrgang 6*), 8 (*5ème*, *Jahrgang 7*) and 9 (*4ème*, *Jahrgang 8*)[1] were chosen for analysis, using statistics produced by publishers and ministries of education. The topic of 'directed numbers' was selected for this more detailed analysis, because this topic was regarded as relatively self-contained and likely to be taught as a new topic, particularly in years 7 and 8. There was also reference to years 9 and 10 in terms of follow-up of topics and coherence through the years.

The textbook analysis generally draws on a range of ideas from the literature, some of them, for example the Cummins matrix (Cummins & Swain, 1986), had not been used before in the mathematics education area. Cummins developed a simple two-dimensional model, where one axis represents the "context-embeddedness", and the other the degree of conceptual demand of the task. More particularly, we brought together concepts and analyses that linked cognitive demand and contextualisation in mathematical tasks in order to 'map cognitive processes' in textbook tasks.

For example, in order to map the cognitive demand of tasks in textbooks, we superimposed Bloom's taxonomy onto the Cummins matrix. Bloom's categories are generally regarded as levels of difficulty and reflected in student action – for the sake of textbooks in verbs that supposedly reflect student behaviour:

– Knowledge: write, list, name
– Comprehension: describe, summarise
– Application: use, solve, apply
– Analysis: compare/contrast, analyse
– Synthesis: design, invent, develop
– Evaluation: critique, justify.

These were subsequently used to define cognitively demanding or undemanding tasks.

The conception of mathematical textbook task used in this article is similar to Doyle's (1983) and Stein et al.'s (1996) notion of academic and mathematical task respectively. It includes what students are expected to do and 'perform' (i.e. verb

[1] The following textbooks were chosen for analysis
– Germany: Lambacher-Schweizer (*Gymnasium*); Einblicke Mathematik (*Hauptschule*);
– France: Cinq sur Cinq;
– England: Keymaths.

analysis). It does not however, involve the resources used for the task, or indeed the length of the task. Neither does it include teachers' mediation of the task in a real classroom setting. For teacher mediation, please see Pepin and Haggarty (2001, 2004). It is 'simply' analysed as a textbook activity, designed for the purpose of focussing student attention to a particular mathematical idea.

The particular foci explored in this presentation are the following:

- What are the general features of English, French and German textbooks?
- What kinds of connections are made in those mathematics textbook tasks?

RESULTS AND FINDINGS

In terms of general background information, in lower secondary years in a mathematics classroom in *France*, the textbook is given to pupils for one year; it 'belongs' to the pupil for the academic year and is therefore available for their use at home and in school. In *Germany* there are different textbooks for each school type (*Gymnasium, Realschule, Hauptschule*). Textbooks are not supplied by the school and it is expected that parents buy them, unless they have financial difficulties. Thus, pupils have access to the textbook both at school and at home since it literally belongs to them. There is an approved list of textbooks from which schools can choose. Pupils in *England* are often given a textbook for the duration of a lesson. They hardly ever are allowed to take books home and they are provided with different books for different topics. There is no expectation that pupils would buy their own except, occasionally, if they are aiming for high GCSE passes in Key Stage 4.

What are the general features of English, French and German textbooks?

Textbooks in the *Gymnasium* and *Hauptschule* have a very formal appearance, with grey hardback covers. Given that they are for particular school forms, they look remarkably similar and even a cursory glance confirms that they follow essentially the same core curriculum. Pages themselves are densely packed with information or exercises, and the exercises are laid out in two columns on a page with extensive use of language and some use of diagrams. There are few worked examples.

French textbooks are all of A4 size and have a soft cover, making them lighter for pupils to carry – as French teachers and inspectors mentioned. They are of 'lively' appearance, colourful text interspersed with cartoons and pictures.

English textbooks are smaller in size and less 'dense'. What is particularly distinctive about the popular selling textbooks analysed in this study, compared with the French and German equivalents, is the avoidance of language in general and in the exercises, therefore, the lack of questions set in context.

In order to develop an understanding of the textbooks in the three countries, we have conducted a statistical analysis of the chapter contents in terms of proportion of space devoted to introductory exercises, explanations, worked examples, exercises and 'others'. Under 'others' we noted quizzes or revision parts, amongst others, at the end of chapters.

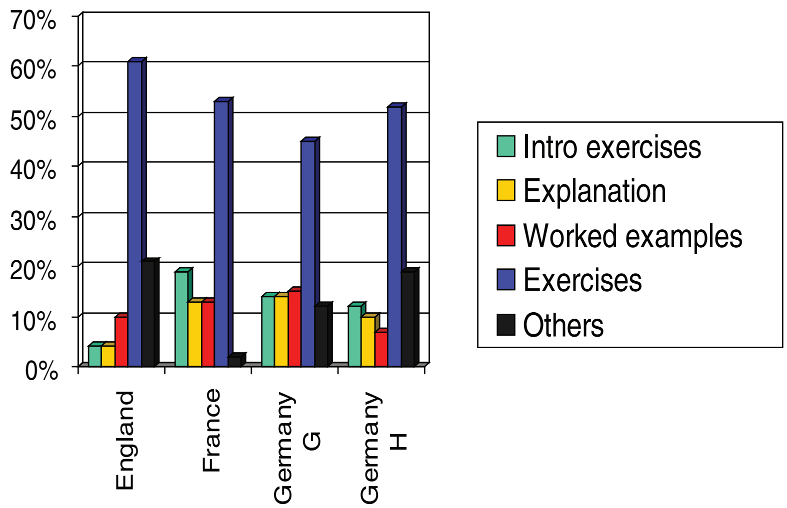

Figure 1

It is clear from the chart that exercises are the predominant feature of all textbooks analysed in the three countries. It is also remarkable that in English mathematics textbooks exercises take up – comparatively – the highest percentage in terms of space, and compared to other parts, this percentage is lowest in German *Gymnasium* books. The emphasis on exercises supported us in our view that mathematical tasks (in introductions or as exercises) are an important feature of textbooks and likely to have a significant influence on student learning.

Referring to what we had previously found out (Pepin & Haggarty, 2001, 2004), that is that French and German teachers spend a considerably larger amount of time on explanation and class discussion, it was not surprising that the introductory exercises section and the explanation section of textbooks reflect that practice, and French and German textbooks devoted more space to the first two categories than the English textbooks. Research on instructional methods has highlighted the role of meaningful explanation rather than relatively unguided 'practical' symbol manipulation activities in promoting problem-solving competence (Mayer, 1987).

The worked example section in textbooks does not seem to differ significantly from country to country, in terms of space devoted to it. However, it is lowest in England. Another feature of English textbooks was that they had the worked examples interspersed into and between the exercises, whereas in French and German textbooks the worked examples were provided at the start, usually after the explanation section. Interestingly, in French textbooks there was a clear distinction made, and this was highlighted with different headings, between introductory activities (*activités*), the essential part to be learnt (*cours*), the worked examples (*apprendre à résoudre*), and the exercise section (*exercises et problèmes*). Thus, in French textbooks worked examples were a category of their own and on 'equal level' with other sections of the book. Research on multiple representations and case-based reasoning has demonstrated the crucial role of worked examples and concrete analogies in supporting students with their problem-solving skills (Mayer, 1987). Worked-out examples help to model problem-solving processes, and concrete analogies provide a means for connecting procedures to familiar experience (Mayer, 1987, in Mayer et al., 1995).

It is evident, and we expected this from previous research (Haggarty & Pepin, 2002), that introductory exercises, such as activities, are a particular feature of French textbooks. We analysed the three best selling textbooks in France and noticed that they each had sections outlining how they were structured, with each chapter having, typically, 'activités, l'essentiel, exercices'. The purpose of each was explained, and in particular the reader was told, that the exercises had four sections, each with its own objectives: to consolidate; to apply the ideas in the 'core' or essential part of the chapter; to read, understand, and communicate the ideas and to become familiar with specific vocabulary, expressions and conventions; and finally to solve problems which required more reflection and initiative.

This can be understood in the light of pedagogical developments in France. In France textbooks are mainly written by mathematics inspectors (who regularly inspect mathematics teachers and, partly on the basis of this inspection, determine the speed at which each teacher moves up the salary scale). These features thus reflect the pedagogical concerns and emphases of those inspectors. The traditional 'cours magistral' (lecture types of lessons) has almost entirely been replaced by lessons organised in three parts: the activity (cognitive activities that are given to introduce the main notion to be studied); the cours (the essential part of the lesson); and the exercises. All textbooks analysed reflected this structure, whichever publisher. It could be argued that it was in the publishers' (and teachers') interests to follow the advice offered by inspectors. Thus, textbooks represented best pedagogical practice amongst a significant group of mathematics educators, and teachers based most if not all of their teaching on them.

What kinds of connections are made in textbook tasks?

In general, it can be said that both French and German textbooks provide more verbal explanations, in particular in tasks. This is not only evident from the 'density' of pages, and use of page space for explanations, but also when analysing the particular tasks. In particular, English textbooks excel in devoting page space to unsolved exercises.

An interesting feature of French textbooks was the emphasis on explanation of problem-solving procedures, in fact one subchapter was named 'savoir a resoudre' – to know how to solve. An emphasis on understanding the process of problem-solving is reflected in the use of worked-out examples which model the problem solving process in various ways, usually in words, symbols and illustrations. In the French case, this was often extensive, but not always consistently done throughout the different chapters. Research on the teaching of problem-solving processes indicates that successful programmes rely on the use of cognitive modelling techniques – such as detailed descriptions of worked-out examples (Mayer, 1992). This, it might be argued, helps students to link to their own models, or help them to develop those. Worked-out examples model appropriate problem-solving processes and analogies provide a means for connecting procedures to familiar experiences (Mayer et al., 1995).

In all three countries similar analogies are used for the representation of 'Directed Numbers' in tasks. These refer most commonly to temperature (rise and fall), lifts, water level and historical (pre-historical) events. In English textbook tasks the analogy of temperature (changes) was dominant. Indeed, in relation to one of the English textbooks (KM), the teacher file explicitly identifies the "Use of temperature and thermometer scales" as one of its cross-curricular aspects.

However, the use of the concrete analogies, and their connections to worked-out examples, were infrequent and usually at the beginning of a chapter. Only in the German case did we find several different analogies used in connection with worked-out examples. Research on multiple representations and reasoning has shown the important role of worked-out examples and concrete analogies in helping students to improve their problem-solving skills (Mayer, 1987).

Moreover, in most French and all English cases different analogies were employed for different operations, and none of the analogies were represented in a multiframe illustration that depicted changes that corresponded to those operations. Only in German textbook tasks were mostly the same concrete analogies used throughout the lesson tasks of directed numbers, and these related to 'change of state' and in different contexts.

In introductory tasks (which may help to introduce the concept) there were more meaningful instructional methods and explanations provided in French and German textbooks than in English textbooks. Research in mathematics education emphasises the importance of helping students build connections among multiple representations of a problem and of helping students induce solutions based on experience with familiar examples (Grouws, 1992; Hiebert, 1986).

It was interesting to note that in the English textbook the concept of the number line was not used. One might expect, for example, that the first chapter in 7[2] entitled 'extending the number line' would refer – albeit in some accessible way – to the abstract number line and the move from natural numbers to integers. The reason for this might be found with reference to the 'early' research literature in England where the use of number line was discouraged (Hart, Kerslake, Brown, Ruddock, Küchemann, & McCartney, 1981).

> the number line should be abandoned, despite its proven effectiveness for addition, in favour of a more consistent approach, for example one in which the integers are regarded as discrete entities or objects … (p. 87)

More recently Kilpatrick et al. (2001) have promoted the number line as 'a picture that lets you think about rational numbers geometrically' (p. 87). They claim that is provides a link between arithmetic and geometry and that "The potential for organising thinking about number and making connections with geometry seems not to have been fully exploited" (p. 87).

They argue that the potential of the number line does not stop at providing a simple way to picture all rational numbers geometrically, but it also lets us form geometric models for the operations of arithmetic. For example, in order to add negative numbers, one must provide the segments an 'orientation', usually represented as arrows (that point into different directions for positive and negative numbers). In another method some numbers are interpreted as points and others as arrows. Thus, it can be said that numbers on a number line have a dual nature: they are simultaneously points and oriented segments (represented as arrows). Kilpatrick et al. (2001) contend that it requires flexibility in using each interpretation and that this does not make it easy in using it despite its pictorial nature

Turning to *concepts used in introducing negative numbers in introductory tasks*, French and German textbooks built connections among symbolic, verbal and pictorial representations for some of the steps in problem-solving. However, they were far too few and not consistently used throughout the chapters, that is different instructional methods for different cases. In German tasks the lesson was often organised inductively, beginning with a familiar analogy and ending with a formal statement of the solution rule. In contrast, the English textbooks often failed to connect the verbal and symbolic representation of the problem to a pictorial representation. Here often a deductive approach was used, stating a rule and telling the learner to apply the rule to exercise problems and tasks. In other cases English textbooks used symbols and pictures to present the procedures, but failed to connect them to words. There was no general description, and the book then moved directly to exercise tasks without presenting the solution rule. In summary, French and German books presented mostly complete explanations of at least one example of the concept application (operations: addition, subtraction etc). In these examples, steps in the procedure were presented symbolically, verbally and often pictorially, and the solution rule was clearly stated at the end. Most French and German textbooks presented multiple representations of example problems and in particular German books presented material in inductive ways, whereas most English books did not provide meaningful instructional methods.

In terms of *context embeddedness and cognitive demand* of textbook tasks, it was surprising to see such low cognitive demand in most textbook tasks in all three countries. Putting them on a spectrum, German tasks were more cognitively demanding than those in French and English textbooks, with the majority of English tasks being of relatively low cognitive demand. Indeed, an important question here is, and this is the difference between tasks, whether tasks are likely to engage students at a surface level or whether they engage them at a deeper level.

Furthermore, it appears that textbooks put different emphases on context embeddedness in different countries. Whereas in English and German textbooks approximately half of all tasks were context embedded, in French books this was reduced to approximately a third. This means that for any German, French or English student at least every second task is situated totally in the abstract world of mathematics, thus not likely to connect to the world students live in. It has to be seen to what extent the remaining context embedded tasks may help student appreciation of mathematical ideas and thinking.

To exemplify this aspect we have identified two similar questions/tasks from one English and one French textbook. In the English textbook and in the case of early work on negative numbers, and as noted before, questions are all asked in context – usually that of temperature – but also of lifts in buildings, quiz shows and altitudes, for example. Many of these contexts are suggested in the Key Stage 3 National Strategy for mathematics (DfEE, 2001). But without a generalised understanding of negative numbers, each question is tackled in isolation and, in the textbook, can be answered using common-sense techniques, which are in danger of remaining disconnected and context dependent. An example in year 7 KM textbook is given in the following:

Here is the control panel in a lift.

4	Fourth floor
3	Third floor
2	Second floor
1	First floor
0	Ground floor
-1	Underground car-park

What number is used for the ground floor?

Where do you go if you press -1 ?

You go from the first floor to the fourth floor. How many floors do you go up?

You go from the third floor to the car-park. How many floors do you go down?

(KM 7.2, p. 11)

It appears that the underlying ideas about negative numbers are subordinated to the context, and the context largely points pupils towards a context-bound and common-sense solution. The context hides the fact that it relates to negative numbers. Indeed, the whole exercise is no more than a number of context-bound questions which relate neither to each other nor to the underlying mathematical ideas of negative numbers.

A task with the same context was found in a French textbook. On page 218 of ‘5 sur 5’ (6eme) (no2) is says:

> A tower consists of 12 upper levels and 4 levels below ground. How do you imagine the lift board to look like? (p. 218) (personal translation)

This task was illustrated with a cartoon where a man waits with two heavy suitcases and a backpack in front of the lift which is 'out of order'.

Whilst outwardly quite similar, the difference between the two textbooks were that in the English textbook the learner was left to solve the problem using common sense, whereas in the French textbook the student was referred to a worked example earlier in the chapter, thus giving the learner guidance in terms of problem solving.

Referring to Skovsmose's *real-life examples* of tasks (Skovsmose, 2002), there were none in the English textbooks, and only very few in the German and French textbooks. Skovsmose talks about 'learning milieus' and distinguishes between three different 'paradigms of exercises': those with references to pure mathematics (context unembedded we called them); those with reference to a semi-reality (context embedded tasks, albeit contracted realities); and 'real-life references' (Skovsmose, p. 119). Considering that real-life exercises and tasks might give students meaningful contexts and connections to familiar experiences, it is interesting that there are so few in any of the textbooks we analysed.

In terms of *connections to other areas of mathematics* and interconnectedness of mathematical content knowledge, it is interesting to note that hardly any connections were provided in English textbooks, in comparison to French and German textbooks. However, there were differences in connections between German and French textbook tasks.

In French textbooks, whenever the notion of Directed Numbers was extended beyond what had been learnt previously, there was a review which linked the previous knowledge to what was to come. In additions, it appeared that, and this was mentioned earlier, there was a progression from chapter to chapter and from year to year, firstly in terms of building on previous knowledge, and secondly in terms of mathematical challenge.

In German textbooks perhaps the most distinguishing example of how the mathematical knowledge was connected 'within' is given by chapter 13 where the number system is explained and how the rational numbers fit into it (p. 108). Although the German *Hauptschul* book (Einblicke Mathematik) treats 'Rational Numbers', and they use this term as the chapter heading, in less depth and less mathematically, they nevertheless show its place in the number system.

Although the Mathematics National Curriculum for England (DfEE, 1999) does not itself make explicit connections between topics in mathematics, it recommends that:

> At key stages 3 and 4, teaching should ensure that appropriate connections are made between the sections on number and algebra; shape, space and measures; and data handling. (p. 6)

When introducing negative numbers to pupils in year 7, one might therefore expect to find connections being made with pupils' existing understanding of numbers and the number line, as well as possible connections with co-ordinates, equations, graphs and so on. As the topic progresses, additional connections are likely to emerge depending on the particular route chosen through the curriculum.

According to our analysis of English textbooks, few connections are made explicit in relation to negative numbers and it appears to be left to the teacher to ensure that 'appropriate connections are made'. The textbooks use the device of 'replay' to remind pupils of previous knowledge, and this is mainly done through exercises and tasks. The emphasis is on re-teaching rather than encouraging the pupils to try to remember what they had previously learnt. There are disappointingly few opportunities provided for pupils to identify for themselves what they already know and then build on from there.

CONCLUSIONS

The review of the literature in the different areas (mathematics textbooks internationally; mathematical tasks; connections) and the analysis of English, French and German textbooks has deepened our understanding of mathematical learning in several ways. Firstly, it reinforces the idea that mathematics learning is a complex activity and it is likely to be influenced by learning cultures of different settings, whether these are nationally specific or context bound in other ways. One could imagine that some of the factors influencing the learning cultures are country specific (influenced in turn by inspectorial guidelines and language boundaries, for example). Others are likely to be trans-national in particular cultural 'learning spheres' (Anglo-American, for example), and again others might not be country-boundary limited at all, perhaps based on cultural regional values (Wang & Lin, 2005).

Secondly, and more particularly, we argue that different countries provide different (mis)representations of mathematics for their students in school textbooks, in particular with respect to tasks and connectivity. Despite perhaps gaining proficiency at certain kinds of procedures and tasks, English students – compared to their French and German colleagues – are likely to have gained at best a fragmented sense of the mathematics and understood few if any connections that tie together the procedures they had studied in textbooks. It can be argued that through these disconnected activities students are likely to develop perspectives on mathematics that may impede them in their use and acquisition of other mathematical knowledge. Whereas in some countries students are inundated with skills, procedures and disconnected mathematical knowledge, in others students are allowed to develop an appreciation of its interconnectedness and generalisable nature.

Finally, it is argued that textbooks, and tasks in textbooks, play an important part in the mathematics education process, and that tasks in textbooks (as well as their mediation) need to be understood in terms of a wider context, in order for shared understandings, principles and meanings to be established, whether for promotion of classroom reform or simply for developing a better understanding of this vital component of the mathematics education process.

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Workshop on classroom research

WORKSHOP ON CLASSROOM RESEARCH ANALYSIS OF TRANSCRIPT DATA

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This paper is a report of the workshop initiated by the Nordic Graduate School in Mathematics Education. The paper focuses especially on the decisions taken in the planning of the workshop and how the plans were implemented. The value of a workshop lies particularly in the participation, knowledge emerging from activity rather than being communicated directly by a presenter. Consequently this paper is not prepared with the intention of communicating techniques of transcript data analysis.

FOREWORD

A successful workshop will provide opportunities for all participants to contribute and ideally the report will reflect the many voices. In that all participants did have the opportunity to participate in the workshop reported here, and it is believed by the authors that almost all participants made use of this opportunity, thus this is a report of a successful workshop. However, it is admitted from the outset that this report does not achieve the 'ideal' by adequately reflecting the many voices. The three authors of this report made planned contributions in the workshop and this report includes details of these separate presentations. The authors have not sought to speak with a 'single voice'[1], on the contrary as will be evident in the following discussion, the complementariness of 'voices' is seen to be of crucial importance in the workshop and this 'separated complementariness' is an important feature of this report. The authors take joint responsibility for the report even though it is written in a style in which each part can be attributed to a single author.

Simon Goodchild was invited by the Nordic Graduate School in Mathematics Education to organise the workshop. Subsequently Barbara Jaworski and Heidi

[1] We claim to speak with separate voices but we work within the same broad compass of a naturalistic paradigm, we each write from our individual positions and understanding, it is thus important for us to make it clear which of us is 'speaking' at any time and that we make clear the personal nature of our understandings and interpretations by use of first person pronouns.

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S. Måsøval were invited to contribute as outlined in the report that follows. The report begins with Simon Goodchild explaining the conception of the workshop and discussing the decisions taken during the planning stage.

INTRODUCTION

Workshops should be an enjoyable activity within a conference. They provide an opportunity to get together with other stimulating colleagues and collaborate on a task. The activity of working together, with a common focus is intended to stimulate discussion and create opportunities to learn from each others' experience as well as the task in hand. When I was approached by the board of the Nordic Graduate School in Mathematics Education to plan and run this workshop I was both pleased to have the opportunity and challenged by the demands which the task entailed. In the following pages I outline how I responded to the challenge. First I describe my experience, beliefs and some of the constraints that guided me in the initial planning. I follow this with an explanation of the aims that I devised for the workshop and then the planning phase that translated the aims into activity. In the programme that was eventually devised, at my request, Heidi S. Måsøval and Barbara Jaworski made significant contributions, they helped me to formulate the programme and drew my attention to crucial operational details. Significantly they both made valuable contributions through presenting some examples from their own classroom research. There follows a brief description of the task in which small groups worked and some details about how this was organised. The paper concludes with some remarks about the final plenary session of the workshop.

INITIAL PLANNING

There are a number of important issues to consider when planning a workshop at a major conference. Foremost amongst these is the very notion of 'workshop'. I guess many have had the unfortunate experience of going to a workshop to find that the convenor uses the occasion as an opportunity to present a 'lecture' that did not have to go through the usual rigorous review process. People attend a workshop with the expectation that they will be actively engaged in working on a task and that their understanding and knowledge will develop through the activity. At least, that is, I believe, the key principle of a *workshop*. In a major conference such as NORMA05 this presents a particular challenge because the conference brings together a large number of researchers with a great variety of experiences.

The intention of this workshop was to focus on classroom research, particularly on the interpretation of transcripts that arise from classroom situations. Inevitably there will be participants who have a great deal of experience in this field, and it is important to give everyone the opportunity to learn from each others' experience. There will also be participants who are at the very beginning of their research career who may feel overawed by the experience of others and that they, themselves, have little to offer. A workshop should offer the opportunity for everyone to engage *actively*. No-one should be made to sit on the periphery as a mere 'observer', although I would not deny anyone the right to take that position if that is what they *wanted*. There should be opportunities for the experienced to develop their knowledge through fresh reflection on a familiar task and opportunities for the newcomers to realise that, despite their limited experience of research, they too have something of value to offer.

I have found in my own experience of classroom research that some of the most valuable comments have come from undergraduate students on teacher education courses with whom I have occasionally shared data on which I am working. It is possibly because they see things from such a completely different perspective. My years of experience in the classroom as a teacher and classroom observer/researcher, and knowledge of the literature in the discipline and theory of teaching and learning in classrooms can prevent me seeing the apparently simple or naïve questions or observations that have the potential to take the exploration of the data further. Thus in planning a workshop on classroom research it is important not just to indicate that everyone has an important contribution but to organise the activities so that everyone feels able to contribute. These two criteria, active engagement in a task and creating a context that makes participation easy, suggests that a substantial part of the workshop should be spent in small group activity.

NORMA05 attracted over 100 participants and I was led to expect, possibly, up to 80 who would want to participate in the workshop. I believed that the optimum group would have 5 to 8 participants so it was necessary to prepare for up to ten groups. These numbers present further challenges. The notion was that participants would work on data, real data that has arisen from real classrooms. First, the issue of how the data will be presented must be considered. Is there sufficient equipment to show videos, or play audio recordings to ten groups? I worked on the assumption that this was not a practical possibility. Thus a decision to use only printed transcripts was made. Second, because participants were drawn from several countries, and the language of the conference was English, I decided to use transcripts arising from an English classroom. It might be suggested that any language could be translated into English. There are great problems in making interpretations of students' activity based upon translations of their conversations; not least because the translation is, already, an interpretation. Should each word be translated to its closest equivalent (not a clear cut decision to make) or should the translator try to capture 'meaning' and use expressions that would be likely to be used by students of similar age and development in that language? In the research project in which I participate at Agder University College we have taken the decision that all analysis will be based upon data in its original language, data is only translated for the purposes of publication in international journals and conferences.

AIMS

The board of the Nordic Graduate School in Mathematics Education (NoGSME) that had invited me to run the workshop at NORMA05 left me to interpret the theme of analysing data from classroom research as I wished. I was informed that the workshop would occupy three and one quarter hours including a short 'tea break'. To be fair to the board they were not, in anyway negligent in this, the director of the board was always available and interested to discuss the plans as they evolved and I am grateful for her input and support. My first step was to develop the theme into a number of aims that I felt were achievable:

- To demonstrate that classroom episodes can be interpreted in a variety of ways, each with a validity that arises from whatever theoretical perspective (or set of beliefs, values and assumptions) held by the interpreter.
- To demonstrate that 'we' learn about these multiple perspectives by listening to each other, and from questioning the beliefs, values, etc. upon which interpretations are articulated.
- To demonstrate the role of theory in making interpretations and the role of empirical evidence in the development of theory.

The first of these aims I believe to be central to all research, in my own development as a researcher I am grateful to Leone Burton who drew my attention to the importance of being clear about ones own beliefs, values and assumptions. For further reading consult (Burton, 2002; also Lincoln & Guba, 1985), Leone Burton also referred me to a powerful research report by Carrie Herbert in which Herbert is at pains to explain her own position in the research (Herbert, 1989). Thus I start from a position in which I am interested to learn from the possibility of multiple perspectives and I am very wary of anyone who suggests there is a right or a wrong interpretation, I can accept 'more likely and less likely' but in classroom research we need to be prepared for different interpretations and the impossibility of arriving at a single truth.

The second aim follows closely from the first, and my own experience which I have mentioned above. I do not want to deny that we can learn from those with much more experience than ourselves, or who have a much deeper knowledge of the literature. However, I do believe that each researcher comes to a set of data with a unique set of personal experiences that has the potential to shed new light on data. Thus I want to encourage the novice to speak up and the experienced to have the space to listen. Discussion needs to take place in a critical but mutually supportive context. I believe we are fortunate in mathematics education in that we belong to such a critical and supportive community.

The third aim arises from a deeply held belief of my own. In classroom research I do not believe it is possible to conduct 'proper' ethnography, that is to act as the 'observant stranger(s) whose ignorance they themselves take to be a condition for eliciting from informants explicit accounts of the obvious and basic aspects of culture and everyday practice' (Lave, 1988, p. 185). This belief arises from the (reasonable) assumption that no classroom researcher enters a classroom without many years of experience of working in classrooms in a variety of guises. In the same way, I do not believe it is possible to approach data with a 'blank sheet', so to speak, free from any prior experiential or theoretical influence. It is important then for the research to make his or her own theoretical position explicit and use that as the starting point of analysis. Thus I do not believe that a strictly 'grounded theory' approach to data analysis as described by Glaser and Strauss (1967) is possible in classroom research. However, this statement of belief must be balanced by a similar statement that acknowledges that theory should be rooted in and developed from and on the basis of empirical evidence. Richard Pring (2000) takes this further when he argues for the dynamic interrelationship between theory and research on the one hand and practice and policy on the other. Thus I am neither dismissing the importance of intelligent interpretation of data nor the intelligent use of theory, both are significant in classroom research. The fact remains, however, that there are different theoretical perspectives that will lead to alternative interpretations of data. I believe that classroom researchers need to be open to explore and learn from different perspectives even when, perhaps especially when, the alternative theoretical perspective is inconsistent with one's own.

PROGRAMME

The next challenge in planning the workshop was how to turn these aims into achievable objectives. There needed to be something stimulating to set up the activity in smaller groups. My first thought was to start the workshop with three brief presentations made by people experienced in classroom research who would demonstrate different approaches arising from different theoretical positions. I knew something of the work of Barbara Jaworski, and I had recently read a very interesting paper by Heidi S. Måsøval based on the application of a theoretical perspective developed by French researchers. I decided to ask them if they would be prepared to participate in this opening session. I was very pleased that both agreed to do this, I planned to make a similar presentation myself. The planning for the workshop continued through our collaboration. Before long it became apparent that my initial idea of restricting each of these presentations to 20 minutes was unrealistic and an unhelpful restriction. For reasons expressed above I did not want to make this plenary presentation any longer than one hour and thus I decided that I would not present my own analysis but rather use the transcript I had intended for my presentation as the substance for the group work.

The programme took the following form:

Part 1: Introduction to the Workshop. Simon Goodchild
Presentation of analysis of classroom episode 1. Heidi S. Måsøval
Presentation of analysis of classroom episode 2. Barbara Jaworski

Break

Part 2: Introduction of group activity. Simon Goodchild
Group activity – working on transcripts of episodes with two 14–15 year old students.
Feedback from group activity including analysis by Simon Goodchild.
Plenary discussion in which all participants have the opportunity to question and respond to issues arising from both parts of the workshop

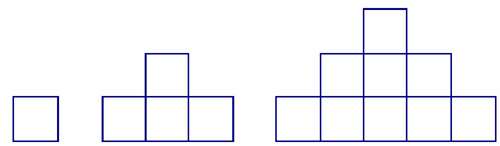
PRESENTATIONS

Heidi S. Måsøval now summarises her presentation of an analysis based on her current doctoral research into students' 'authoring' of mathematics.

The episode focused on three students, Alise, Ida, and Sofie (pseudonyms) who are student teachers in their first year of a programme of teacher education for primary and lower secondary school. At the time the data was collected, they had been collaborating on several tasks in different topics during the five months they had been on the programme. Along with his colleagues, the teacher educator[2] (who teaches mathematics to the group of students) is concerned about development of relational understanding (Skemp, 1976) for students in mathematics.

The transcripts used in the presentation were made from a video recorded small-group work session, in which the students are supposed to collaborate on a generalizing problem in algebra. The teacher has designed the task, aiming at developing competence in conjecturing, generalizing, and justifying for the students. The first part of the task handed out is set out in the figure below.

[2] The student teachers will be referred to as 'students', and the teacher educator as the 'teacher'.



Here the first three figures in a pattern are illustrated.
You may use centicubes to illustrate.

How many cubes are there in the fourth figure? In the fifth?
How many do you think there will be in figure number 10? And in figure number n?
What kinds of numbers are present in these figures? In each stripe, and totally in the figure?
Can you express, as a mathematical statement, what the figures seem to show?
– With words? – With symbols?

The students collaborate for 6 minutes. They find out that the stripes in the pattern consist of odd numbers, and that each whole figure consists of a square number. They have agreed on $F(n) = n^2$ as a representation of the general term of the sequence of staircase towers, but have not revealed any connection between (sums of) odd numbers, and square numbers. There is uncertainty connected to the concept of 'a mathematical statement'. Sofie has focused on an implicit relation between consecutive terms in the sequence of staircase towers, asking Alise and Ida if they are supposed to show the increase (from one figure to the next).

There are multiple interpretations of what a mathematical statement in this context might look like, and there is no requirement in the task that precludes an implicit or recursive relationship to be formulated. A mathematical statement expressing an implicit relationship between two consecutive terms of the sequence would be legitimate, and could be formulated for instance as, "Square number (n+1) equals square number n, plus odd number (n+1)".

The teacher's intention with the task is the formulation of the fact that the sum of the n first odd numbers equals n^2, an explicit relationship between the position of a term and the value of the (same) term of the sequence. The evidence which indicates this is a *funnel pattern* (Bauersfeld, 1988) in the interaction between the teacher and the students. The (from the teacher) expected notion 'square number' from Alise brings the teacher to the presentation of the solution of his interpretation of the task. He reveals that the sum of the ten first odd numbers equals the tenth square number, representing in natural language an explicit relation between the position and the representation of the general term of the staircase tower sequence. When the teacher characterizes the outcome of the dialogue as "almost a little discovery", it is an example of the Jourdain effect, which is a form of what Guy Brousseau (1997, p. 25) calls Topaze effect. The Jourdain effect is characterized by the teacher's disposition to "recognize the indication of an item of scientific knowledge in the student's behaviour or answer, even though these are in fact motivated by ordinary causes and meanings" (Brousseau, p. 26).

Now the task implicitly is reformulated or narrowed, so that the problem is interpreted to be what the teacher originally had in mind when setting up the task:

The students are supposed to represent with symbols what the teacher has stated generally; that the sum of the *n* first odd numbers equals the *n*-th square number. This process of step by step reduction of the teacher's presumption of the students' abilities and self-government is "quite opposite to his intentions and in contradiction even to his subjective perception of his own action (he sees himself 'providing for individual guidance')" (Bauersfeld, 1988, p. 36).

Brousseau (1997, p. 30) defines an *adidactical situation* to be a situation in which the student is enabled to use some knowledge to solve a problem "without appealing to didactical reasoning [and] in the absence of any intentional direction [from the teacher]". The *devolution* of an adidactical learning situation is the act by which the teacher makes the student accept the responsibility for an adidactical learning situation or for a problem, and (the teacher) accepts the consequences of the transfer of this responsibility (Brousseau, p. 230). The teacher has designed an adidactical situation which presupposes the students' mastering of a technique of expressing a general odd number. Because the students do not master this technique, there is a didactical problem. The teacher has not "contrived one [adidactical situation] which the students can handle" (Brousseau, p. 30); the devolution of the problem specific to the construction of the target knowledge has not worked well.

When Alise asks for the teacher's thinking, she takes into account features related to the *didactical contract* (Brousseau, 1997). This, together with the funnel pattern of interaction mentioned above, and the fact that the teacher does not succeed in hiding his will and intervention as a determinant of the students' focus and action, causes the collapse of the adidactical situation. For an account of evidence in the data supporting these interpretations, and further analysis, see (Måsøval, 2005).

The aim of my presentation was to give an example of how theory had been helpful in order to understand and describe why a small-group collaborative session in mathematics had been less successful with respect to mathematical learning. Brousseau's (1997) theory of didactical situations in mathematics had been a tool to explain how mathematical rationality had become secondary to the striving for fulfilment of the didactical contract.

I used the video of the session to illustrate my presentation. This posed some interesting ethical issues. Firstly, in giving permission to film the session the students had not been made aware of the possibility that the film might be used in a conference presentation. Consequently they were contacted and the necessary permission was given. Secondly, the teacher who featured in the film was also a participant at the conference. Was it fair to use this video clip in this circumstance? I am very grateful to the teacher, who raised no objection to the film being used. Video records of classroom episodes are powerful data sources and I think we in the mathematics education research community are generally careful to get the necessary permissions from the people filmed; at least we most certainly should be careful in this respect. However, I wonder if permission would be so readily granted if at the initial stage we outlined all possible uses of the data.

Next **Barbara Jaworski** presents a summary of an analysis of an episode from a class of 12 year old students in which they work on a task set by their teacher, George.

The task involves fitting together four squares, each of side one unit, in various configurations (all squares touching at an edge or a corner, but not overlapping: e.g. Figure 4a) and finding out what perimeters are possible in the resulting figures. After doing this with four squares, the class is asked by George to extend their thinking to other numbers of squares, 2, 3, 5, 6, ... , and to try to generalise. One group of girls concludes that perimeters will all be even numbers, and the girls justify this conclusion. George then asks the girls whether they can find an *odd* perimeter. In tackling this challenge they arrange squares so that the edge of one square touches just half the edge of the adjacent one (Figure 4b).

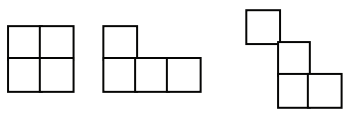


Figure 4a

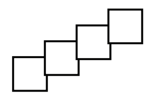


Figure 4b

We see the girls' activity and hear their conversation as they explore this situation. One of the research questions that was addressed through analysis of the video data was: *What sense are the pupils making of the mathematics they are working on?*

I draw attention here to some of the issues in interpreting the video data to approach an answer to this question.

The video pans from face to face, showing close-ups of facial expression and physical activity. The girls speak animatedly, interrupt each other, listen to each other, ask unfinished questions, doodle on paper; one girl keeps raising her hand presumably to call the teacher. We see finely highlighted details that can be replayed over and over to elicit evidence of sense-making and mathematical construction. However, the video also excludes. It has no peripheral vision, so we gain no sense of the wider classroom setting, what others in the class are doing, even the other students sitting at the same table. When we analyse data encapsulated in video we have to be aware both of what it affords and the ways in which it constrains analysis.

In the video extract, viewers see and hear George ask whether it is possible to find an arrangement of four squares that gives an *odd* perimeter. They explore various configurations, counting squares around the edge in each case. Finally they come to the arrangement shown in 4b and count the whole and half sides around the perimeter, coming to 13 units in total. The following transcript from the video captures to some extent the discourse involved.

14.	Kath:	So, because, this would be, let's start there, 2, 3, and there's the 4 and then there's the half. That makes it 4. No?
15.	Kate:	Have you got one inside there?

16.	Kath:	Yeah. Because we've got all these halves (other girls speaking simultaneously and indistinctly). Count whole sides. How many?
17.	Girls:	(All looking at Katherine's sheet while she traces the outline of the diagram) 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 and then.
18.	Kath:	Wait, wait. Look. (All count) 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13.
19.	Lucy:	So you can. If you have an odd number of halves. You know, if you have..
20.	Kath:	Yes.
21.	Lucy:	You can have 6 halves adding up to 3.
22.	Kath:	(Taps her pencil) Yeah. That would be it.
23.	Kate:	Yes. That means if you had 4, that using 4 you could make loads of different, couldn't you?
24.	Kath:	Yeah.
25.	Kate:	That'd be (indistinct, then laughter).
26.	Kath:	So you can make odd numbers if you use halves.
27.	Lucy:	If you use halves.
28.	Kate:	If you use halves. But if you don't use halves then you can't.

The video gives us access to concentrated engagement as the girls count and recount. How are they counting? Are they missing something? Count again. Lucy and Kate watch as Katherine's pencil moves around the figure denoting sides as she counts, and they count with her. They count the whole perimeter twice, and get 13 both times. The means of counting, whole square first (10) and then the six halves (making three) provides a rational for their conviction that the answer is correct. Statements 19 to 28 show their articulation of what they have just found, convincing themselves. For example, in statements 26–28, we have
So you can make odd numbers if you use halves.
If you use halves.
If you use halves. But if you don't use halves then you can't

All three girls contribute to these strong statements; there is no doubt in their voices. They *know* that an odd number perimeter is possible, and moreover, they know exactly how to generate it. They subsequently convince George, which provides further evidence of their knowing.

30.	Girls:	You can get odd numbers. (Girls all showing George how this works by tracing the outlines on their working sheet).
31.	Kath:	That's 13.
32.	Lucy:	Because if you have 6 halves then they add up to 3. So you make …
33.	Geo:	Right. Let's start then. 1, 2, $2^1/_2$.
34.	Kate:	No. You count, we counted the whole sides first and then added the halves. (Pointing repeatedly at sheet of paper).
35.	Geo:	Whole sides. What 16?

36.	Girls:	No. No. Count all the whole sides first.
37.	Geo:	Oh, all the whole sides, that's right. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. (He counts the whole sides in their diagram.)
38.	Girls:	Yeah.
39.	Geo:	Half sides.
40.	All:	1, 2, 3, 4, 5, 6. (Together they count the half sides.)
41.	Geo:	Hey, well done. (writes 10+3 = 13).

George starts to count in a way different from the girls. Confidently, they stop him. He has to count in their way.

An interpretative conclusion from just the video (represented in statements 1 to 56, transcribed for the purposes of the workshop, of which two extracts are shown above) is that the girls showed that an odd perimeter was possible and justified the validity of their claim through the mode of counting. However, they showed *more* than this. Although implicit in their articulation, we see a proof that arranging in half squares as they showed leads to the possibility of an odd perimeter. We see them do this only for four squares, one special case, but the explanation carries a generic sense of being true for all 'such' cases. Mathematically we would want to ask about odd numbers of squares, prime numbers, non-square numbers, and to generalize further. However, in their limited case, I claim that the girls have produced a wider reaching generality than just the perimeter for four squares, and that the video provides evidence of their insight into this generality. In terms of concepts of perimeter and area, it is possible to conjecture with some degree of certainty that they know the meaning of "perimeter", and can apply it to differing shapes. Less clear is their concept of area, although they are dealing implicitly with area as they consider shapes with differing numbers of squares. We see also their development of problem-solving expertise; perhaps in terms of confident application of Polya's four stage process: *understand*, *plan*, *carry out* and *look back* (Polya, 1945/1957, pp. 5–6).

Analysis of one small piece of video here provides some insights into classroom activity and cognitive processing, but the case is too particular for wider generalizing in these areas. A broader study would require many more such cases and extension to the wider (sociocultural) contexts in which such activity is situated (see for example Jaworski, 1994). However, we can see here the potential power of such recording and analysis for teachers and teaching, and for didacticians, to gain deeper understandings of students' and student-teacher interactions in the classroom milieu. Such analyses can be a powerful tool for clearer understanding of the contribution of teaching to learning and for the development of teaching, as well as providing research evidence for generalization in these areas.

The purpose of this report is only to offer an outline the workshop and thus only brief outlines of the presentations made by Heidi S. Måsøval and Barbara Jaworski are included. Simon Goodchild now continues to provide a report of the remainder of the workshop.

GROUP WORK

Following a break the task to be worked on in small groups was presented. The task was based upon transcripts (originating from my own research in mathematics classrooms) of two short episodes that occurred about four months apart. In these episodes it is the same two students who discuss their approaches to exercises from their regular text book. The following information was given to help set the context:

The transcripts were made from audio recordings of students working in a 'regular' mathematics classroom in England. Students engaged in the tasks set by their teacher and in the recorded discussions there was no 'researcher intervention' until after the reported discussion. The students are in their tenth year of school, they are 14–15 years old. In the class it is normal for students to collaborate in their activity. Students had a free choice about where they sat and with whom they worked. Gill and Jacky sat with each other in every mathematics lesson throughout the school year and did not collaborate with anyone else in the class, when one of the two was absent the other would work alone. In terms of measured attainment the students appeared to be reasonably well matched, during the course of the year there were three mathematics tests, the results for (Gill, Jacky) in each test were (60, 58), (36, 44), (64, 70); for each test the class statistics (lowest, median, highest) were (18, 55, 74), (12, 30, 58) and (10, 38, 80) respectively. Gill missed a large number of lessons (about 25 % throughout the year) due to poor health. The conversations take place in December and April respectively.

Experienced researchers were asked to act as chairpersons for the small group activity. Each was given the following set of instructions that were prepared with the intention of facilitating the discussion and achieving the aims of the workshop.

1. Try to focus discussion on interpreting the transcripts.
2. Try to 'enable' the discussion – try to avoid being halted by arguments that claim unknown factors are crucial to the interpretation (of course they are!) – encourage the group to make whatever assumptions are necessary to continue (and note the assumptions because they reveal issues that need to be considered when conducting any form of classroom research).
3. Appoint a member of the group (not yourself) to provide feedback from the discussion in the final plenary part of the workshop. Each group will have approximately 4 minutes to feedback.
4. Try (very hard!) not to dominate the discussion, try to draw every participant in the group into the discussion.
5. Pay particular attention to different interpretations – try to ensure that each of these is valued and that participants have an opportunity to explain or justify their interpretation.
6. Try to communicate that this is a task in which there is no 'correct' answer, the aim of the task is to expose a range of possible interpretations and engage participants in the analytic processes that support or refute particular interpretations.

Elsewhere in this volume (see paper 'Students' Goals in Mathematics Classroom Activity') I have drawn attention to the complexity of the classroom. In that paper I quote Paul Ernest who writes, 'the mathematics classroom is a fiendishly difficult object to study'. Students' activity within a mathematics classroom cannot, in any way, be separated from that complexity or the concomitant difficulty in making sense of what is going on therein. It follows that the analysis of a transcript of students' conversation within a classroom is a complex and challenging task. To present such a task in the context of a workshop, as has been described in the forgoing, places huge, maybe impossible, demands upon participants if they are expected to make sense of the transcript. Students' discussions about their mathematics tasks within a classroom context requires a considerable body of contextual information about the historical, socio-cultural arena that indeed would be impossible to digest within the time scale of a short working session. Thus groups were not expected to come forth with polished interpretations.

Any interpretation of a classroom event can be contested, thus the more supporting evidence that exists for an interpretation the more likely it is to be accepted. The supporting evidence will be found through a detailed historical, socio-cultural analysis of the class in which the students featured in the transcript are situated. Support for any interpretation should also be taken from existing research output and the growing body of literature that focuses upon students' mathematical knowledge, learning, discussion and classroom activity in general.

As the complexity of the classroom context becomes realised within a complex body of data one further challenge faces the analyst, this is the choice of a suitable unit of analysis. I am unfolding this discussion in a fairly linear, sequential manner so it is necessary to express my own opinion that the choice of unit of analysis should be made before any data is collected, otherwise, I ask, how does the researcher know what data to collect? Further, the unit of analysis emerges from the theoretical framework in which the research is placed, it does not arise when the researcher surveys the data and then asks: what on earth am I going to do with this?! In the previous two paragraphs I have made my own position regarding the inseparable nature of cognition and socio-cultural theory quite evident, it should not be surprising that in my own work I seek a unit of analysis that allows me to both represent the socio-cultural context of activity as well as the transformations that might be recognised as learning. For this I find the 'complex model of an activity system' that Engeström (1999) develops from Vygotsky's (1978) account of 'a complex, mediated act' very useful. Engeström's model allows the problematisation of the web of dialectical relations between student, task, resources, community, rules and division of labour which are characteristic of activity in mathematics classrooms.

Thus given the complexity of the task, the analysts need for a broad body of data, a clear theoretical framework and a well defined unit of analysis there was little expectation that the groups in the workshop would produce a range of viable interpretations. The best that might be achieved would be that the transcripts would form the starting point for a discussion of the issues involved in analysing this type of data.

FINAL PLENARY SESSION

In workshops of this kind the final plenary reporting session is important. It helps to focus group activity on the task in hand, it also helps to ensure that everyone benefits from the discussion in other groups and it provides an opportunity for more general discussion on issues that have arisen. Groups were asked to appoint someone to report back, this to be a person other than the chairperson with the intention of promoting wider involvement of participants. More recently I have been told of a technique used by one teacher who asks groups to assign a number to each person and only at the reporting back stage does the teacher indicate the number of the person in each group who should report. This seems a good idea in a school classroom, as the teacher explained, it ensures that every member of the group engages in the task and that the whole group has a responsibility to ensure everyone understands. I can see this as an effective technique in a school classroom. I am not sure whether it would work in the type of workshop situation that I describe here but I think it is worth trying.

The reporting back session is important but there are concerns. First with many groups to report back each group has only a limited time to give their report and inevitably this will be insufficient to do justice to the discussion that has taken place. It is important, therefore, for groups to prepare summaries that draw attention to key points emerging from the discussion. Second there is a danger that every group gives a similar report and thus the session can become very repetitive. There is also the challenge faced by the person leading the plenary session when a report goes significantly beyond the allotted time. Despite these concerns, this final plenary session is seen to be sufficiently important to include it in the workshop.

Anyone who has engaged seriously in the task of analysing classroom episodes, whether it be from video, audio or transcript data, will know that analysis demands a great deal of time; far more time than is available in an afternoon workshop. I set out arguments at the beginning for the form the workshop should take, that is participants should have the opportunity to *work* on data. It would be naïve to expect anything concrete to come from the group activities, the best that might be hoped is that participants within the groups would be made more aware of the demanding nature of this form of analytic activity. The discussion that took place in the plenary session focused especially on the difficulty of making any sense of the transcript provided without rather more contextual details than had been given at the outset. This type of outcome had been anticipated, nevertheless it was evident that the groups had been engaged in productive and meaningful discussion that had the potential to illuminate and inspire. In this respect, and given the valuable contributions of Heidi S. Måsøval and Barbara Jaworski I believe it is reasonable to believe that the workshop achieved the fairly modest goals that were set.

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Paper presentations

FOCUS ON TEACHERS' REFLECTIONS AND THE ROLE OF THE RESEARCHER IN COLLABORATIVE PROBLEM SOLVING IN ALGEBRA: BUILDING A LEARNING COMMUNITY

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This paper is related to my ongoing research concerning the possibility to enhance the learning and teaching of elementary algebra through the creation of a learning community. It explores teachers' reflections when engaging in problem solving related to elementary algebra in collaboration with a researcher; and also the role played by the researcher. The data presented here illustrate the way a community of teachers and a researcher engaged in a mathematical task through their reflections emerging from this process. Preliminary results suggest that engaging in such activity increases teachers' awareness of their own ways of working in mathematics and of the complexity of the teaching situation.

INTRODUCTION: RESEARCH FOCUS

A learning community consisting of three teachers and a researcher is studied and the participants' emerging reflections are analysed and presented. The aim of the research is to look at the way in which developing a learning community can lead towards a deeper understanding of algebraic thinking for the teachers. Through the data presented in the paper the reader can follow our group working on a particular mathematical task chosen by the researcher. An analysis of teachers' reflections, as they emerge through the workshop, is proposed.

The notion of learning community is rooted in Wenger's (1998) community of practice, and is closely related to the notion of "community of inquiry" (Jaworski, 2004a) which may be characterized as:

> In a community of inquiry, inquiry is more than the practice of a community of practice: teachers, develop inquiry approaches to their practice and together use inquiry approaches to develop their practices. This indicates a reflexive relationship between inquiry and development (where development implies learning and deeper knowing). (Jaworski, 2004a, p. 25)

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Relating Practice and Research in Mathematics Education. Proceedings of Norma 05, Fourth Nordic Conference on Mathematics Education (pp. 151–162). Trondheim: Tapir Academic Press

The nature of the learning community can be defined in the following way: certain tasks are created or found by the researcher, through which the community can think and address algebraic thinking. Since the concept of reflection is central in this study, I ask, as research questions: What is the nature of teachers' reflections, and how do these relate to the creation of a learning community? How are these reflections implemented in the teaching of algebra, and what issues does implementation raise for the teachers? Data from one of our workshops address the question of the nature of the teachers' reflections. Analysis of data concerning the implementation of these reflections in the teaching of algebra will not be presented in this paper.

THEORETICAL PERSPECTIVES

The focus on algebraic thinking development through building and studying learning communities is inspired by Wells' (1999) perspective of dialogic inquiry, in relation to communities of inquiry. According to Wells (1999), the concept of inquiry may be characterized as:

> … a stance towards experiences and ideas – a willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them. (Wells, 1999, p. 121)

Furthermore issues concerning the possibility for teachers to implement their reflections in their own teaching, and linking inquiry to learning, are addressed in the following way:

> … the aim of inquiry is not "knowledge for its own sake" but the disposition and ability to use the understandings so gained to act informedly and responsibly in the situations that may be encountered both now and in the future. (Wells, 1999, p. 121)

Within this theoretical approach learning is understood as social participation, and the components necessary to characterize social participation as a process of learning and of knowing are:

> *Meaning*: a way of talking about our (changing) ability – individually and collectively – to experience our life and the world as meaningful.
>
> *Practice*: a way of talking about the shared historical and social resources, frameworks, and perspectives that can sustain mutual engagement in action.
>
> *Community*: a way of talking about the social configurations in which our enterprises are defined as worth pursuing and our participation is recognisable as competence.

> *Identity*: a way of talking about how learning changes who we are and creates personal histories of becoming in the context of our communities. (Wenger, 1998, p. 5)

In this sense the notion of learning community is deeply related to the processes of "belonging" and "becoming". Learning and identity are closely interwoven "because learning transforms who we are and what we can do, it is an experience of identity" (Wenger, 1998, p. 215). In my study, the three teachers and the researcher create and belong to a learning community through the activities proposed during the mathematical workshops, and this community may offer to the teachers the opportunity to become more knowledgeable in elementary algebra.

One of the central features of the design of this research is the creation and development of mathematical tasks, by the researcher, which may provoke teachers' reflections concerning algebra, and enhance their awareness concerning the learning and teaching of algebra. These tasks, proposed during the workshops, are created or found both as a means to provoke teachers' reflections, and at the same time the tasks allow the community to work together. In other words, the problems are instruments both to the development of algebra and to the building of the community.

In order to study the collaborative learning in algebra and to identify different layers of teachers' reflections, a six-level developmental and analytical model has been elaborated and consists of the following levels: at the first level, I propose mathematical tasks related to algebra to the teachers and observe (as a researcher) the way they cooperate in solving these. During the second level, the emphasis is on teachers' reflections emerging from this process. In the next step (level 3), the researcher observes the way the teachers plan what kind of tasks they can offer to their pupils in their respective classes in order to foster the same kind of reflections that they experienced in level 2. At level 4, the researcher follows each teacher into his/her class in order to observe the possible implementation of the reflections in the practice of each teacher. During the fifth level the research studies teachers' evaluations of and reflections on the teaching period. Finally, the last level gives the possibility to the three teachers and the researcher to exchange reflections in common after the observation in the class.

The focus in the iterative cycle is on the development and refinement of teachers' reflections and the possible implementation of teachers' reflections in their respective practices. The plan in this study is to follow the teachers during one year.

METHODOLOGY AND DATA COLLECTION

The research is situated within the co-learning inquiry paradigm (Jaworski, 2004a, 2004b; Wagner, 1997) in which the engagement of both practitioners and researchers is one of the main features. In this way, being involved in action and reflection together enables the participants to achieve a deeper understanding of both their world and the world of the other. The research project is based on the design-based research paradigm and according to Kelly (2003) research design can be described as an emerging dialect whose operative grammar is both generative and transformative (p. 3). It is generative by creating new thinking and ideas, and transformative by influencing practices. This new research approach addresses problems of practice and leads to the development of usable knowledge (The Design-Based Research Collective, 2003, p. 5). According to Wood and Berry (2003), design research can be characterized as a process consisting of five steps: the creation of a physical/theoretical artefact or product; an iterative cycle of product development; the deep connection between models and theories and the design and revision of products; the acknowledgment of the contextual setting of development and the fact that results should be shareable and generalisable; and the role of the teacher educator/researcher as an interventionist rather than a participant observer.

The goal of my research is not the development of a special type of mathematical tasks, the tasks proposed to the teachers during the workshops have to be considered as tools whose purpose is to provoke, enhance, and give the opportunity for deepening of teachers' reflections concerning the learning and teaching of elementary algebra. Therefore both an *a priori* and an *a posteriori* analysis of each task are needed. The *a priori* analysis consists of choosing tasks relying on the following criteria: the task is easily understandable in order to motivate and engage all participants; the task can be solved using different approaches, at least in an algebraic way; the task offers the opportunity for widening and deepening of mathematical understanding with focus on algebra; and the task may offer some insight in the history of mathematics and in this way it encourages the community to see mathematics as a continuous process of reflection and improvement over time, and provides an opportunity for developing participants' conception of what mathematics is. Teachers' reflections, as emerging from our workshop, are evaluated in the *a posteriori* analysis. Did the task motivate and engage all the participants, and in this way address issues concerning the "belonging" to the community? Did the participants solve the task using algebraic notation? Did the analysis of the workshop show any evidence for some enhancement of teachers' algebraic thinking, addressing issues concerning the "becoming" more knowledgeable in elementary algebra? These kinds of questions are central of the *a posteriori* analysis and are related to the step of "an iterative cycle of product development" in design research (Wood & Berry, 2003), where "product" here addresses the concept of "reflection".

As mentioned earlier, the intention of this study is to follow the teachers during one year. All the workshops and classroom observations are audio-recorded. Interviews with teachers both before and after class are also audio-recorded.

COLLABORATIVE PROBLEM SOLVING IN ALGEBRA: EXPLORING VIVIANI'S THEOREM

The problem-solving approach

In a historical review on problem solving in the mathematics curriculum, Stanic and Kilpatrick (1989) point to the fact that, since antiquity, problems have occupied a central place in the school mathematics curriculum, but it is only recently that the idea of developing problem solving abilities has been emphasised. The view of problem solving as an art emerged from the work of Polya (1945/1957), who placed focus on heuristic reasoning, as the art of discovery. In his seminal book "How to solve it", he introduced the following four-stage model structuring problem-solving processes: understanding the problem; devising a plan; carrying out the plan; and looking back. This model has been further developed by Borgersen (1994, 2004) by expanding Polya's model into seven steps which are: analysing and defining; modelling and drawing; qualified guessing by trying and failing; finding a hypothesis; development of a proof; characterization of the solution and interpretation; and unifying ideas, generalisations and applications. Researchers such as Schoenfeld (1985, 1992) and Mason and his colleagues (1982, 1991) have been inspired by the work of Polya and have made important contributions to the development of problem solving in relation to school curriculum. These models for heuristic reasoning offer guidelines for pupils, teachers, and researchers who are interested in developing their own mathematical thinking, exploring the art of discovery, and analysing students' approaches to non-routine tasks.

Viviani's theorem

This theorem was presented to the three teachers during one of our workshops. We usually met at one of the schools of the teachers, using the teachers' meeting room, and working for about two hours. The three teachers (Mary, Paul, and John) and I sit around a table and we have the possibility to use a flip-chart. Mary and Paul work at the same school with pupils at grade 9 (14–15 years). John works in another school with grade 10 (15–16 years). All the dialogues during our workshop are audio-recorded. Our meetings have the following pattern: first I propose a mathematical task to the teachers and we try to solve it, then the teachers are invited to reflect on the way the task has been solved and on the possibility to use similar tasks in their class or to implement similar reflections during their teaching.

The task, as exposed below, was presented during our fourth workshop. Here both the task and a reflection concerning the possibility to use the same task in classroom are addressed.

60. CB: … If you have an equilateral triangle and a point inside the triangle, then the sum of the distances from the point to the sides of the triangle is equal to the height of the triangle. (pause). So I thought we could explore this, what is it? and when we have (know) a little more, when we see what it is

> about, we could see if it is something that could be useful in teaching? Part of it, perhaps not the whole (task), but only part of it, I mean to explore somehow this task. Is this ok for you? (translated from Norwegian by the author)

In order to facilitate the understanding of the progression of the work in our workshop, teachers' reflections on the solving and learning process are divided into five different subsections which are: analysing and defining, modelling and drawing, exploring the task from a teacher's perspective, qualified guessing by trying and failing, finding hypothesis and development of a proof. These categories have emerged from the analysis of the data and they follow Borgersen's (1994, 2004) terminology except for the one called "exploring the task from a teacher's perspective". The last section addresses the role played by the researcher during the workshop.

Analysing and defining

64. Mary: yes, I understood, but, hmm, just how to start …
65. CB: yes, how do we start with such a task?
66. Mary: hmm, the first thing I will do is to draw a triangle, an equilateral triangle, and then draw the point, yes, …

The difficulty of "getting started" is illustrated in Mary's utterance (64). In the first part of her sentence she points to the fact that the mathematical task is understood, and therefore there is no need for additional indications. The difficulty lies in the transition from understanding the question to the elaboration of a strategy in order to explore the task, and this is underlined in the second part of her sentence where the emphasis is on "how to start". This step corresponds to the two first stages in Polya's (1945/1957) model on heuristic reasoning (understanding the problem; devising a plan) and to what Borgersen calls "analysing and defining". I repeat the question (65), offering Mary and the other teachers the opportunity to elaborate and reflect on this point. For Mary, a way to handle this difficulty is first to draw a picture of an equilateral triangle and then to choose a point P inside the triangle (66). This is not yet an elaborate strategy for solving the task, but having a drawing may help her to look for useful properties within the triangle in order to guess, try, and reflect on different hypotheses.

Exploring the task from a teacher's perspective

67. John: yes, here indeed it is about to look for and maybe to play with the properties of an equilateral triangle and then it could be one of the things that come, will orient it (the activity) in that direction, if it is what is wanted. I cannot tell it is desired yet, isn't it, but let us suppose we want there, that we want to see how it is to construct the perpendicular, take the middle

perpendicular for all the sides and all that stuff, and then we can see what we end up with and then we can try to direct them to find it, and so on, it will be like to try and fail, most like a game. Because, I mean, I have never seen this (task) before, don't think so, so I feel it is outside what we are supposed to teach …

68. CB: hmm, hmm

69. John: now, you are going to learn how to construct a perpendicular in a way, but is it something we can look at here? Some (pupils) like it, some like to make

70. Mary: construct

71. John: yes, some like to make patterns and like this, and maybe they can find something here …

John follows up (67) by looking at the task, reflecting on which possibilities this task offers to the pupils, while Mary (64, 66) was focusing on exploring Viviani's theorem. Following John's long utterance step by step, he sees first this task as an opportunity to explore (both looking for and playing with) the properties of an equilateral triangle. Second, among all the observed properties it may be that one of them will lead to a deeper understanding of this particular task. In a sense, he turns around the task, starting with all the properties an equilateral triangle has, to the focus on one particular that could be helpful in relation to Viviani's theorem. This view is developed by explaining that it could help pupils to practice how to construct the perpendicular, the middle perpendicular, and John sees it in an atmosphere of playing. His last remark concerns both the fact that the theorem is new for him and therefore he has not had the opportunity to reflect in it before, and as a teacher he feels that the skills needed to solve the task may not be part of the curriculum ("I feel it is outside what we are supposed to teach"). In the next utterance (69), John speaks as if he was in a class talking to one of his pupils, and inviting not only to construct a perpendicular, but also explore the task in more detail. His way of speaking is of "didactical mode", following by a general remark that some pupils enjoy working in these ways. His last utterance (71) goes back to his description of the task (67) as one offering both a way of looking for and playing with the properties of an equilateral triangle, just interrupted by Mary's precision (70).

Modelling and drawing

109. CB: yes, and you, Paul, are you finished? Is it right?

110. Paul: hmm, hmm, yes …

111. John: approximately, (laugh), yes, one millimeter or something like this, two perhaps …

112. CB: so this is correct for one point?

113. John: yes, I think it (the result) lies within what we can accept at least within error margin …

114. John (to Paul): did you find it exactly?
115. Paul: yes, (pause)
116. John: I will say this is a kind of play, but what are we supposed to do with it? (laugh)

This discussion occurs after a short discussion concerning the use of CABRI and a pause where each of us, taking a randomly chosen point, measure the distances to the sides and add up. I remember being aware of the fact that Paul has been quiet a long time, and inviting him to participate (109), but his answer is short (110). My concern about Paul's participation and Paul's reaction are closely related to issues of "belonging" as exposed earlier. My role in the learning community is slightly different from the other participants in the sense that I am both focusing on the mathematical task and on the fact that participating in the workshop may increase the "belonging" to that community. Paul's answer shows his participation to the task and at the same time he feels confident enough within our group to decide not to engage in further discussion, asking indirectly for some more time to reflect on the task. John's answer (111) to my question (109), that was originally meant to Paul, addresses also issues of "belonging" concerning John and his wish to participate and share the results, and at the same time utterances (111) and (113) show his experience as a teacher, being aware of the fact that one has to accept some loss of accuracy in such task. John's question (116) presents a shift in the level of reflection, addressing the purpose for this activity. But his challenge is not followed by the other participants.

Qualified guessing by trying and failing

127. John: in order to accept (the result of the theorem), then I would have directed a little bit where the points should be, isn't it, and not as similar as Paul and I have done, isn't it?
128. Paul: hmm, yes …
129. John: we have already done it, but I think we should have some more points …
130. CB: yes?
131. John: so if you had ten (pupils) in the class that had got it (the result) correct, then it is ok in a way, shall we take one …
132. Mary: in the middle
133. John: in the middle too, is it correct?
134. Mary: yes

After trying Viviani's theorem on one particular point, I had reminded the participants of the fact that the result has to be valid always, and in that way I indirectly called for possible strategies to explore the theorem in more detail and possibly to elaborate a proof using algebraic notation. In other words, to use algebra to be able to prove Viviani's theorem means to be able to deduce that $h = h_1 + h_2 + h_3$ (where

h is the height of the triangle, and h_1, h_2, h_3 are the distances from the point P to each side of the triangle). In (127) John follows up by elaborating a strategy for the other participants, where everybody may take different points. He also proposes another strategy (131) in relation to a teaching situation.

After having tried several different positions for the point P inside the triangle, in particular having P in a corner, and verified the Viviani's theorem, our group looks for a deeper understanding of the theorem, using algebraic thinking and notation.

Finding hypothesis and development of a proof

191. CB: … now let us see, we have, when we consider the distance from the point, let us call it the point P, when we take the distances from the point P to the sides here, I am looking at the small triangle now, one of the three, so this is an equilateral triangle, how can we name the sides? The length of the sides, how can we name it?
192. John: name it "a"
193. CB: "a", so we get "a", "a", "a" (for the three sides of the equilateral triangle)
194. John: are we going to calculate area or something like this?

.
.
.
.

205. Paul: like this, it was smart!
206. CB: did you arrive at something?
207. Paul: yes!
208. John: same for me
209. Mary: yes, for me
210. John: then we agree that the area of the three small triangles is as big as the area of the whole (triangle) and we put the following, yes, we can call it an equation or a claim, yes, I name it "a" (the side of the triangle), "a" multiplied by h_3 divided by two, plus "a" multiplied by h_2 divided by two, I mean these are the small heights, plus "a" multiplied by h_1 divided by two is equal "a" multiplied by h_4, which is the big height, divided by two.
211. CB: hmm, hmm
212. John: common factors outside, then we get "a" divided by two, I have h_4 there, and a parentheses with the other three heights in there, and then they (h_1 plus h_2 plus h_3) have to be equal to it (h_4) in order to be correct.

In (191) I try to direct the awareness of the group to the possibility to divide the equilateral triangle in smaller triangles and to introduce algebraic notation. John (192) proposes to use the symbol "a" for the length of the sides of the equilateral triangle, and makes the hypothesis to calculate the area (194). From utterances (194) to (205) we discussed ways of comparing the area of the big triangle with the area of the three smaller ones. Just before Paul's remark (205) it has been a long pause during which each participant has tried to use algebraic notation to prove Viviani's theorem. I can remember Paul's satisfaction by the expression of his face, smiling, leaning back in his chair, and claiming he had proved the theorem (205, 207), but without giving more detail concerning his calculations. Immediately after, the same agreement is made by John (208) and Mary (209), still without any detail. It is only in (210) that John takes the initiative to explain what Mary, Paul and himself agree about, he uses "*we* agree" speaking for the whole group. The dynamic shown by the group in this episode reveals some issues concerning the "belonging" to our learning community. Paul has been relatively quiet, with few utterances, but still fully engaged in the task and coming first to the algebraic proof for the theorem, as this episode shows. John gives some more insight into his reflections, both as a participant to the workshop and as a teacher, and also takes the role as a "speaker for the group" as shown in utterances (210) and (212). Mary's position is more like Paul's one, but she also fully is engaged in the task and is able to deduce an algebraic proof.

John's explanation (210, 212) corresponds to the proof of Viviani's theorem using the notion of area. By dividing the big triangle into three smaller triangles and considering the area of the big triangle as the sum of the area of the three small ones, it is possible to write (following John's notation from 210 and 212):

$$\frac{a \cdot h_4}{2} = \frac{a \cdot h_1}{2} + \frac{a \cdot h_2}{2} + \frac{a \cdot h_3}{2} = \frac{a}{2}\left(h_1 + h_2 + h_3\right)$$

John's conclusion (212) is that after taking common factors outside it is possible to get the result: $h_4 = h_1 + h_2 + h_3$ which proves Viviani's theorem. The fact that our group was able to find a proof using algebra is important. In an earlier workshop the teachers (and Paul specially) had expressed the fact that they found it extremely difficult, if not impossible, to prove general statements using algebra. Now during this workshop, the group managed to come to a proof using algebraic notation, revealing issues concerning "becoming" more knowledgeable in elementary algebra and algebraic thinking, as explained earlier.

FOCUS ON THE ROLE OF THE RESEARCHER

Looking critically at the role played by the researcher includes examining the *a priori* analysis, the participation in the workshop, and the *a posteriori* analysis. I decided to choose this task concerning Viviani's theorem according to the *a priori* analysis: the task is easily understandable and can be explored by different approaches. Proving the theorem may give the opportunity to widening and deepening algebraic thinking, and since this theorem was found by a mathematician who lived from 1622 to

1703 issues concerning the history of mathematics might be addressed. A critical examination of the role of the researcher during the workshop reveals a focus on different layers: on issues concerning the "belonging" to the learning community (see utterance 109), and on the possibility for the teachers to address elementary algebra through exploring and proving Viviani's theorem. The fact that both the three teachers and the researcher are fully participating to the workshop is related to the notion of "co-learning" in research partnerships (Wagner, 1997; Jaworski, 2004a). The *a posteriori* analysis shows that the task offered to the teachers gives the possibility of exploring the properties of an equilateral triangle and the distances from a point to a line. It was easily understandable and did engage and motivate all the participants; our group approached the task through looking at different points inside the triangle, referred as "modelling and drawing" (Borgersen, 1994, 2004) or "specialising" (Mason & Davis, 1991); and our group made qualified guessing and found a hypothesis by considering the area of the big triangle compared to the area of the three smaller ones. Actually the group solved the task by using algebraic notation, and in this way it clearly offered to the teachers an opportunity for widening and deepening of algebraic thinking.

CONCLUSION

This article presents an account of the activity in a learning community consisting of three teachers and a researcher and gives an analysis of the way they engage in problem solving related to algebra. Theoretical perspectives including the notion of a learning community and the relation to community of inquiry are presented and linked to issues concerning the "belonging" to the learning community, and the "becoming" more knowledgeable in algebraic thinking. A six-level developmental and analytical model is introduced and data from one of the workshop are presented. The analysis of the different episodes from this workshop shows some elements of heuristics thinking (analysing and defining; modelling and drawing; qualified guessing by trying and failing; finding hypothesis and development of a proof) using Borgersen's (1994, 2004) terminology. Therefore it seems appropriate to characterize this workshop as offering a "problem solving-like-situation" to the teachers and the researcher. Through the analysis of dialogues different layers in teachers' reflections are made visible. These reflections concern ways of exploring the task, but also issues related to the possibility of using this task in their respective classes. These results illustrate the complexity in teachers' reflections and suggest that engaging teachers in such activities may increase awareness concerning algebraic thinking, heuristic reasoning, and issues presented by the implementation of a particular task in the teaching situation.

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THE CONCEPT OF UNDERSTANDING IN MATHEMATICS TEXTBOOKS IN ICELAND

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The importance of understanding concepts and procedures is emphasized in most mathematics textbooks written in Iceland, from the earliest writings in Icelandic up to present day. Simultaneously, assertions are found at regular intervals that the emphasis on understanding is a novelty, not promoted in earlier times. In order to analyze what lies behind these statements, the literary meaning of the Icelandic word "skilja", usually translated as "understand", will be compared to the corresponding words in Danish, English and Latin, and related to the modern sense of understanding in mathematics.

INTRODUCTION

A citation from a recent master's thesis (Angantýsdóttir, 2005) gave rise to the following reflection upon the verb *skilja* in Icelandic, translated as "understand" when used in its metaphorical meaning, and *skilningur*, the corresponding noun, translated as "understanding". The author of the thesis cites a young teacher: "Today *skilningur* / understanding is the most important. That's for sure. It was not so in the past."

Having examined most printed mathematics textbooks written in Icelandic, intended for primary and lower secondary level, I was astonished to see that quote. This led me to collect quotes where the concepts *skilja* and *skilningur* are discussed in widespread textbooks at various times. The research question concerns what the authors meant by these words and if the meaning was influenced by the literal meaning of the words. In order to search for an answer to this I will consider in what context they were said, what remedies were suggested to the perceived lack of understanding and relate to modern theories about understanding.

Skilja and skilningur

The verb *skilja* has existed in the Icelandic language at least since writing began in the 12th century. In Cleasby's Icelandic–English dictionary (Cleasby, 1962), its literary meaning is to part, separate, divide, while its second metaphorical meaning is to distinguish, discern and understand, and in old English, to *skill*. The original

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sense is to cut, in Latin *secare.* This appears also in Gothic where for example *skilja* means a butcher and in Anglo-Saxon in the word *scylan*: to separate.

In an etymological dictionary by Falk and Torp (1991)*,* the Danish word for understand, *forstå*, assumes the meaning *blive stående, standse, nemlig for at undersøge noget*, that is stay for a while or stop in order to investigate something. This is compared to the Anglo-Saxon *understandan* which means *stille sig hen under*, that is position oneself below, while in Mid Low German *understand* is to *træde under noget, forstaa*, that is step below something, understand.

According to *Webster's New World Dictionary* (1982), *understand* is derived from Old English *understandan*, literally to stand under or among. It means to get or perceive the meaning of, know or grasp what is meant by, comprehend, and so on. The verb to *skill* [archaic] means to matter, avail, make a difference. *Comprehend* is explained as literally to catch hold of, seize, while its present meaning is 1) to grasp mentally, understand 2) to include, take in, comprise. *Perceive* is explained as to grasp mentally, take note (of), recognize, observe, to become aware of through the senses.

In Cassell's *Latin–English, English–Latin Dictionary* (Simpson, 1990), the English word *understand* in the meaning to comprehend, grasp with the intellect, is translated into Latin by the following verbs, which are translated back into English as shown:

– *intellegĕre*: to distinguish, discriminate, perceive 1) by the senses; 2) mentally: to understand, grasp, come aware of
– *comprehendĕre*: literally: 1) to take together, to unite; in metaphorical sense: to embrace, comprise, include, 2) to take firmly, seize. In a metaphorical sense with the senses or intellect: to comprehend, perceive
– *amplecti*: to twine around, clasp or grasp
– *complecti*: to embrace, circle, surround
– *percipĕre*: to lay hold of, take possession of, seize; literally: to collect, gather, harvest.

Summing up, there seems to be three basic literary meanings to European words related to understand:

– Icelandic: *skilja*: to part, separate, divide. Latin: *intellegere*: to distinguish, discriminate
– Danish, German, English: *forstå, verstehen, understand*: to stay for a while (in order to investigate), stand under or among
– English, French and Latin: *comprehend, comprendre, comprehendĕre*: to take together, seize; *perceive, percipĕre*: to collect, gather.

I will consider whether the literary meaning of *skilja*, to separate, divide or even analyse, has an effect on the meaning each author puts into the expression in the context of learning, or if *skilja*, *forstå*, *understand* and even *comprehend* are in fact synonyms.

UNDERSTANDING IN MATHEMATICS

Skemp (1978) distinguished between *instrumental understanding*, based on routines and rote learning, and *relational understanding*, where an understood idea is associated with many other existing ideas in a meaningful network of concepts and procedures.

Anna Sierpinska (1994) distinguishes between acts of understanding and processes of understanding. Experiencing acts of understanding is mentally relating the object of understanding to another object, the basis of understanding (Sierpinska, pp. 28–29). Processes of understanding may be regarded as lattices of acts of understanding, linked by reasoning. Acts of understanding and reasonings in one process of understanding constitute quite a dense network (Sierpinska, pp. 72–73). A relatively good understanding is said to be achieved if the process of understanding contained a certain number of significant acts, namely acts of overcoming obstacles specific to that mathematical situation (Sierpinska, p. xiv).

In a textbook on mathematics education for student teachers (Van de Walle, 2001) understanding is defined, referring to Hiebert and Carpenter's research from 1992 and others, as a measure of the quality and quantity of connections that a new idea has with existing ideas. The greater the number of connections to a network of ideas, the better the understanding (Van de Walle, pp. 28–29). Understanding with rich connections is identified with Skemp's *relational understanding*, while understanding, where ideas are completely isolated, is identified with *instrumental understanding*.

In the following investigation of textbooks I will refer to instrumental and relational understanding in the above sense.

INVESTIGATION OF TEXTBOOKS

The textbooks chosen for investigation were all influential and dominating in Icelandic mathematical education for some period of time. They indicate the rough outlines of trends in Icelandic mathematical education. The collection was as follows:

- *Algorismus*. A treatise about the seven arts of computing, preserved in the manuscript collection *Hauksbók*, dated in 1306–1308. It is believed to be written in the 13th century.
- Two textbooks from the 1780s by Ó. Olavius (Olavius, 1780), and Ó. Stefánsson (Stefánsson, 1785). They were the first complete arithmetic textbooks written in Icelandic.
- An arithmetic textbook by the Rev. E. Briem, published in 1869–1880 (Briem, 1880). It was the most widespread arithmetic textbook for adolescents from the 1880s, when the first legislation on requirements in writing and arithmetic for the general public came into effect, until the 1900s.
- An arithmetic textbook by Dr. Ó. Daníelsson (Daníelsson, 1906), first published in 1906. Its third edition from 1920 became the most influential and dominating textbook for the lower secondary level until the 1950s.
- A textbook on numbers and sets, written by G. Arnlaugsson (Arnlaugsson, 1966) in 1966 as an introduction to the "modern" mathematics reform wave. It was a required textbook for college-bound pupils for a decade, 1966–1976.

The textbooks by the Rev. Briem and Dr. Daníelsson may be said to have dominated the arithmetic textbook market in Iceland in the 1870s–1950s, with a short interlude in the 1900s and 1910s when there was a choice of a variety of arithmetic textbooks.

Furthermore, I will elaborate upon the comment of the young teacher, that in the past understanding was not emphasized.

Algorismus

The text *Algorismus* is found in several manuscripts, the oldest dated 1306–1308. It is an almost word-for-word prose translation of the Latin hexameter *Carmen de Algorismo*, a kind of a textbook for schoolboys to learn about the decimal place value system. However, the translator has added words and explanations of his own. When the multiplication of a multi-digit number has been explained, the author feels compelled to add his own explanation:

> Og að þú <u>skiljir</u> þetta margfalda .vii. og níu. Níu skortir einn á .x. Því taktu eina .vij. af .vij. tigum. Þá verða eftir .iij. og .vi. tigir, það eru .vij. sinnum níu.
> So that you <u>understand</u> this multiply .vii. and nine. Nine lacks one to .x. Therefore take one .vii. from .vii. tens. Then .iij. and .vi. tens will be left, that is .vij. times nine. (my emphasis)

The translator drops out from the original text to explain multiplication of two one-digit numbers, which he feels that his readers might need to become better acquainted with to *skilja* / understand. He therefore explains one of the most complex parts of the multiplication table by *parting the problem into more details* in order to create connections to simpler ideas. The literary meaning of *skilja* emerges here.

Textbooks from the 1780s

In his arithmetic textbook, *Greinileg vegleiðsla til Talnalistarinnar / Clear Guidance to the Art of Numbers* (Olavius, 1780) (374 pages + introduction), the author, Ólafur Olavius, discusses in his foreword his means to promote learning. His book is a reaction to a total lack of arithmetic textbooks and teachers in 1780s Iceland. The only previous booklet was printed in 1746 as a handbook in purchases, with the main emphasis on exchanges between measuring units and currencies.

The author stated that he himself learnt more from one solved problem than ten unsolved problems (Olavius, p. xv). About understanding he says:

> Sumir <u>skilia</u> fljótarr en sumir, en því meigu þeir eigi, at skjótnæmir eru eðr áðr vitu, kalla þat úþarfa mælgi, err eg segi greiniliga frá því sem þeim virðizt auðvellt. hinum er hún til nota er lítið <u>skynia</u>. ne en helldr megu þeir, at veikari <u>skilníngu</u> hafa, láta ser þegar í augu vaxa þat er eigi fá numit at bráðgirni sinni, en leiti helldr at sigrazt af slíku með iðuligum lestri og athyglli.
> Some <u>understand</u> quicker than some, but therefore those who learn quickly or know beforehand, must not call it unnecessary verbosity when I expound clearly what seems easy to them. It is of use for others who <u>perceive</u> little. Neither must those who have weaker <u>understanding</u> let it overwhelm them what they can not learn for their impatience, but rather try to overcome such things by avid reading and attention. (Olavius, p. xviii, my emphasis)

The author believes in explaining, that is analysing, <u>parting</u> into details, rather with more words than fewer, while he also promotes <u>patience</u>. In his opinion, understanding would emerge through step-by-step demonstration of examples of increasing complexity, rather than by working on exercises. One senses here both the need to break into smaller parts, and to "stand under" or "stay for a while to investigate". The author also uses the word *skynja*, in the sense of perceive.

Olavius's contemporary, Ólafur Stefánsson, published a textbook *Stutt Undirvísun í Reikningslistinni og Algebra / Short Teaching in the Art of Arithmetic and Algebra* of similar size, (Stefánsson, 1785, 248 p. 8vo + foreword etc., 16p.) where he says:

> þó væntir eg, at þeim, er skrif þetta les með aðgætni frá upphafi, og síðan frameftir, muni auðvelt veita, að <u>skilia</u> og nema, serhvat er í henni fyrikemr, en hinn, er lætr ser nægja, at grípa her og hvar niðr í bókina, en nennir ei at lesa neitt samanhángandi með athygli, lærir þarvið lítit eðr eckert í reikníngi, hvat auðvelldr sem hann vera kynni, en dæmir þó strax: at bókin sé þúng<u>skilin</u> og óhentug únglíngum.

> Still, I expect that a person who reads this carefully from the beginning, and thereafter goes on, will find it easy to understand and learn everything that happens there, while the one who contents himself with dropping into the book here and there, but does not feel disposed to read anything continuous with attention, learns thereby little or nothing in arithmetic, however easy it could be, and then judges it difficult to understand and unsuitable for adolescents (Stefánsson, foreword, my emphasis).

In this quote the author emphasizes coherence and patience, which points to a "take together" and "stay for a while" attitude. After the author has explained how to do multiplication of multi-digit numbers, he takes this example for clarification:

$$\begin{array}{r} 3456 \\ \underline{\cdot\ 4} \\ 24 \\ +\ 200 \\ +\ 1600 \\ +\ \underline{12000} \\ 13824 \end{array}$$

The author says that this method should not be copied but is meant to

> gefa þeim, sem nema vill, góðann skilníng, sem mest á ríðr, og röksemdir fyrir øllu því, sem honum er kennt.
> to offer those, who want to learn, a good understanding, which is the most important, and reasons for everything that is taught to him (Stefánsson, p. 36, my emphasis).

Here the author wants to support the understanding by parting into details and providing reasons, thus creating connections to previously known facts.

The two 1780s arithmetic books were intended for farmers and growing youth. The 40 boys who attended a learned school at any time were given Stefánsson's *Short Teaching*. They received no instruction, and had to study arithmetic on their own. These were the only textbooks available in Icelandic until 1841.

The 19th Century

For various reasons, education was at its nadir in the early 19th century Iceland, while from the 1860s there was growing interest in public education. An influential arithmetic textbook *Reikningsbók*, by the Rev. Eiríkur Briem was published in 1869 (Briem, 1880), and in a more detailed edition in 1880. Its main emphasis was on memorizing rules. Reasoning was only to be presented if it could support the memorizing (Briem, foreword). The Rev. Briem favoured rote learning or, according to Skemp's definition, supported instrumental understanding. However, the author warned that in order to be able to understand each chapter it was necessary to have been through everything that preceded it, except in the case of decimal fractions, where one page of common fractions would suffice (Briem, foreword). Thus the author emphasized coherence, that is "to take together" in order to understand. The philosophy of the book was patience, that is rather "staying, in order to investigate" than "parting" or "separating".

This textbook, intended for adolescents, was dominant until the 1900s, during which time the predominant means of education was independent study in the absence of schools for the general public.

The Early 20th Century

Dr. Ólafur Daníelsson, whose textbooks were dominant up to the 1960s, published his first arithmetic textbook in 1906. Legislation on public education and schools appeared first in 1907, so that the book was published in the context of the traditional home-schooling and self-education. In the foreword to his *Reikningsbók* / *Arithmetic* he said:

> Þetta litla kver á ... að bæta úr tveimur göllum, sem mjer þykja vera á flestum eða öllum reikningsbókum vorum; er annar sá að þær gefa alls engar skýringar, jafnvel ekki á einföldustu reikningsaðferðunum, og læra því margir aðferðirnar utan að, án þess að skilja hvernig á þeim stendur ... En hinn gallinn er sá, að dæmin í þeim eru yfir höfuð helzt til ljett, og er hvert þeirra optast nær miðað aðeins við eina reikningsaðferð. Nemandinn getur því getið sér til aðferðarinnar án þess að skilja dæmið. ... yfir höfuð að tala hef ég leitast við að velja dæmin þannig, að þau ekki verði reiknuð, nema þau sjeu skilin til fulls.
>
> This booklet is intended … to compensate for two drawbacks which I think characterize most or all of our arithmetic textbooks; one is that they give no explanations at all, not even of the simplest computation methods, and therefore many learn the methods by heart without understanding their reasons … But the other drawback is that their exercises are generally too easy, and each of them is most often aimed at only one computation method. The pupil can therefore guess the method without understanding the problem. … generally speaking I have tried to choose the exercises so that they may not be solved unless they are fully understood (Daníelsson, 1906, pp. iii–iv, my emphasis).

The author is concerned with explaining basic arithmetic procedures, parting them into details. He is also concerned with preventing routine and rote learning by composing problems where the pupil has to choose his/her own solution methods. In Sierpinska's terms, he wants to introduce the pupil into new problem situations with the aim that the pupil will overcome obstacles and thus perhaps make the necessary reorganizations of previous understandings (Sierpinska, 1994, pp. 121–122), that is new connections or links.

The author explained common algorithms clearly, by taking examples from the daily life of the early 20th century. When it came to adding fractions he emphasized the necessity of finding a common denominator divisible by all the denominators.

Often it sufficed to multiply all the denominators together. However, for the least common denominator in the case of for example

$$\frac{1}{5}+\frac{5}{6}+\frac{2}{3}+\frac{4}{9}+\frac{5}{16}+\frac{3}{12}+\frac{7}{12}+\frac{3}{10}$$

the denominators were to be written in a line: 5 – 6 – 3 – 9 – 16 – 4 – 12 – 10. Then every denominator that divided another one was ruled out, in this case 5 (dividing 10), 3, 4 and 6 (dividing 12). Thereafter the remaining denominators were to be divided by the lowest number dividing two or more of them, and the others copied unchanged:

$$\begin{array}{l} 2\underline{)\ 9 - 16 - 12 - 10} \\ 2\underline{)\ 9 - 8 - 6 - 5} \\ 3\underline{)\ 9 - 4 - 3 - 5} \\ \quad 3 - 4 - 1 - 5 \end{array}$$

By now all the remaining numbers and the divisors were to be multiplied together: $2 \cdot 2 \cdot 3 \cdot 3 \cdot 4 \cdot 1 \cdot 5 = 720$ Done.

This totally unexplained procedure in Dr. Daníelsson's lower secondary textbook was taken up in a primary arithmetic textbook written by one of his student teachers, and it prevailed as a standard procedure in Icelandic schools for half a century. Many of those who worked on teaching never became acquainted with anything else. Thus the tradition of instrumental understanding of unexplained procedures in this respect was carried on for decades. This is the more regrettable as for example another text-book from the same period (Gíslason, 1911–1914) introduced prime factoring in order to compose the least common denominator and the greatest common divisor. Dr. Daníelsson first introduced prime factoring at the back of a succeeding book, when he had proved rigidly that the number of prime factors could not exceed the number itself, a fact that one might expect pupils to find unnecessary to prove. Thus, in spite of Dr. Daníelsson's intention to avoid rote learning, his rigid mathematical training hindered him in introducing comprehensible procedures.

Dr. Daníelsson does explain many of his procedures thoroughly, thus conceiving understanding depending on separating or dividing into details, connecting to previous knowledge. However, it seems that Dr. Daníelsson's conception of *skilja /* understand was mainly "to stay in order to investigate", when he emphasized that the pupils were to choose their own solution methods to problems. But also here, teachers promoted rote learning by testing "seen" problems, that is having pupils to solve Daníelsson's problems on examinations by the solution methods the teachers themselves had worked out, in the hope that it would help the pupils to recognize the similarities when other problems were posed to them.

The 1960s Reform Movement

The intention of the participants in the "modern" mathematics reform movement was to promote understanding. The OEEC organized a meeting for mathematicians and mathematics educators in Royaumont, France, in November 1959. The Royaumont meeting can be seen as the beginning of a common reform movement to modernize school mathematics in the world (Gjone, 1983, Vol. II, p. 57).

Understanding was to arise from guided discovery under the supervision of teachers who understood the implied mathematical relations (OEEC, 1961, p. 109). Rote learning, that is instrumental understanding, was to be eliminated. In the final section of a summary of the report from the seminar it says that

> ... we see a *change in purpose* that a reform must emphasize – namely, a building of mathematical concepts, structure and <u>understanding</u> as paramount to manipulative skills, although the latter must be adequately developed. (OEEC, p. 124, my emphasis).

Dr. Daníelsson's student, Guðmundur Arnlaugsson, was the next influential mathematics educator. Arnlaugsson brought "modern" mathematics into the Icelandic school system. His book *Tölur og mengi / Numbers and Sets* (Arnlaugsson, 1966) is a good introduction to basic number theory, such as divisibility. There he introduced for example primes and prime factoring as an aid to finding the least common denominator and the greatest common divisor. Set algebra was introduced as a unifying structure and was to precede ordinary algebra. (Landsprófsnefnd, September 1968). The idea of set theory unifying various branches of mathematics in a coherent structure originated in a French group of mathematicians, acting under the pseudonym Bourbaki.

Arnlaugsson wrote in the foreword to his *Number and Sets*:

> Áhersla á leikni og vélrænum vinnubrögðum hefur þokað fyrir kröfum um aukinn <u>skilning</u>. Þessi þróun hefur ýtt nokkrum grundvallarhugtökum úr rökfræði, mengjafræði og algebra niður á barnaskólastig. Reynsla víðs vegar bendir til þess, að börn – og það jafnvel á unga aldri – eigi tiltölulega auðvelt með að tileinka sér þessi hugtök, sem áður voru eigi kynnt fyrr en á háskólastigi, og hafi gaman af þeim. Enn fremur virðast þau stuðla að auknum skýrleik og nákvæmni í hugsun og reikningi.
>
> The emphasis on skills and mechanical methods has given way to demands for increased <u>understanding</u>. This development has pushed several basic concepts from logic, set theory and algebra down to primary level. The experience from many places points to children – even very young children – being easily able to adopt these concepts, which previously were only introduced at university level, and to enjoy them. Furthermore, they seem to support increased clarity and exactness in thinking and arithmetic (Arnlaugsson, p. 4, my emphasis).

Here Arnlaugsson advocates deviation from mechanical skills, associated with instrumental understanding, and suggests that the unifying concepts of logic and set theory would facilitate understanding, even for small children. Here, understanding may be interpreted as comprehend, "taking together", connecting to other areas of knowledge. One must also note that the origin of implementing Bourbakian ideas into school mathematics was in the French-speaking countries in Europe.

In Arnlaugsson's article in the educational journal *Menntamál* in 1967 he reacts again against mechanical methods and skills, resulting in instrumental understanding:

> Aðalgalli reiknings- og stærðfræðikennslu hérlendis hygg ég hafi verið sá að hún hefur verið of vélræn, beinzt um of að vissri tegund leikni, en ekki lögð nóg áherzla á yfirsýn og skilning. Aðferðir sem ekki hafa nægan skilning að bakhjarli gleymast fljótt aftur og koma að litlu haldi.
>
> The main drawback in arithmetic and mathematics teaching in this country I think has been that it has been too mechanical, too much aimed at a certain kind of skills, while not enough emphasis has been placed on overview and understanding. Procedures, which are not backed up by enough understanding, are quickly forgotten and are of little use (Arnlaugsson, 1967, pp. 40–42, my emphasis).

Arnlaugsson's interpretation of *understanding* is connected to the notion of "overview" in the sense of connecting various areas of mathematics, observing their similarities, again referring to "taking together".

The experiment of "modern" mathematics, which in Iceland became extremely widespread, did not yield the expected results. The reason may not have been solely the inability of the children to absorb the new ideas, but rather a clash between two different cultures. On one hand there were university mathematicians with ideas of implementing advanced mathematical concepts into the education of small children, and on the other hand there were teachers trained at the Teacher Training College, mainly in the culture of instrumental understanding of mathematics, and in most cases unfamiliar with the new concepts and their remedial role.

The reaction to the disappointments following the set-theoretical experiment in primary schools in Iceland was to introduce a second wave of "modern" mathematics, this time with a more investigational approach, that is to promote understanding by "staying in order to investigate", stimulating children to create connections to their previous understandings and knowledge. A varied choice of a limited number of exercises on each topic was expected to prevent emphasis on mechanical exercising of skills. During the quarter of a century when this material was provided to schools by a state-monopoly Textbook Publishing House, a number of extra skill-training exercises were published. In the 1990s, "back-to-basics" training of mechanical procedures had largely become prevalent.

PRESENT TIMES

The young teacher, who maintained that "Today *skilningur* / understanding is most important. That's for sure. It was not so in the past", graduated from the Iceland University of Education in 1999 without taking mathematics as an elective. In an interview, this teacher said that she had done well in mathematics in primary school. Mathematics in lower secondary school had not gone as well as she had hoped, while the mathematics she took in a social science stream in upper secondary school had been fine.

She conceived the mathematics instruction she had at school as performing routines, that is that it was concerned with instrumental understanding. She knew what routines to use in primary and upper secondary schools, but had never thought of or felt that she had not been taught what for example keeping or borrowing procedures in addition and subtraction meant. This had also been the case with the first primary class she taught. She taught for three years, in grades five to seven, the conventional syllabus she herself had studied at school. The pupils had come from various schools from the capital area into the fifth grade in a new school in a new district. They knew the procedures, but not the reasons for them. When she looked at the national test her pupils were expected to take in seventh grade, she realized that they would not do well on that, so she began to collect appropriate material wherever she could find it, for example on the Internet.

In 2002 she took over a fourth grade to lead through the sixth grade with a new syllabus with emphasis on invented strategies. She was fortunate enough to be a member of a team led by one of the textbook authors, who was carrying out action research on implementing the new syllabus together with new ways of work. She delved deeply into the material, as did her pupils. They could generally reason why they had chosen the procedures they used and explain the meaning of them. She had learnt that she herself should not reveal too much, and that she should challenge her pupils to find out for themselves. In her opinion, their number sense was much deeper than her previous group's. However, she knew that some of her colleagues, who had not had similar support, found the new syllabus pathetic.

CONCLUSIONS

The authors of the six textbooks, dominating at their time, were concerned with conveying their ideas and knowledge of arithmetic to their young readers. The first five textbooks were intended for people who had to study them by themselves with minimal assistance. The authors considered it necessary to explain their procedures, that is part them into details for their prospective pupils to help them make connections to earlier knowledge.

One can sense the meaning of *skilja* as separating, dividing into smaller pieces, as the original meaning in the Icelandic term, while the meaning "stay" or "stop to reflect" as implied in *forstå* or *understand* had already emerged by the 18th century.

The notion of "coherence", mentioned by at least two of the authors, points to "to take together" rather than "separating" into smaller items. The authors were the learned persons of their time. The 18th-, 19th- and 20th-century authors were educated in Copenhagen. They knew well the term *forstå,* "stopping to reflect", and the Latin terms *comprehendere*, "to take together", and *intellegere,* "distinguish". This knowledge is reflected in their use of the term *skilja*, so it may be considered as a synonym with these European words, while the Icelandic meaning, "to separate", seems to be clearly underlying in the use of the word in many cases.

The ideas were, if possible, conveyed with the aid of an intermediary, some kind of a teacher. In the case of the writings from the 13th, 18th and 19th century and 1906 no class of mathematics teachers existed, and in 1966 they were very scarce. The ideas had to be conveyed by people who did not have professional training in mathematics or mathematics teaching. Even in the case of the teacher in 2004, there was only limited professional training in the field of mathematics.

The mathematician G. Arnlaugsson and his collaborators were among the few that had opportunities to know other approaches than the procedures of the early 1900s. They realized the deficiencies in routine skills without regard to the meanings of the procedures used. However, it is not at all a simple matter to dissolve traditions or to unite traditions developed in the different teacher subcultures.

Traditions are difficult to change, even if an agreement exists that a permanent change has to take place, except with an effective system of supervision or apprenticeship. It is difficult to part with traditional working methods, where training in mechanical skills, with minimum obstacles to overcome, often not achieving more than instrumental understanding, is performed in a peaceful and quiet atmosphere. It is like entering an insecure world where the teacher does not have full control of all what is said and done. This is hard for teachers to cope with, and many of them will need more support and apprenticeship than traditional teacher training has to offer.

On the other hand there is a need to challenge the lack of purpose and unrelatedness that many pupils sense in their routine work, all the way from keeping and borrowing up to the application of the sine and tangent functions mentioned by the promising young teacher, who feels now that she has discovered meaning in mathematics.

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COLLABORATIVE PROBLEM SOLVING IN GEOMETRY WITHOUT TEACHER INTERVENTION

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This paper reports research that focuses on student teachers working collaboratively in a problem-solving context without teacher involvement. One episode from the group discussion has been chosen to illustrate how elements of the students' reasoning process are revealed in dialogues. The students' use of different strategies has also been identified by taking a situational, socio-cognitive approach to studying communication and cognition. In their attempts at coming up with a solution to one of the problems, the students discuss different ideas and geometrical concepts. The analysis puts emphasis on how they attribute meaning to the concept of similarity. It is also natural to ask when it is suitable for a visiting teacher to become involved in the group discussion.

INTRODUCTION

The aim of the paper is to contribute to the understanding of how elements of reasoning are revealed in mathematical discussions. The concept of reasoning has been defined as five interrelated processes of mathematical thinking categorised as *sense-making, conjecturing*, *convincing*, *reflecting*, and *generalising.* It is derived from the contributions of Schoenfeld's (1985, 1992) framework for analysing problem-solving behaviour and from three models of problem solving (Borgersen, 1994; Mason, Burton, & Stacey, 1982; Mason & Davis, 1991; Polya, 1945/1957).

In Bjuland (2002), I have focused on two groups of student teachers with different mathematical backgrounds and experience as they work on two problems from classical geometry. In this paper, I focus on the group of students with limited background while they work on an open form of the geometrical problem originally introduced by the Italian mathematician Vincenzo Viviani (1622–1703). Viviani's theorem states that the sum of the distances from any interior point to the sides of an equilateral triangle equals the altitude. From this historical problem, the students are challenged to find the distance sum based on drawings, measurements, conjectures, and maybe come up with a proof.

From these introductory comments, the following research question has been posed: Which elements of reasoning can be identified in student communication while working on a geometry problem without teacher involvement?

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RELEVANT RESEARCH RELATED TO PRACTICE

A major concern for the teaching and learning of mathematics from the 1970s onwards at Agder University College in Norway has been to stimulate students to work on mathematical problems in collaborative small groups. This teaching philosophy has been strengthened by the foundation of the MA-programme and the PhD-programme in Mathematics Education at the college in 1994 and in 2002 respectively. This philosophy is exemplified by the studies carried out by Borgersen (1994, 2004) which show that problems in geometry are promising for use in small groups in order to stimulate mathematical discussions.

In the context of my work, collaborative mathematical problem solving is defined as the cognitive, metacognitive, socio-cultural and affective process of figuring out how to solve a problem. When students work on a real problem, they do not have a readily accessible mathematical algorithm in order to find a possible solution (Schoenfeld, 1992).

Reasoning, the cognitive and metacognitive component of the problem-solving process, is defined as five interrelated processes of mathematical thinking. *Sense-making* becomes the whole process of ideas and heuristic strategies that may help the students to approach and make sense of a problem. It is related to the importance of understanding (Polya, 1945/1957), analysing and defining the problem (Borgersen, 1994). The students come up with some ideas or plans (Polya) but these might not be the ones that will be a constructive direction for the reasoning process.

The two mathematical thinking processes, *conjecturing* and *convincing* refer to the exploring part of the solution process. The students are guessing and testing possible conjectures, and these processes form the basis of the argumentation. According to Mason (in Sfard, Nesher, Streefland, Cobb, & Mason, 1998), it is important to establish a conjecturing atmosphere in order to have a constructive mathematical discussion. In such a context "there is a shared struggle to find ways to express and convince which can be understood and appreciated by others, as well as challenged, exemplified, amplified, varied, generalised, etc. by them" (Sfard et al., p. 48). Mason and Davis (1991) suggest replacing the word *proof* with the word *convincing*, indicating three steps which change the orientation to the process of proving: from convincing yourself, to convincing a friend, to convincing a sceptic.

The process of reflecting is related to Polya's looking-back step (1945/1957). In my work, looking back on the solution process is a heuristic strategy used in order to find a successful solution, while *reflecting* refers more to the conscious consideration of examining and getting a deeper understanding of the solution (Bjuland, 2002). In Bjuland (2004), I have focused on how student teachers reflect on their experience as learners of mathematics and as future teachers of mathematics. That perspective of the reflective activity will not be addressed here.

In Bjuland (2002), I have distinguished between the process of *generalising,* which is related to the students' attempts at creating new problems, and the more local strategy of generalising. The process of *generalising* means that a conjecture or a problem can be transferred or modified to new situations by stressing some aspects of sameness. According to Mason and Davis (1991), the term *generalising* is used as a label for the process when you gradually try out more general cases. This is the corresponding process to specialising, referring to any act of simplification or of trying out special cases.

In his social constructivist model of learning and teaching, Wells (1999) focuses on the importance of creating a classroom community which shares a commitment to caring, collaboration, and a dialogic mode of making meaning. Following Wells, the term inquiry does not refer to a method or to some generic set of procedures for carrying out activities. Inquiry indicates a stance towards ideas and experiences. When students in a particular small group have created a community of inquiry, they show willingness to wonder, to pose questions, and to seek to understand by collaborating with each other in order to come up with a solution on a particular mathematical problem.

METHODOLOGY

This qualitative study (Bjuland, 2002) is carried out within a naturalistic research paradigm (Lincoln & Guba, 1985; Moschkovich & Brenner, 2000) which assumes, as a theoretical premise, that meaning is socially constructed and negotiated in practice by the participants. This paradigm relies largely on an ethnographic stance (Chambers, 2000) towards research. The empirical material consists of fieldnotes, verbal data registered on a cassette recorder and written transcripts of the dialogue of three small groups of students (8 lessons in each group).

I have chosen the dialogical approach to communication and cognition (Cestari, 1997; Linell, 1998; Marková & Foppa, 1990) in order to conduct the data analysis. According to Cestari, this approach "allows one to analyse the co-construction of formal language among participants in a defined situation" (p. 41). This means that it permits me to identify interactional processes, which, in the analysis of this particular episode, are the verbalisations expressing the students' elements of reasoning.

I follow Wells (1999) and his way of analysing episodes at three levels. At the first level, the episode has been divided into thematical sequences (segments). At the second level, each sequence has been divided into exchanges, and the third level consists of moves, either initiating, responding or follow-up moves. It is the exchange that constitutes the most appropriate unit for the analysis, according to Wells. Inspired by Lambdin (1993), I am especially interested in monitoring moves. These are verbalisations which stimulate the metacognitive awareness among the students.

Procedure, subjects, and problems

My research project was carried out on a problem-solving course in geometry in the students' first semester at a teacher-training college in Norway. The first part of the course consisted of teaching over a month including group work assisted by me and a colleague of mine. We focused on basic classical geometry, prepared students to work on problems in small groups, and stimulated them to experience mathematics as a process, as described by Borgersen (1994). The students were also introduced to metacognitive training in combination with cooperative learning. For more details, see (Bjuland, 2002, 2004).

The second part of the course consisted of collaborative small-group work without teacher involvement over three weeks. The empirical material was collected over four meetings throughout this period. Since the situation was in an educational context and the group work was presented with certain objectives to the students, it is possible to argue that the collaborative small-group work in the second part was not free from teacher involvement. However, the students had to solve the problems without any help from us. As an observer, I was present throughout the meetings, while my colleague would only visit the project groups in order to find out that the students were present.

I am here concerned with one group comprising one male student, Roy, and four female students Unn, Mia, Gry and Liv (ages: 24, 20, 21, 21 and 22 respectively). Names are pseudonyms. Four of the students have only attended the compulsory course in upper secondary school (1MA), and Gry and Liv have low marks from this course. Roy has attended voluntary courses the second and third year in upper secondary school, preparing for further studies in natural sciences (2MN, 3MN). Even though he has middle marks from those courses, he makes clear in the group reflection at the end of the fourth meeting (see Bjuland, 1997, p. 201) that he had poor background knowledge in geometry when he started at the teacher-training college. Based on the background knowledge about these subjects, I have chosen to categorise this as a group of students with limited mathematical background.

During the four group meetings in the second part, the students had to work on two problems, a compulsory one and one chosen from two other problems (see Bjuland, 2004). In this paper I am concerned with part B of the compulsory one, Problem 1B:

Choose an arbitrary equilateral triangle ΔABC. Let P be an interior point. Let d_a, d_b, d_c be the distances from P to the sides of the triangle (d_a is the distance from P to the side opposite of A, etc.)

a) Choose different positions for P and measure d_a, d_b, d_c each time. Make a table and look for patterns. Try to formulate a conjecture.
b) Try to prove the conjecture in a).
c) Try to generalise the problem above.

APPROACHING AND MAKING SENSE OF PROBLEM 1B

The aim of the following episode is to illustrate how the students come up with elaborated attempts at proving the conjecture found in problem 1Ba. The analysis is especially concerned with how the students attribute meaning to the concept of similarity. The episode is taken from the second group meeting when all five group members are present.

The discussion preceding this episode (see Bjuland, 2002) has shown that the students were uncertain about how to do the measurements in problem 1Ba since the problem does not tell them how to measure the distance from *P* to the side of the triangle. Some of the students suggested that they had to measure the lengths of the perpendiculars from *P* to their intersections with the sides of the triangles, but one of the group members had an alternative way of doing it. She thought that they should measure the distance along the line through *P* parallel to the base of the triangle. This idea challenged the other students to focus on an alternative way of doing the measurements. However, by debating the two different perspectives, the students established agreement about the fact that they had to choose a particular and unique distance in order to find a pattern. The reconstruction of the solution process on problem 1Ba has shown that the students come up with the following conjecture $d_a + d_b + d_c = constant.$ For further details, see (Bjuland, 2002).

One of the students has, earlier in the solution process, been concerned with three cyclic quadrilaterals located inside the equilateral triangle (see figure 1 below). This idea has been reconsidered and brought into the discussion as a possible direction for the reasoning process. The students have also discussed the idea of cyclic quadrilaterals as kites. The concepts of kite and cyclic quadrilateral have been discussed

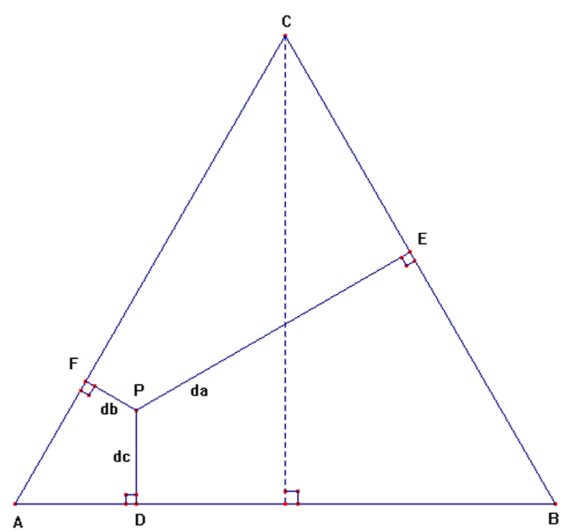

Figure 1

and used in a variety of examples, helping the students to clarify what is meant by these particular concepts. In figure 1 the cyclic quadrilaterals are called *ADPF*, *DBEP* and *FPEC* respectively. This was not the notation used by the students.

Conjecturing about cyclic quadrilaterals being similar

As the continuation of the dialogue, the students are still in the process of making sense of the conjecture by trying to find a direction for the reasoning process.

847	Gry:	Liv … Liv … how would you do it?
848	Liv:	I think maybe that those are similar and
849	Roy:	What?
850	Liv:	That these (cyclic) quadrilaterals
851	Roy:	Those here?
852	Liv:	Mmm … (8 sec.) …
853	Roy:	No they aren't … (8 sec.) …
854	Liv:	How can you say that?
855	Roy:	If we place that point there … then you get that quadrilateral … then you get that quadrilateral … and then you get that quadrilateral … you can see that those aren't similar
856	Unn:	Now you have to use that number to something … that was the difficult thing … (Unn helps Gry, she speaks simultaneously with Roy in a low voice)
857	Roy:	Maybe that one and that one could be similar? … and that side is a bit longer than that side
858	Liv:	But … well those could be similar … even if they don't have equal lengths … the point is that the angles are equal … and that there is a common ratio
859	Roy:	Yes it's … actually they are similar … since each of those have two *90*-degree angles … one *60*-degree (angle) and one *120*-degree (angle)
860	Liv:	Mmm
861	Roy:	That will be the case for all of those … (6 sec.) … to every point (*P*) … then we get three similar triangles … no … quadrilaterals, yes
862	Liv:	Cyclic
863	Mia:	Is it a cyclic quadrilateral when it is inside a triangle? … isn't it only in circles we find those?
864	Roy:	Yes, to draw a circle around
865	Liv:	It's possible to draw a circle which goes through all the points … then it is called a cyclic quadrilaterals
866	Mia:	Yes … (6 sec.) …

Gry's question (847) invites Liv to explain her work to the other students. This is an important *monitoring question,* bringing the idea of the cyclic quadrilaterals being similar (848) into the discussion. By monitoring and controlling what Liv is concerned with, Gry makes it possible for all the students to focus on the new idea and participate in the discussion. Graumann (1995) emphasises the importance of establishing mutual relationships between participants in dialogues. The common ground among the students is one crucial condition for learning to take place. Roy's brief questions (849), (851) ask for a repetition of the idea, inviting Liv to point out her cyclic quadrilaterals, which are also similar.

Roy's verbalisation after the silence is a negation, claiming that the cyclic quadrilaterals are not similar (853). Liv's idea has then been challenged by Roy, and the students are confronted with two opposite interpretations for the idea of considering the cyclic quadrilaterals as similar. A conjecturing atmosphere (Mason, in Sfard et al., 1998) has been established, and the analysis will focus on how the students elaborate on these two opposite perspectives.

Roy is challenged by Liv's *how question* (854) to elaborate on and explain his perspective. He focuses on an arbitrarily chosen location for *P* and the corresponding three quadrilaterals inside the equilateral triangle, and he claims from the figure that those quadrilaterals are not similar (855). Unn's verbalisation (856) indicates that not all students participate in the discussion whether the quadrilaterals are similar or not. It seems to be a discussion between Roy and Liv (857)–(862). Roy does not stick to his own perspective, suggesting that there are maybe some similar quadrilaterals on the figure (857).

From a mathematical point of view, Liv's next initiative points to the essential characteristics of similar quadrilaterals (858). She makes clear that similar quadrilaterals do not need to be congruent, which means that corresponding sides do not necessarily need to have equal lengths. She also mentions the condition for similar quadrilaterals, that there is a common ratio between corresponding sides. However, she also points to the fact that corresponding angles are equal, which is not a sufficient condition for quadrilaterals being similar. Even though the condition for similar quadrilaterals is pointed out in the conversation, the students make a wrong, local conclusion (859)–(862) by suggesting that the cyclic (862) quadrilaterals are similar. They focus on corresponding angles of equal size (859), neglecting the condition of the common ratio of corresponding sides.

Liv's brief verbalisation (862) is a supplement to the local conclusion, pointing to the fact that the quadrilaterals are both similar and cyclic. But it also triggers a question (863) about the connection between cyclic quadrilaterals and circles (863)–(866). The students focus on one important characteristic of cyclic quadrilaterals, pointing to the fact that it is possible to construct a circles which goe through all four vertices of cyclic quadrilaterals.

Later at the same meeting when the other students have started to work on another problem, Liv is still concerned with the proof of the conjecture, focusing on the idea whether the quadrilaterals are similar or not. She cuts out the quadrilaterals from her figure by using a pair of scissors, observing that they are not similar. She informs her colleagues about her discovery. However, Roy does not agree with her, claiming that the cyclic quadrilaterals must be similar. The dialogue below finishes this discussion. I observe that the students stick to their wrong conclusion by pointing out that the cyclic quadrilaterals are similar since corresponding angles are equal.

1604 Liv:	The point is that they (the quadrilaterals) all have equal angles
1605 Roy:	Yes … and they are therefore similar
1606 Liv:	Yes

This indicates that the students transfer the condition for similar triangles to similar quadrilaterals by just focusing on corresponding angles. Even though Liv has cut out the cyclic quadrilaterals from her figure and observed that they are not similar, the students stick to their wrong conclusion. They seem to make an overgeneralisation from similar triangles to similar quadrilaterals.

After having spent about *25* minutes working on problem 1Bb, the students decide to end the reasoning process without coming up with a convincing argument. However, the analysis of the episode has shown that the students have been concerned with important mathematical concepts. The conjecturing atmosphere in the dialogue has stimulated the students to clarify what is meant by the concept of cyclic quadrilateral and the concept of similarity, explicitly expressed by similar quadrilaterals.

DISCUSSION

During the 90s, quite a few studies in mathematics education have put emphasis on peer interaction in small groups as an important arena for learning (Cobb, 1995; Healy, Pozzi, & Hoyles, 1995; Kieran & Dreyfus, 1998). Brodie (2000) makes it clear that peer interaction is seen to provide support for the construction of mathematical meaning by pupils since it allows more time for pupil talk and activity. According to Farr (1990), the dynamics of a three-person group change dramatically compared to peer interaction since it is possible to form coalitions in a group of more than two members. The dynamics of our particular group are even more complex since the perspective of five students could be brought into the dialogue. Following Wells' (1999) perspective of creating a community of inquiry, I believe that it is crucial that student teachers in their teaching-training programme get the opportunity to experience collaborative problem solving in small groups. The importance of teaching mathematics through problem solving has also been stressed by Lester and Lambdin (2004). They emphasise that this instructional approach could stimulate students to develop a thorough understanding of mathematical concepts.

The detailed analysis of the dialogue of this particular episode puts emphasis on the identification of elements of reasoning expressed in one group of student teachers while working on the open form of the Viviani's theorem. The students came up with a conjecture for equilateral triangles: $d_a + d_b + d_c = constant$ (problem 1Ba), but they failed in their attempts at proving the conjecture. On this particular part of the problem (1Bb), the reasoning process consisted only of *sense-making*. However, in this sense-making process, I have also identified *conjecturing* elements. The students tried to find a direction for their solution process by suggesting different ideas. The analysis has illustrated one of the main ideas from the sense-making process, the students' *conjecturing* about cyclic quadrilaterals being similar.

In Bjuland (2002), I have also identified the students' reasoning process on problem 1Bb for another group, categorised as a group of students with average mathematical backgrounds. These students also have difficulties in coming up with a proof for the basic conjecture $d_a + d_b + d_c = constant$ for interior points in equilateral triangles. However, in this particular group, the identified elements of reasoning could be summarised in *sense-making, conjecturing*, *convincing*, *reflecting on the solution* and *generalising*. This finding is an indicator of the poor reasoning process on problem 1Bb for the student group analysed above.

The analysis has also revealed that the students are aware of the essential characteristics of similar quadrilaterals, but they transfer the special condition for similar triangles to similar quadrilaterals by just focusing on corresponding angles. One possible reason for their wrong conclusion may be found in Norwegian textbooks in lower secondary school (pupils aged 14–16) in which the concept of similarity is often linked to triangles only. Triangles are characterised as similar when corresponding angles are equal. This result cannot be generalised to quadrilaterals since there is not necessarily a constant ratio between corresponding sides of quadrilaterals to corresponding angles being equal.

This particular finding in my analysis is problematic and challenges my argumentation about having students working in collaborative working groups without teacher involvement. If the verbalised reasoning process comes up with an overgeneralisation on a particular geometrical concept and the students conclude their conversation with a wrong conclusion, should they remain in this situation without being helped by a teacher?

In one respect, it is natural to have a teacher visiting a group in order to challenge and try to correct partial conceptions. A supervising teacher could also have guided the students with open questions, stimulating them in the solution process to find a solution. It is also considered by Laborde (1998) that visual evidence in a computer-based environment could be a catalyst for generating mathematical questions. Unfortunately, the students did not have a dynamic geometry software package like Cabri at their disposal. However, I think it is important to emphasise the fact that student teachers with limited mathematical backgrounds are able to make progress with unfamiliar problems without teacher intervention if they are given sufficient time to work on them. The conjecturing atmosphere (Mason in Sfard et al., 1998) established in the group indicates that the students are inspired by their own mathematical conversations in this particular problem-solving context. Instead of being involved in the group discussion, I suggest that a visiting teacher could observe the students during their work and identify their difficulties with the problems. These observations could then be brought into the classroom after the collaborative small-group work for a discussion to the whole class.

Even though the students in my group do not have a rich knowledge base in their reasoning process and they fail to solve problem 1Bb, they have used a great variety of strategies (see Bjuland, 2002) particularly expressed in their ways of posing open questions. The one episode, analysed above, has illustrated how questions, categorised as *monitoring* and *how*, have stimulated the students' discussion about bringing the idea of the cyclic quadrilaterals being similar into the discussion.

Problem 1Bb is not an easy one for students with poor mathematical backgrounds, particularly without any help of a visiting teacher. It is therefore possible to argue for the fact that this part of the problem is too challenging for them. However, the students have really struggled with that particular problem, indicating that it has provoked an interest in them to come up with a convincing argument. The analysis has shown that they have been concerned with the concept of cyclic quadrilaterals. More specifically, they have identified and argued why the three quadrilaterals in figure 1 are cyclic. From a mathematical point of view, this is not obvious, and students need quite a lot of geometrical expertise to see this. I believe that the students' discussions on this particular problem can function as a starter for an elaborated conversation, in the spirit of a Lakatosian dialogue, with their teacher afterwards.

CONCLUDING REMARKS

This paper addresses adult students' reasoning processes while solving a problem in geometry. Its original character rests in considering the analysis of these processes from both a social and a content perspective. It has reported student teachers' collaborative work in small groups without teacher involvement which is quite a radical position with respect to the teaching and learning of mathematics. However, I suggest that it is easier to take this radical position with student teachers since they often are more mature and motivated for learning than pupils at lower levels in the school system. I have pointed out that a supervising teacher could help students by posing open questions, stimulating new suggestions in the solution process. However, I claim that teacher involvement should not be prominent if students should have the opportunity to work on problem solving in collaborative small groups to experience the whole process of doing mathematics. After the group meetings, teachers could prepare for an elaborated discussion in the whole class.

A possible direction for future research would be to focus more closely on the following question: What kinds of reasoning, heuristic strategies or questions can be identified in pupils' communication at lower levels in the school system during collaborative problem solving in small groups?

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IDENTITY AND AGENCY IN MATHEMATICS TEACHER EDUCATION

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The paper discusses theoretical frameworks presented in the book "Identity and Agency in Cultural Worlds". I use the framework to analyse texts produced by one student teacher during the compulsory mathematics course in the pre-service teacher education. The student's texts are about her learning. I see them as reports of her perception and understandings of the "figured worlds" of mathematics education in which she participates as learner, to indicate her "positionings" and reflections of her "authoring of identities" as learner and performer as teacher of mathematics.

INTRODUCTION

During their teacher education it is presupposed and expected that student teachers will develop an identity as teachers, and a view on students, school subjects and teaching/learning that will give them sufficient confidence and competence to enter the Norwegian primary and lower secondary schools as teachers.

The students' texts are seen as utterances within the teacher education genre. It is necessary to see the different and shifting contexts and genres, including ideologies that surround the utterances (Ongstad, 1997). Ideology is broadly defined as unspoken premises for communication (Braathe & Ongstad, 2001). It is something we think *from*, not on (Ricoeur, 1981). Genres are therefore carriers of ideology (Bakhtin, 1986).

The student teachers are exposed to different educational ideologies, and these will influence their development of identities as teachers (Braathe & Ongstad, 2001). Essentially these are ideological conflicts within which the student teachers are struggling to form their teacher identities.

The paper will present "Analysis of Positioning" of student teachers' texts as a methodology for investigating their identities as mathematics learners and becoming mathematics teachers. The purpose of the paper is to see how this methodology can be validated by relating to the theoretical framework developed by Holland, Skinner, Lachicotte, and Cain (1998).

In the examples I point to the genres, and hence underlying ideologies, that can be identified both from students' previous experiences and within the teacher education.

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THEORETICAL CONSTRUCTS AND CONCEPTS USED

Positioning as a triadic discursive concept

Genres can be described as kinds of communication. Genres are ideological, that is they give tacit premises for the participants' positioning in the communication (Ongstad, 2004). Positioning as it is developed and used by Ongstad (1997, 2004) is taking its initial inspiration from Bakhtin's essay "The problem of speech genres" (Bakhtin, 1986, pp. 60–102) where he identifies Bakhtin's communicative elements necessary for an utterance to communicate in a dialogic relation. These elements are: delimiting (from former utterances), positioning (the utterance as such by expressing, referring a semantic content, and addressing someone), and finalising (the utterance as a whole by finalising forms, semantic exhausting, and ending speech will) (Ongstad, 2004).

Ongstad (2004) further holds that the genres are also to be seen as triadic in the same sense as the positioning of the utterance, that they simultaneously give potential for the expressing, referring and the addressing. The three aspects are seen as parallel, inseparable, reciprocal, simultaneous processes.

The concept 'positioning' is used by sociologists, educational sociologists and educationalists as a more flexible and complex concept to describe the meaning and sense of personal acts than for instance 'motivation', 'strategy', 'role', 'agency', and 'voice'. In these works the notion of positioning generally refers to some kind of active, conscious strategy, as *discursive roles*, which is related to power or *agency* in particular situations. "Positioning Theory" has been discussed and developed among others by Harré and van Langenhove (1999).

Analysis of positioning

Seeing every utterance as an inseparable triadic unit theoretically inspires the methods I apply, hence I analyse the texts both from 1) expressive, 2) referential and 3) addressive aspects. This 'triangulation' can give insight in how the student (respectively) position him/her self in relation to 1) own emotions and experiences, 2) the mathematics and the teaching and learning of mathematics and 3) the classroom, teachers and others. Seen this way a specific focus is to apply this 'triangulation' and to develop methods for analysing the influence on students' identities as teachers of mathematics through their writing of mathematical educational texts.

In the examples this 'triangulation' is used as a critical reference to the concepts *figured worlds, positioning and authoring* as they are developed by Holland et al. (1998).

They develop their theoretical framework within socio-cultural tradition using Bakhtinian genre theory to explain how subjects through their perception and understandings of the *figured worlds* in which they participate, indicate their *positioning(s)* and reflect their *authoring* of identities. I read their use of 'positioning' as discursive role and/or agency developed in an anthropological context. For not to mix these meanings of positioning I will in this paper refer to discursive role or agency where I interpret Holland et al. (1998) to use 'position' or 'positioning' in this meaning.

Identity

Holland et al. (1998) state that: "Identity is used as a concept that figuratively combines the intimate or personal world, with the collective space of cultural forms and social relations" (p. 5). Identity within mathematics education research has been connected to systems of beliefs and values (Leder, Pehkonen, & Törner, 2002). These beliefs have been identified and differentiated between beliefs about the self, mathematics education and the social context (Eynde, De Corte, & Vershaffel, 2002), and seen as underlying the identities of teachers of mathematics and as such influencing their practice (Lerman, 2001, 2002).

As indicated above both Holland et al. (1998) and Eynde et al. (2002) are seeing identity and beliefs respectively with the expressive, referential and addressive aspects.

I see identities and subjectivity as dialogically situated in and formed by genres/figured worlds, and so can have many expressions dependent on the context. I further follow Ongstad (2004) in that the genres are also to be seen as triadic in the same sense as the positioning of the utterance, that they simultaneously give potential for the expressing, referring and the addressing. Identity can then be seen in the same way as Holland et al. dynamically combining the personal, the cultural and the social.

Figured worlds

According to Holland et al. (1998)

> [f]igured worlds rest upon people's abilities to form and be formed in collectively realized "as if" realms. … People have the propensity to be drawn to, recruited for, and formed in these worlds, and become active and passionate about them. People's identities and agency are formed dialectically and dialogically in these "as if" worlds. (p. 49)

By further pointing to the abilities of humans to manipulate their worlds and themselves by means of symbols, they adhere to the pragmatic turn focusing on language and symbols in communication as primary means for development and learning. Taking a communicational view on learning and teaching as a theoretical framework includes seeing how a personal *self*, a *world* of phenomena and social *others* are embodied and inseparably connected *in* communication and *by* communication in all of us (Ongstad, in press).

The Bakhtin circle was influenced by Gestalt theory (Brandist 2002, 2004; Holquist, 1990). The concept of figured world, inspired by Bakhtin (1981, 1986), then indicates the idea of what communication as focusing matter dynamically produces, that is to problematise the relation between the focused *figure*, its *background* and the focusing *position.*

Giving attention to a phenomenon by focusing, and thus directing any of the senses towards sounds, signs, smells or movements, will create some kind of gestalt or mental figure. The focusing, or the figuring, gives priority to the object as a possible mental category and our mind is accordingly more likely to accept this 'something' not only as mental reality but as *reality* and even as *the* 'reality'. It favours a conceptualizing and tends to build up our mind-world by and as *nouns*, which by their fixedness tempt us to believe, nominalistically that they are 'real' and (separate) 'things'. This phenomenon is described by Sfard (1991) as the process of reification in the development of concepts in mathematics.

Holland et al. (1998) point to this competence that makes possible culturally constituted or figured worlds and, consequently, the range of human institutions, and "They become the cultural resource that mediates members' identities" (pp. 51–52).

We communicate through utterances. Utterances are any sufficiently closed use of sign that makes sense. We communicate by uttering and by giving utterances meaning. All utterances are uttered and interpreted related to expectations of genres, i.e. contexts that helps us to understand the utterance, as indexing and pointing to a culturally figured world.

By "figured world", then, Holland et al. (1998) mean a socially and culturally constructed realm of interpretation in which particular characters and actors are recognized, significance is assigned to certain acts, and particular outcomes are valued over others. These collective "as-if" worlds are sociohistoric, and so, from the perspective of heuristic development, inform participants' outlook. Figured worlds in their view could also be called figurative, narrativized, or dramatized worlds. Mathematics and mathematics education will be seen as such "Figured World".

Drawing links between identities and activities Wenger (1998) explores the consequences of self-development in and through activities. Identities are formed in the process of participating in activities organized by figured worlds. Figured worlds provide the loci in which people fashion senses of self – that is, develop identities. The texts analysed are results from participating in these organized activities. Leone Burton (1999) has pointed to the same phenomena within mathematics education as the narrative approach to the learning of mathematics.

Cultural artefacts, like mathematical subgenres, gain a kind of force by connecting to their social and cultural contexts, to their figured worlds. The actions, the deployments of artefacts, evoke the worlds to which they were relevant, and relate individuals with respect to those worlds.

> It is their pivotal role, as Vygotsky called it – *their capacity to shift the perceptual, cognitive, affective, and practical frame of activity* – that makes cultural artefacts so significant in human life. … The conceptual and material aspects of figured worlds and of the artefacts, through which they are evinced, are constantly changing through the improvisation of actors. This context of flux is the ground for identity develop-

ment. It sets the conditions for what we call the authoring self … "space of authoring". (Holland et al., 1998, p 63, my emphasis)

Relational and figurative identities

Holland et al. (1998) make an analytic distinction between aspects of identities that have to do with figured worlds – storylines, narrativity, generic characters, and desire – and aspects that have to do with one's discursive role/agency relative to socially identified others. But they underline that "these figurative and relational aspects of identity interrelates in myriads of ways" (p. 125).

Relational identities have to do with behaviour as indexical of claims to social relationships with others and have to do with the day to day and on the ground relations of power, social affiliation and distance – with the social interactional, social relational structures of the lived world. Relational identities, as they have used the term, is a person's apprehension of her social role/agency in a lived world: that is depending on the others present, of her greater or lesser access to spaces, genres and, through those genres, authoritative voices, or any voice at all. Relational identities are likened to indexicality, to the pragmatic facet of speech, and therefore normative.

Narrativized or figurative identities, in contrast, have to do with the stories, acts, and characters that make the world a cultural world. Figurative identities are likened to propositionality, to the referential, semantic facet of speech.

Holland et al. (1998) are not explicitly pointing to emotional or expressive aspects of identity, but I read them to see these aspects implicit in both relational and figurative identities. They however point to this aspect in a note:

> … makes the point that emotional experiences are also collectively developed in certain activities. Those who are barred from these activities are unlikely to develop the emotional experiences to the same degree of facility as those who fully participate. If expressing a certain emotion completely is treated as a claim to a certain relational identity, then one is unlikely to be able to claim that identity if one has been barred from the formative activity (Note 9, p. 135).

People develop different relational identities in different figured worlds because they are afforded different discursive roles in those worlds. Discursive roles and agencies, in other words, become dispositions through participation in, identification with, and development of expertise within the figured worlds. "The formation of identity in this posture is a byproduct of doing, of imitation and correction, and is profoundly embodied" (Holland et al., p. 138).

Authoring Selves

Holland et al. (1998) take Bakhtin's vision of self-fashioning through appropriating genres and discourse as a point of departure for their theory for the authoring self.

> Bakhtin's concepts allow us to put words to an alternative vision, organized around the conflictual, continuing dialogic of an inner speech where active identities are ever forming. […] The figured worlds of dialogism is one in which sentient beings always exist in a state of being "addressed" and in the process of "answering". People coexist, always in mutual orientation moving to action; there is no human action which is singularly expressive. (p. 169)

Understood this way in the making of meaning we "author" the world. We do this by using others' words, like the bricoleur who builds from preexisting materials. "One is more or less condemned, in the work of expression, to choices because "heteroglossia", the simultaneity of different languages and of their associated values and presuppositions, is the rule in social life" (Holland et al., 1998, p 170). That is, the author works within, or at least against, a set of constraints that are also a set of possibilities for utterance. These are the social forms of language that Bakhtin summarized: dialects, registers, accents, and "speech genres" (Bakhtin, 1981, 1986; Volosinov 1986).

> The whole utterance is no longer a unit of language (and not a unit of the "speech flow" or the "speech chain"), but a unit of speech communication that has not mere formal definition, but contextual meaning (that is, integrated meaning that relates to values – to truth, beauty, and so forth – and requires a responsive understanding, one that includes evaluation). The responsive understanding of a speech whole is always dialogic by nature. (Bakhtin, 1986, p. 125)

The validity is also seen by Bakhtin as connected to the dialogic situation and related to the possible response to the utterance:

> But from the very beginning, the utterance is constructed while taking into account possible responsive reactions, for whose sake, in essence, it is actually created. As we know, the role of the *others* for whom the utterance is constructed is extremely great. […] From the very beginning, the speaker expects a response from them, an active responsive understanding. The entire utterance is constructed, as it were, in anticipation of encountering this response. (Bakhtin, 1986, p. 94)

It is not only being addressed, receiving others' words, but the act of responding, which is already necessarily addressed, that inform our world through others. Like Mead (1934), Bakhtin insists that we also represent ourselves to ourselves from the vantage point (the words) of others, and that those representations are significant to our experience of ourselves. Holland et al. (1998) elaborate their understanding of Bakhtin's notion of what the self is to existence as the pronoun "I" is to language.

> Both the self and "I" designate pivotal positions in the stream of (language) activity that goes on always. In explaining what an "I" is, position, rather than content, is important. … In Bakhtin's system the self is somewhat analogous to "I". The self is a position from which meaning is made, a position that is "addressed" by and "answers" others and the "world" (the physical and cultural environment). In answering (which is the stuff of existence), the self "authors" the world – including itself and others. (p. 173)

This can be interpreted as they, similar to Ongstad (in press), hold that

> these three aspects [the focused *figure*, its *back-ground* and the focusing *position*] will form an … logic triangle that will be relevant for any research on culture as meaning. That is what positioning is about – to search for and to try to comprehend a complicated, dynamic balancing between sub-aspects and general relations, between processes (position/-*ing*) and products (position-ing/***s***).

Seeing "I" as position implies how unlikely it is that one's identities are ever settled, once and for all. Dialogism makes clear that what we call identities remain dependent upon social relations and material conditions. If these relations and material conditions change, they must be "answered", and old "answers" about who one is may be undone. This at the same time gives room for agency, or positioning as a triadic discursive concept, which saves identities from being totally determined by discourse.

EXAMPLES OF EMPIRICAL DATA AND ANALYSIS

Below I will give examples of analysis of student teachers' texts to illustrate how Holland et al.'s concepts, together with a Baktininan understanding, can give insight into the development of students' identity as mathematics teachers. The methodological focus here is to illustrate how validating of the analysis of positioning can be supported by use of this understanding.

The following examples are from an ethnic Norwegian student, here called Kari. All examples are translated from Norwegian by me. There is one and a half year between the two examples in which she has been active in learning about mathematics and mathematics education.

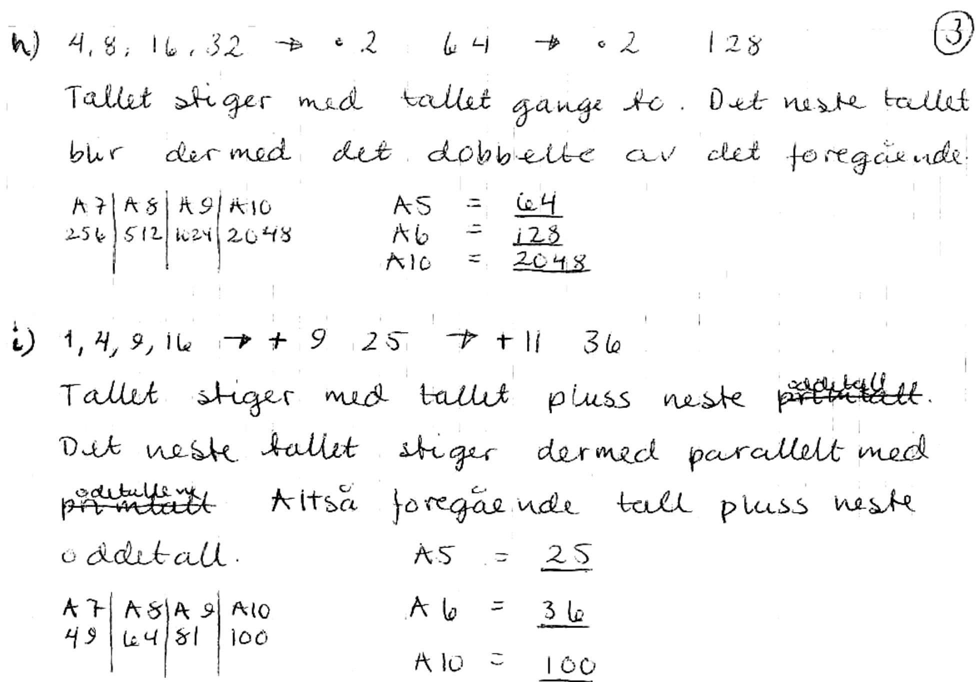
Task 1

After finishing the first semester the students got an individual mathematical task to solve within a week. The task was divided into two parts. The first part was to read and comprehend an introductory text about number sequences and some common number patterns, figurate numbers, like square, cubic, and triangular numbers, in which there were some tasks to solve. The second part was about using figurate numbers to create artistic decorations. Below is an extract from Kari's work.

The task illustrated below was to find and describe the pattern of the number sequences given and explicitly to find the 5th, 6th and 10th number in the sequences.

h) 4, 8, 16, 32,….

i) 1, 4, 9, 16,….



h) 4, 8; 16, 32 → ·2 64 → ·2 128 (3)

Tallet stiger med tallet gange to. Det neste tallet blir dermed det dobbelte av det foregående.

A7	A8	A9	A10
256	512	1024	2048

A5 = 64
A6 = 128
A10 = 2048

i) 1, 4, 9, 16 → + 9 25 → + 11 36

Tallet stiger med tallet pluss neste ~~primtall~~ oddetall.
Det neste tallet stiger dermed parallelt med ~~primtall~~ oddetallene Altså foregående tall pluss neste oddetall.

A5 = 25
A6 = 36
A10 = 100

A7	A8	A9	A10
49	64	81	100

The figure above shows Kari's explanations to these tasks.

The written text in h) is, in English: *"The number increases multiplied by two. The next number then becomes the double of the previous number"*.

The text in i) is: *"The number increases with the number plus the next ~~prime~~ odd number. The next number then increases parallel with the ~~prime~~ odd numbers. That means the previous number plus the next odd number."*

The "figured world", or mathematical sub genre, are number sequences in the genre of answering tasks in mathematics teacher education. The focused object in this analysis is Kari's utterance in each part about the sequence at hand and the background is the didaktical[1] context of teacher education in mathematics.

The expressive aspects of the utterance are related to form and what this form symptomatically can express. One could read how she uses the dots and arrows in the beginning either as a (rough) draft she does to help her own thinking, and/or it can be read as a communicative utterance where she explains how the next number in the sequence is constructed. In both cases she uses an informal, nearly oral, genre. The same is the case with the written texts that also are in an informal genre, although we can identify it as a "rule giving" genre; it is written in an impersonal voice, in present tense and in general terms (it is about "the next number"). In the last part she is setting up a table for the next numbers in the sequence, this can be identified as from a technical genre, as in her mathematics textbooks. This mix of genres could be seen as voices from her experience from school and also from the teaching of sequences at the teacher college. The introductory text, and the textbooks she should read, uses a more formal mathematical style.

My reading of the form in this utterance, related to her subjective/emotional aspect of identity as teacher of mathematics, connects to both the figurative and relational aspects. According to Holland et al. (1998)

> … indices of relational identities, however, become conscious and available as tools that can be used to affect self and other. In other words, some relational identities and their associated markers are clearly figured. Perhaps these figured aspects of relational identities become relatively conscious for anyone successfully recruited into the figured world where such qualities are deemed important. The everyday aspects of lived identities, in contrast, may be relatively unremarked, unfigured, out of awareness, and so unavailable as tools for affecting one's own behaviour. (p. 140)

The early phase of the teacher education she now is part of, and consequently the widening gap she experiences to her "everyday aspects of lived identities" until now dominated by her experience of school mathematics as student, can be read as the vague "unfigured" use of the formal mathematics style.

The referential, or figured, aspects of the utterance are related to true or false related to the mathematics she is uttering about. We see that she has got the answers correct, even the square numbers in i), but this written explanation is not easy to evaluate. It is not obvious that we get the next square numbers by adding the next odd number to the previous square number. She has not in her text tried to explain this and it is therefore uncertain if she has understood the mathematics behind it; or if she has just seen that it hold for some examples and concludes from this that it holds for all cases; or if she just remembers it as a fact from either the textbook or from

[1] I prefer using the German spelling to differentiate from the English didactic which have a different meaning. This is becoming an established practise within European literature on education. See Gundem and Hopmann (1998).

the classes; or if she remembers it as a geometrical pattern from the demonstration in the class.[2] When it comes to the referential aspects of texts/utterances it must be held open that they may contain many possibilities depending on the researcher's focus and also on possible *intentions* that the utterer may have. From the perspective of Bakhtinian (and Holland et al., 1998) dialogism one would expect that she would be able to give reason for this statement, and therefore it must be kept open which references she would argue for.

The addressive, or relational, aspects of the utterance are related to normativity, here in the sense of usefulness related to role of mathematics teacher in the primary school. Usefulness here includes ethical values concerning teaching and learning, including beliefs on mathematics, children, teachers and others. Her explaining text can be identified as "rule giving" genre within mathematics, and as such a part of the repertoire of the becoming teacher. The normative claim could be read as part of an instrumental view on teaching and learning mathematics, which could be seen as an element of her "everyday aspects of lived identities" as her belief of school mathematics teaching. That is an ideology within the genre of teaching mathematics.

As I position this utterance of Kari I read it as dominated by the expressive aspect as its form symptomatically express her insecurity with the formal mathematical genre, and at the same time her incomplete explanation of the mathematics. Her written texts are all describing processes, which again connects to an oral genre, and can indicate a diffuse figuring of the mathematics "objects". The analysis of positioning then will evaluate the utterance as dominant expressive, but the two other aspects are present and as important when it comes to understanding her appropriating the mathematical genre/register. Her utterance as act is giving an answer to a task sat to her, and therefore her reference point is answering this within the 'task' genre in (school) mathematics, and seen as her authoring herself as a student of (school) mathematics.

Task 2

The second example is in the same mathematical subgenre as the first example, but done one and a half year later than the first. It is presented here to illustrate possible development when it comes to her positioning as teacher of mathematics.

As a condition for passing the final exam the students have to do an individual mathematics test, a school sitting for six hours. This test contains both didaktical and mathematical questions. In this test there was a task on number sequences:

Two number sequences are given:

2,7,12,17,….

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[2] For the reader of this paper I will show that her rule is correct:
Look at:
$n^2 = ((n-1) + 1)^2 = (n-1)^2 + [2(n-1) + 1]$, or in her words: *"The number increases with the number plus the next odd number. The next number then increases parallel with the odd numbers. That means the previous number plus the next odd number.»*

1,3,9,27,….
Find the next two numbers in the sequence.
Find the recursive and the explicit formula for the two sequences.
Explain why the formulas are correct.

2 a) 2, 7, 12, 17 ..... +5 b) 1, 3, 9, 27 .... · 3
22, 27 81, 243

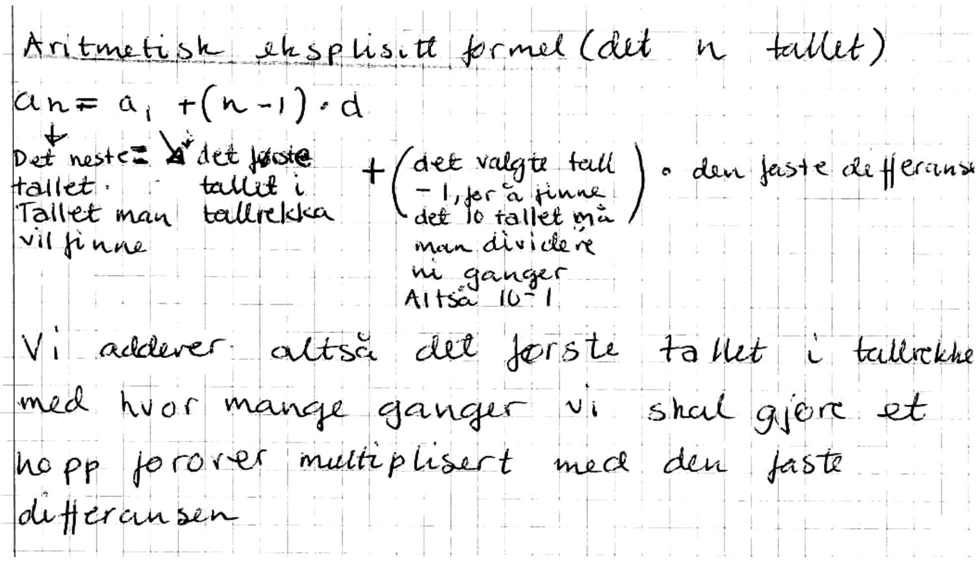
b)
Rekursiv (Aritmetisk) Rekursiv (Geometrisk)
$a_n + 1 = a_n + 5$ $a_{n+1} = a_n \cdot 3$
Eksplisitt Eksplisitt
$a_n = a_1 + (n-1) \cdot d$ $a_n = a_1 \cdot k^{(n-1)}$
$a_n = 2 + (n-1) \cdot 5$ $a_n = 1 \cdot 3^{(n-1)}$

Kari's texts on a) and b) are similar to her texts from the individual task on number sequences referred to earlier. However, to give the recursive and explicit formulas for the sequences she is giving the explicit formula initially in its general form. This could be seen as a symptom of her having some experience with the recursive formula, but less with the explicit one and therefore she has to look it up in her textbook and then translate it into the sequence at hand. This whole utterance is without any explanatory text; all explanations are implicit. Here she is drawing both on her experience as mathematics student in the class, and also from theory she has been reading. However I see the overall form much as part of her experience of the school mathematics genre. She is still not into the genre of the mathematics in the textbooks or able to explain her thinking in that genre.

Below is her answer to c):

c) Aritmetisk rekursiv formel (det neste tallet)
$a_{n+1} = a_n + d$
Tallet vi vil finne. Neste tall i rekken = forrige tall + differansen (tallet du øker med)
Her tar du det forrige tallet som du vet og adderer med det tallet du vet er differansen.

To explain the recursive formula for the arithmetic sequence she writes: "Here you take the previous number which you know and add the number you know is the difference."



Aritmetisk eksplisitt formel (det n tallet)

$a_n = a_1 + (n-1) \cdot d$

Det neste tallet. Tallet man vil finne = det første tallet i tallrekka + (det valgte tall −1, for å finne det 10 tallet må man dividere ni ganger Altså 10−1) · den faste differanse

Vi adderer altså det første tallet i tallrekke med hvor mange ganger vi skal gjøre et hopp forover multiplisert med den faste differansen

To explain the explicit formula for the arithmetic sequence she first explains the different parts of the formula by translating the mathematical symbols into everyday language.

To explain the (n-1) part she writes: *"The chosen number –1, to find the 10th number one has to divide nine times. 10 – 1 then."* She explains the general case by an example. To explain the formula she writes: *"We therefore add the first number in the sequence with the number of times we shall make a jump ahead multiplied by the difference."* In this utterance she is nearly tactile in how she explains the jumping ahead along the number sequence.

This utterance, as in the previous example, uses a mix of genres. However one genre seems dominant, the "Explaining" or "Introduction" genre as she uses "*Here you take…",* addressing a individual "you", and also an inclusive "we" in *"We therefore add…",* and by both explaining the general by an example and the tactile metaphor she uses in explaining the explicit formula. This is a genre which is used frequently in the mathematics texts she is meeting in the study, and explaining by examples are used a lot both in educational texts and also in teaching sessions both at the college and in the practice schools. One could see this as a sign on her appropriating the voices of didaktical genres. Her explaining of the formulas for the geometric sequences is similar. Her utterance this time must also be seen as an act giving an answer to a task sat to her, but here I read her giving an answer to this within the 'task' genre in mathematics education, and as such positioned with an addressive dominance. The normative claim identifiable in her didaktical voice can again be read as part of an instrumental view on teaching and learning mathematics, which could be seen as an element of her figured world as her belief of mathematics as a subject where you have to learn the rules, and where you have true or false answers. The teacher education in mathematics is arguing for a more relativistic and context sensitive view on mathematics and mathematics education.

As she is appropriating the mathematical genre/register she could be said to author herself towards an identity as teacher of mathematics to communicate subject matter. From the examples above she still has an oral, elementary school mathematics way of presenting her mathematics. She is describing processes and is only partly focusing the figured mathematical objects as such.

Summing up

The analysis of positioning, as it is inspired by Bakhtin (1986) and developed by Ongstad (1997, 2004), has here been seen with the concepts within Holland et al.'s (1998) concepts of figured worlds, agency and authoring selves. A set of triadic validity claims have been used as "tools" for validation by focusing the expressive, the referential, and the addressive aspects, separately in the first example, but trying to see them from the dominant aspect in the second. The three aspects will always simultaneously be present, but the dominance of the three will vary dynamically. The dominance of the utterance will also be related to the triadic dominance of the genre that the utterance is uttered within.

These analytic frameworks may reveal how complex different identities are uttered on the way to become an agentic subject as a teacher in mathematics. By showing how a student tries to communicate her mathematical tasks in the teacher education genre as "I", as position, it is possible to place how agency works on the way of becoming a mathematical teacher. This follows the recommendation to research in mathematics education given both by Eynde et al. (2002) and Lerman (2001, 2002), that is to see the three aspects as simultaneous processes, and the utterance and genre as inseparable.

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GEOMETRIC SERIES AND MATHEMATICAL MASTERY AND APPROPRIATION

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This paper discusses upper secondary students' mastery and appropriation of geometric series. Analyses of student dialogues in a collaborative problem-solving small group are conducted to identify what the students are doing together mathematically when working on and discussing mathematical problems. The analytical approach puts emphasis on the words the students use, how they use concept-related definitions and formulae, and the students' argumentation when discussing and applying the concept. The analyses show both mastery and appropriation of the concept of geometric series, although the students' use of ordinary and formula-related words and argumentation indicate some appropriation difficulties.

INTRODUCTION

Earlier conducted research to investigate upper secondary students' mastery and appropriation in mathematics seems to be rare. As part of a doctoral study this paper's aim is to contribute to the comprehension of how students in upper secondary school develop mastery and appropriation of geometric series when working with and discussing mathematical problems in small groups. Following Cobb (2002), analyses of two dialogues try to identify what this group of students are doing together mathematically. The following research question has been formulated:

Which aspects of mastery and appropriation of geometric series are identifiable in small-group discourses?

THEORETICAL FRAMEWORK

The study takes a sociocultural perspective on learning (Säljö, 2001), a perspective heavily built on the theories of Vygotsky (1978, 1986) and the outlines of Säljö (2001). In his book, Säljö is arguing for understanding human beings' learning in a communicative and sociohistoric perspective. Knowledge exists first of all through interaction between humans and secondly becomes part of the individual and its thinking and actions.

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The basic assumption in a sociocultural perspective on learning and development is that learning is viewed as resulting from social and interactional processes in which humans actively participate and contribute with ideas and arguments (Säljö, 2001). An important tenet is that how people learn and take part in knowledge is culturally dependent. Humans are cultural beings who interact with and think together with other humans in everyday activities using material resources.

A sociocultural perspective on learning and human thinking and action focuses on how people acquire knowledge in activities. Furthermore, focus is put on how individuals and groups take advantage of physical and cognitive resources. The interaction and communication between the individual and the collective is therefore fundamental (Säljö, 2001).

This study builds on the assumption that mathematical learning is not different from learning in general. Learning in a sociocultural perspective is very much related to the concept of internalisation (Vygotsky, 1978), understood as labelling the process of how human beings acquire meaning and knowledge in sociocultural practices. Vygotsky defines the concept of internalisation as 'the internal reconstruction of an external operation' (p. 56). This concept has been elaborated on by researchers within the Vygotskian tradition (Säljö, 2001; Wertsch, 1998). Wertsch distinguishes between internalisation as mastery and internalisation as appropriation. The term mastery is used as equivalent to knowing how in particular situations. Wertsch argues that internalisation, in the Vygotskian meaning of the term, often does not happen. Several mediated actions are carried out externally and are never executed on an internal plane. An illustrative mathematical example of the process of mastery is the problem of multiplying 523 by 768. To a lot of people the process of resolving this task is never completely internalised. In this case it seems, following Wertsch, more suitable to talk about mastering a cultural tool; the multiplication algorithm.

Internalisation as appropriation is related to the process of making something one's own (Wertsch, 1998). An appropriation of the above mentioned mathematical task includes knowing about the underlying operations involved, knowing about properties of the number system and so on. From this description Wertsch's and Säljö's (2001) understanding of the term appropriation is coinciding. Säljö uses the notion appropriation as a kind of substitute to the Vygotskian internalisation, claiming that the concept's connotations potentially recreate the difference between the external, as communication between humans, and the internal, our thinking. This idea is incompatible with a sociocultural perspective. This is why Säljö introduces the term appropriation to denote the process of how people learn to handle artifacts without being deeply influenced.

Nevertheless, Wells (1999) emphasises the importance of focusing on practices and not terms, and he analyses the activities in which knowledge is involved. Following Wells, emphasis is put on analysing student small-group discourses, an activity in which mastery and appropriation are involved. The important thing becomes to put emphasis on what the students comprehend and how they undersstand each other in the context. This is related to Halliday's (1992) term context of situation. The students' problem solving in a small group constitutes a particular situation that guides a particular way of talking mathematically. Furthermore, Halliday's term context of culture relates to the way of talking and reasoning in an institutionalised school context.

According to Lerman (2001), "Learning mathematics or learning to think mathematically is learning to speak mathematically" (p. 107). Lerman concludes that language and discursive practices have to be main research topics within investigations of mathematical learning and teaching. This is because meanings precede us and language and related practices constitute us in the contexts in which we participate.

Mastery and appropriation versus conceptual understanding

Sierpinska (1994) claims that students develop understanding when discussing and verbalising their understandings, when these are confronted with other students' understandings, and when they have to engage in justifications and validations that tell whether or not their understanding makes sense. The analytical focus here on mastery and appropriation (Wertsch, 1998) seems to be adequate according to Sierpinska's description of development of understanding.

Relevant research

Research that emphasises the importance of discourse in coping with students' mathematical meaning making and learning, is extensive. Zack and Graves (2001) focus on making mathematical meaning through dialogue. These researchers examined the problem-solving interaction among three boys that were heterogeneous in ability. Zack and Graves report that differences among the students served to affect learning and triggered elaborated comprehension of the mathematical content. Furthermore, this led to more articulated explanations and clearer understanding of what they had achieved, not achieved, and had to achieve to figure out the mathematical problem.

Partnered problem solving and mathematical discourse is the issue of Kieran (2001). She found that joint creation of mathematical meaning was problematic to accomplish, and students' utterances were mostly procedural in nature lacking explanation and arguments. Kieran concluded that to make one's emergent thinking available to one's peer such that the interaction is mathematically productive for both is a large challenge to learners.

Cobb (2002) claims that conceptual discourses and use of tools and inscriptions are resources for mathematical learning. Inscriptions are exemplified in his study as written notations and graphical displays, but also include for instance drawings, texts, diagrams, and mathematical symbolisations. These inscriptions make thinking public or visible. Hence, the inscriptions can be subjected to discussions, elaborations, and argumentations. In these ways inscriptions mediate human communication and thinking, and they exemplify how learning and symbolic tools are inseparable.

Earlier conducted research on learning of geometric series seems to be rare. Alcock and Simpson (2004) conducted semi-structured task-based interviews with first year university students about convergence of sequences and series. All students in the study had performed tremendously well in previous mathematical courses. These researchers found that some of the students expressed good understanding of definitions and classifications of different series according to convergence and boundedness. Furthermore, the students were able to include these definitions in argumentation. Other students were neither able to understand mathematical definitions nor use them in arguments. These findings coincide with the research of Eade (2003). He claims that in spite of being mathematically well qualified, students have difficulties with elementary concepts such as convergence of sequences. Eade's research focused on secondary trainee teachers' conceptual understanding of convergence and continuity conducted through task-based interviews with seven students. Eade found that it was problematic to decide whether the students' understanding difficulties were about semantics or mathematics.

Mathematical problems and problem solving

It is the person who faces a mathematical exercise that has to decide whether it is a problem or a routine task. A mathematical problem is here understood in accordance with Schoenfeld's (1993) definition:

> For any student, a mathematical problem is a task (a) in which the student is interested and engaged and for which he wishes to obtain a resolution, and (b) for which the student does not have a readily accessible mathematical means by which to achieve that resolution. (p. 71)

The origin of modern mathematical problem solving usually is traced back to Polya (1945/1957), and his four step model of problem solving is widely known within the field of mathematics education. The process of problem solving has been used to mean both "working rote exercises" and "doing mathematics as a professional" (Schoenfeld, 1992, p. 334). Schoenfeld maintains real problem solving as the core of mathematics, if not mathematics itself.

Through analyses of Swedish curricula, Wyndhamn (1998) reports on what problem solving might be. He phrases three trends, teaching and learning *for, about*[1], and *via* problem solving. In this study it seems reasonable to use the phrases teaching and learning *for* and *via* problem solving as describing what the students do. The small-group problem solving focuses on application of recently learned concepts

[1] Teaching and learning about problem solving emphasised teaching of heuristics, strategies and sequences of operations.

and skills (the *for*), but at the same time this work is seen as a means for learning mathematics as such (the *via*).

METHODOLOGY

This study has been done within a naturalistic research paradigm (Lincoln & Guba, 1985). The research paradigm is derived from social sciences that traditionally have studied cultures, communities, and social interactions. Lincoln and Guba put it this way:

> N (the naturalist) elects to carry out research in the natural setting or context of the entity for which study is proposed because naturalistic ontology suggests that realities are wholes that cannot be understood in isolation from their contexts… (p. 39)

A case study design that draws on the work of Stake (1995) and Bassey (1999) has been employed within a qualitative research strategy (Bryman, 2001). A case study concerns an intensive and detailed analysis of a single case, and it deals with the particular nature and complexity of the case in question (Stake, 1995).

The data collection has been done by an ethnographic approach (Tedlock, 2000). Overt passive observation together with videotaping collaborative problem solving in small groups in separate rooms with blackboards has been conducted. Thus, the label *ethnographic* is chosen as a metaphor, as a way of describing the data collection process.

A dialogical approach (Linell, 1998) to the data analysis following Cestari (1997) and Bjuland (2002) has been used. Markovà (1990) defines a dialogue as

> interaction, in temporal and spatial immediacy, between two or more participants who face each other and who are intentionally conscious of, and orientated towards, each other in an act of communication. (Markovà, 1990, p. 6)

Particularly, the analyses of the dialogues put emphasis on mastery and appropriation through four aspects, the words the students use, how they use concept-related definitions and formulae, the students' argumentation when discussing and explaining, and how they are able to apply the concept.

Related to the study's research question relevant and interesting parts of the third videotaped group-lesson are transcribed[2] in detail to do in-depth analyses of the dialogues.

[2] The following transcription codes have been used: (.) Small break, (…) Longer break, = continued utterances, [ ] overlapping utterances, (( )) Non-verbal activity /comments, ((:-D)) Laughter in voice, – sudden break, : prolonged sound or letter, ( ) inaudible fragments, (guess) best guess, . end of sound, Under emphasised words, CAPS loud utterance, low utterance, >< faster or slower talk respectively.

Context

A 3MX class has been observed in a Norwegian upper secondary school. 3MX is a mathematics course that is meant to prepare for university studies in mathematics and thus the most theoretical and demanding mathematics course in upper secondary school. Data collection has been done in a small group consisting of five students (here named Aud, Eli, Pia, Jan and Pål)[3], that is the group members have been picked with their previous marks as a basis[4]. As a researcher I followed the class for every lesson in a period of 12 weeks, both lectures and the students' small-group work in separate rooms, which was an ordinary part of the mathematics teaching.

PRESENTATION OF DATA

In the passages presented in the following the students work with an exercise connected to the concept of geometric series. In the dialogues below the students deal with the exercise 1.31 (Erstad, Heir, Bjørnsgård, Borgan, Pålsgård, & Skrede, 2002)[5], having a formula for the sum of a geometric series at their disposal:

$$S_n = \frac{a_1 \cdot \left(k^n - 1\right)}{k - 1} \qquad (*)$$

It is anticipated that the students will use the formula for the sum of a geometric series in this particular exercise[6]. To use the formula one has to know or evaluate the three parameters involved, a_1, n, and k. In exercises **a** and **b** the three parameters are known, at least implicitly, but in **c** (1 + 3 + 9 + 27 + … + 729) the context slightly changes. The number of terms, the n, is no longer known, the students have to evaluate what the parameter n is themselves, and that seems to be a problematic challenge. The context is new to the students since they neither during class lectures nor in small-group problem-solving sessions, have faced an exercise where the n is not expressed in the exercise text. From the following excerpt the described problem of the unknown n seems to be highlighted.

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[3] Pål does not contribute to the discussions in the transcripts below.
[4] The students with the best marks in 2MX, the forerunner of 3MX, have by the teacher been organised in one group.
[5] See Appendix 1
[6] The students have also been introduced to the formula $a_n = a_1 \cdot k^{n-1}$ for the nth term in a geometric series.

Problems of deciding the value of n

390 Jan: On c?
391 Aud: Yes
392 Jan: Yes, why isn't k equal
393 Eli: k equals one times x, no
394 Aud: It is three times
395 Jan: Three divided by one, nine divided by three=
396 Aud: Yes
397 Jan: =twenty seven divided by nine
398 Aud: And then=
399 Jan: All become three
400 Aud: =Seven hundred and twenty nine divided by three, it is two hundred and forty three. Does n equal two hundred and forty three?
401 Eli: Yes
402 Aud: <u>BUT</u> it is so big, the result

Jan starts off this discussion by asking a clarifying question of which exercise they are going to work with (390), and he gets his assumption confirmed by Aud (391). Jan continues asking another question which is incomplete (392). In that sense it seems as if Jan externalises his thinking by expressing it out loud. It is also possible to interpret Eli's verbalisation (393) as an interruption of Jan's train of thought. What Eli means by her utterance seems not to be clear. Probably it is a start of some kind of argumentation, but she cuts herself off. By introducing the *x* it seems as if she is thinking about making an equation to find the *k*, for instance $a_1 \cdot x = a_2$, where a_1 equals 1 and a_2 equals 3, and use the more familiar letter *x* for the unknown ratio between two following terms. Aud responds (394) by correcting Eli and claims that it is three (and not one) times something. The dialogue does not answer what this something is or means. It might be *x*, as in Eli's argument or something else.

However, Jan, in (395), (397) and (399), gives an argument that indirectly tells the other students what *k* is in this exercise. The argument includes the procedure of how to evaluate the ratio *k* in a geometric series, calculating the ratio between the second and the first term, between the third and the second and so on. Aud (396) obviously follows Jan's reasoning when confirming his way of approaching the exercise. She adopts Jan's strategy and continues the group's problem-solving process by using the recently calculated *k* in her argument about evaluating the unknown *n* (400). She divides 729 by 3 (the *k*) and gets 243, and she asks the group whether *n* then equals 243. This utterance suggests that Aud is mixing up the *k* and the *n*. However, it seems to be a genuine question from Aud's point of view and the words used do not indicate a strong assumption that *n* is likely to be 243. Since this, in addition, is incorrect, Aud's suggestion seems to be somewhat a guess at random. That Eli (401) supports and agrees on the suggested incorrect value of *n*, confirms that the group, at least Aud and Eli, is insecure of how to evaluate *n* or indeed what *n* means in geometric series. There seems to be a lack of appropriation according to the meaning of *n*.

Nevertheless, Aud's verbalisation in (402) indicates some reasoning about the suggested value of *n*. She claims that the value is so big. The word *result* has at least two different meanings. It may indicate, as mentioned, the value of *n*, but it may as well be interpreted as it is the total sum that becomes so big. Anyhow, Aud's utterance seems to have an institutionalised feature. She reasons upon the answer they get and rejects it as an unlikely answer to the exercise. The whole problem-solving process is situated in an educational context; they are doing problem solving in school mathematics. The answer Aud gets seems not to be reasonable within this setting. It is too big. A possible reason for this is that an *n* equal to 243 gives overflow in the calculator, and the students have neither in lectures nor in small-group work faced such a big *n*. The word *but* supports this analysis. It is both pronounced louder than the other words and it is emphasised, indicating a reasoning process going on. The result is too awkward. Subsequently, the students also compare their solution with the answer in the back of the mathematics textbook and find that their solution is not consistent with the textbook's answer to the task. The institutionalised context tells the students that there is a solution to the task and the solution should be somewhat reasonable, but their approach does not fit with these requirements.

In the dialogue the discussion starts off (392) with how to evaluate the ratio *k* between the terms. Jan concludes (399) that *k* has to be three, and Aud is using that information in her utterance (400) to evaluate the number *n* of terms although her approach is unsuccessful. In this sense it seems as if the students have grasped the meaning of *k* but not the meaning of *n* in relation to the concept of geometric series.

Later on in the problem-solving process Jan comes up with an argument and explains why the number *n* of terms is seven.

Explanation of why n equals seven

590 Jan: Therefore we have that a_1 equals one, because there is (nothing left), right?=

591 Pia: Yes

592 Jan: =And thus it is like(.) And here we then have, here we have here here here we ((points at the blackboard[7])) have e:: how many is it? It is one times *k* e:: ((:-D)) *k n* minus one. But what we have figured out is *n* minus one. We have not found *n*. We have found *n* minus one, that is six. That was what we calculated just now, right?=

593 Pia: Yes

594 Jan: =Therefore we have then that e:: that number, that is *n* minus one equals six, and then we just take-move over and then add one on each side, then we get *n*.=

595 Pia: Oh yes

596 Jan: =Therefore n becomes seven, if you know what I mean? ( )

[7] See Appendix 2 for what was written on the blackboard

Basically it is Jan who does all the talking in this dialogue, although Pia contributes with short comments. Jan's argument here seems to be influenced by reasoning based on mastery and appropriation of geometric series and the available general expression for a_n. The whole dialogue is one coherent argument where Jan externalises his thinking of how to evaluate the n, to Pia and most likely to the whole group. Analysing the argumentation or explanation, it seems to include mastery of how to resolve a mathematical exercise about geometric series as well as appropriation of the different constituent parts of the concept. Both in (590) and in (592) Jan asks whether Pia understands his explanation by asking *right?* This might be interpreted as just a dialogical element usual in oral explanations, but it also might be interpreted as if Jan is really concerned about Pia's comprehension of his elucidation. Pia both times (591, 593) confirms that she agrees on the argument. Whether Pia understands the explanation seems impossible to decide from her short comments.

When Jan gives his explanation he has written some mathematical symbols at the blackboard; a representation of the externalisation of his thinking to support his explanation. He has written the task (1.31c) there, and his whole argument is based on the formula for a_n, that $a_n = a_1 \cdot k^{n-1}$. In (590) Jan says that a_1 equals 1. The reason for this is given by the expression *nothing left* that refers to the exponent zero, n equals one and one minus one is zero (See Appendix 2, left coloumn). In (592) he moves on and talks about the number 729. When he expresses the word *here*, he points at this number and writes $1 \cdot k^{n-1}$ below it. Thus, in the utterance Jan reasons upon this inscription. Earlier on the group has solved the equation $3^x = 729$ for x and got $x = 6$. What seems to be the mathematical content of Jan's utterance is that they basically have evaluated the equation $1 \cdot k^{n-1} = 729$, filled in for a_1 and a_n, and substituted the exponent $n-1$ with x. Despite of Jan's inaccurate mathematical terminology, Pia seems to follow his argument. Jan says *k n minus one*, but what he means is *k to the power n minus one*. This is what he has written at the blackboard. The inscriptions, that is the written mathematics, are accurate although Jan's oral argumentation is somewhat blurred.

Jan emphasises the substitution they have done in their calculation, and to get the value for n, they have to substitute the x back to $n-1$ (594) and add one on both sides of the equation. Jan's reasoning in this utterance seems to be typical. In one sentence he explains two procedures to get the n, which actually are identical. He says *move over and then add*, but it seems quite obvious that he means move over the (-1) on the left hand side and get (+1) on the right hand side or just add 1 on both sides (which is what is done in the first approach too).

Pia's utterance in (595) seems to indicate that she at this moment in the dialogue comprehend what Jan talks about. She seems to have appropriated both the conceptual and the procedural elements in Jan's explanation. Thus Jan seems to have mediated his mastery and appropriation of the concept of geometric series in a way such that Pia has appropriated it.

DISCUSSION

The exercise that is discussed in the excerpts is from a mathematical point of view a routine task and thus apparently not a problem according to Schoenfeld's (1993) definition. Nevertheless, the students experience difficulties in resolving it. This correlates to the discussion above about the term *problem*. It is the person that faces an exercise that decides whether it could be called a mathematical problem or not, and following the students' discussion this is a mathematical problem to them.

In the dialogues elements of mediation of mastery and appropriation are seen (cf. Säljö, 2001). The students are doing joint collaborative problem solving through which the mediation certainly appears. In both excerpts the students comment on each others' contributions and ask for clarification and explanation when needed. In the second dialogue mastery and appropriation is highlighted (Säljö, 2001; Wertsch, 1998). From the arguments given by Jan, the assumption is made that he has understood the constitutive parts of the concept of geometric series. In both dialogues the students take advantage of both daily and mathematical language including formulae to mediate and communicate their thinking.

In a sociocultural perspective communication is important to analyse (Säljö, 2001). What students say and do reveal some indications of their thinking and appropriation. Jan's use of the blackboard in the second excerpt made communication easier, the words *here* in (592) and *that number* in (594) in addition to pointing exemplified that. Writing expressions on the blackboard (See Appendix 2) visualise and externalise Jan's thinking according to the evaluation of the parameter *n*. This relates to Cobb's (2002) assertion. These inscriptions mediate Jan's thinking. The students in the group seem to be highly coordinated in their thinking, and their discussions carry elements of a collective practice in a school context. The first dialogue exemplifies this point. This corresponds to the Hallidayian (1992) terms context of situation and culture. The culture context, mathematics in school, and the situation context, small-group problem solving, determine and frame the students' way of speaking mathematically (Lerman, 2001).

There seems to be hardly any hesitation or confusion according to what they were doing together mathematically. This supports the basic assumption in this educational study, that the students are rational and do what they think are the right things to do. They really want to resolve this exercise and they put quite a lot of effort into solving it.

Some findings in these analyses coincide with Cobb's (2002) claim that discussion and usage of inscriptions and tools are resources for mathematical learning. Genuine questions and calls for help are communicated in the first excerpt, which create opportunities for learning to occur. Explanations given, with use of both inscriptions and tools, seem to initiate mathematical learning and growing mastery and appropriation. These findings coincide with the research conducted by Zack and Graves (2001) as well. The dialogues among the students play a crucial part in their appropriated mathematical meaning and knowing, as seen from the conversation between Jan and Pia in the last dialogue.

The conclusions drawn from the analyses slightly differ from the findings of Kieran (2001). Some of the students' utterances are procedural as seen in the first dialogue, but other utterances are supported by arguments and explanations. This did not seem to be the case in Kieran's study.

Even though the problematic part of the present geometric series was not about convergence, it seems reasonable to claim that these students have elementary difficulties about the concept of geometric series. This relates to the findings of Alcock and Simpson (2004) and Eade (2003). The discussed exercise is a routine task and in that sense elementary, but the students still have difficulties in solving it.

CONCLUSIONS

From the above discussion it is possible to conclude that both mastery and appropriation of the concept of geometric series are communicated in the dialogues. Jan reveals mastery and appropriation of the concept. However, the students' use of ordinary and formula-related words and argumentation indicate that they have some difficulties in appropriating the concept completely. They seem to know what formula to apply, but are not able to apply it because of the missing parameters.

It follows from this study that it is important to focus not only on route learning of concept dependent formulae, but on contextual applications and combinations of these as well. Both mastery and appropriation of mathematical concepts need to be emphasised.

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APPENDIX 1

Exercise 1.31

Calculate the following sums:

a

$$500 \cdot 1{,}05 + 500 \cdot 1{,}05^2 + 500 \cdot 1{,}05^3 + \ldots + 500 \cdot 1{,}05^{10}$$

b

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$$

c

$$1 + 3 + 9 + 27 + \ldots + 729$$

d

$$1 - 3 + 9 - 27 + \ldots + 729$$

APPENDIX 2

This is what Jan wrote on the blackboard

1 + 3 + 9 + 27 . . .	+ 729
$1\ k^{(n\text{-}1)}$	$1 \cdot k^{(n\text{-}1)}$
1-1	$(n - 1) = 6$
$k^{(0)}$	$n = 6 + 1$
$1 \cdot 1$	$n = 7$

TEACHERS' REFLECTIONS ON THE USE OF ICT TOOLS IN MATHEMATICS: INSIGHTS FROM A PILOT STUDY

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This paper is based on data from a case study at a secondary school in Norway. The aim of the paper is to present an analysis of the nature and content of two teachers' reflections on the use of ICT tools in mathematics at grade 9. A central method in the study is the use of focus group interviews to gain insights into teachers' perspectives. Here we see two teachers reflecting on use of ICT in their mathematics teaching recalled by a video recording of three students' work in their lesson. A table is presented with questions addressing the reflections from the teachers. The table indicates teachers' ability to address a variety of questions, both linked to the concrete activity recalled by the video and on a more general level concerning ICT teaching in mathematics.

BACKGROUND TO THE PAPER

This paper reports from a pilot study to PhD student Ingvald Erfjord's main research study and is linked to the KUL-ICTML[1] research project managed by didacticians[2] at Agder University College (AUC). The ICTML-project emphasises students' learning, understanding, and development of knowledge in mathematics using ICT; teachers' competence in using ICT in their teaching; and how use of ICT influences their teaching.

The interviewed teachers in this pilot study work at one of eight schools that participate in the KUL-ICTML and/or the KUL-LCM[3] research project for a period

[1] The KUL-ICTML research project (NFR no. 161955/S20) is supported by the Norwegian research council as part of the KUL programme and managed by didacticians at Agder University College. KUL is a shortening for "Kunnskap, Utdanning og Læring" (Knowledge, Education, and Learning) and ICTML for ICT and Mathematics Learning.

[2] Didacticians are people who have responsibility for theorising learning and teaching and considering relationships between theory and practice. In the project we refer to the university educators and Ph.D. students as 'didacticians' in order to recognise that both teachers and didacticians are educators and both can engage in research.

[3] The KUL-LCM research project (NFR no. 157949/S20) is supported by the Norwegian research council as part of the KUL programme and managed by didacticians at Agder University College. LCM is a shortening for Learning Communities in Mathematics.

2007. In C. Bergsten, B. Grevholm, H. S. Måsøval, & F. Rønning (Eds.),
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of 3 years. These two research projects are managed in co-operation and share similar theoretical grounding and goals. The projects emphasise the development of communities of inquiry among teachers in schools and didacticians at AUC. The teachers and didacticians take part in workshops at AUC that emphasise inquiry and collaboration processes. Didacticians take part in teachers' planning of lessons, observe how lessons are carried out and take part in reflection meetings concerning some of the teachers' lessons. Data are collected during this process.

Research questions

In the main study of Ingvald Erfjord, four research questions have been raised which relate to teachers' perspectives on integrating ICT in their mathematics classrooms. The different questions are intended to give insights into how teachers use ICT in mathematics, their reflections concerning students' use of ICT, factors influencing implementation of ICT in classrooms, and how the use of ICT has an impact on teaching and teachers in mathematics.

1. How do teachers use ICT tools in mathematics classrooms at grade 8–10 in some Norwegian schools?
2. What is the nature and content of teachers' reflections to students' use of ICT tools in mathematics at grade 8–10?
3. Which main factors influence teachers' use of ICT tools in mathematics classrooms?
4. How do ICT tools (and other tools) impact mathematics teaching and teachers?

In this pilot study, the main aim is to study the teachers' reflections in relation to their use of ICT in mathematics in order to understand how teachers themselves make meaning of what they do. The research question in this paper is closely linked to question 2 above and is formulated like this: *What are the teachers emphasising when they study a video recorded segment from their lesson showing students' use of ICT tools on a mathematical task, and what are their further reflections in regard to use of ICT in mathematics teaching?* Our analysis in this paper does not aim to explicitly connect teachers' reflections and observed activities to activities in the ICTML and/or LCM project of which the teachers are part. The overall aim of this pilot study is to get experience with methodology and to generate research experience for the main study. Based on findings from this pilot study, we present a list of 16 questions which address two teachers' reflections. These questions give a first impression of teacher reflections in this area, which will be one of the main areas of focus in the main study of Ingvald Erfjord.

THEORETICAL CONTEXT

Teachers' use of ICT tools in mathematics

Use of, for example, fingers, sand, paper, ruler, compass, abacus, calculator or computer in mathematics are all examples of "things" that can be helpful in understanding, teaching and learning of mathematics. We[4] will use the term tool when we refer to such aids; in particular, in this paper we speak of ICT tools which deal with the range of hardware (desktop and portable computers) and software applications available at the school at the time of the research.

The national curriculum for primary and secondary school in Norway (Hagness & Veiteberg, 1999) focuses on integrated use of ICT in mathematics. ICT is not a separate subject in the curriculum; instead ICT is supposed to be used as a suitable tool in other subjects. From the spring 2005, all students have a right to choose whether they want to use ICT in part of the final secondary school examination in mathematics at grade 10. ICT is also planned to be a compulsory part of the national tests in mathematics. A new national curriculum[5] in Norway, that will be carried out in primary, secondary, and upper secondary schools from autumn 2006, seems to maintain a great emphasis on use of ICT. Nevertheless, there are still schools in Norway with teachers not engaging in use of ICT tools in mathematics, schools with quite limited and old, unreliable ICT equipment and access only to a few mathematical ICT tools. At secondary school the main, and often the only mathematical tools is a spreadsheet tool, for instance Microsoft Excel. Yet, according to Kløvstad and Kristiansen (2004) most principals in Norwegian schools emphasise priority on ICT use in schools. Kløvstad and Kristiansen also claim that access to ICT gradually has increased in Norway over the last years and, as an example, there were on average seven students per computer in grade 9 in 2003.

However, how ICT equipment is used by students and teachers in classrooms and reasons why it is used in these ways is not yet properly known and provides vital questions to focus on. Dörfler (1993) focuses on how ICT tools influence and actually change the mathematical content:

> In general, substituting one tool for another one causes a change of the objects to be worked with and upon. Therefore not only the structure and form of the activity but its content are changed by introducing new tools. (Dörfler, 1993, p. 163)

[4] The paper has been written in co-operation but is part of PhD student Ingvald Erfjord's pilot study.

[5] Based on a new school reform named 'Kunnskapsløftet', URL http://www.kunnskapsloeftet.no/, new national curriculum guidelines are developed. Some parts are finalised and others still not finalised. More information on the webpage http://skolenettet.no/laereplaner/login_lp.aspx?id=10723&scope=ScopeLaerAns&epslanguage=NO

In our opinion, teachers' implementations of ICT in mathematics classrooms seem to be of vital importance. ICT changes the mathematical content but are teachers aware of it and with which consequences? And ICT tools do not only change the mathematical content, also the pedagogical approaches are changed according to Goos, Galbraith, Renshaw, and Geiger (2003):

> The introduction of technology resources into mathematics classrooms promises to create opportunities for enhancing students' learning through active engagement with mathematical ideas; however, little consideration has been given to the pedagogical implication of technology as a mediator for mathematics learning. (p. 73)

In a survey on research literature concerning use of ICT in mathematics education, Lagrange, Artigue, Laborde, and Trouche (2001) claim that many studies have been concerned with students' mathematical learning mediated by ICT, but less with focus on teachers' roles. Teachers are often regarded as an obstacle; actually the main obstacle for development of ICT based teaching and learning:

> Classroom teachers are simply using the technology to do what they have always done, although in fact they often claim to have changed their practice. (Hennessy, Ruthven, & Brindley, 2005, pp. 156–157)

This contrast between theory and practice, between what teachers wish to do and perhaps even believe they do, and what they actually do and why they do it, is an important aspect of our paper, too, in relation to the teachers' implementation of ICT in mathematics teaching.

The contribution by Hennessy et al. (2005) focuses on teachers' perspectives on integrating ICT into teaching of subjects like mathematics and science. This is done by studying how teachers implement use of ICT in their teaching of mathematics, how the use of ICT changes the tasks, activities and teaching, and especially on what motivates and constrains teachers' use of ICT. Through interviews they focus on the nature of teachers' reflections with regard to their implementation of ICT and concerns regarding ICT and mathematics in classrooms. Their findings indicate few changes in teachers' practice and an apprehension that teachers emphasise that use of ICT will weaken students "basis knowledge" in mathematics. However, they also report that many teachers are aware of and do reflect on new forms of activity, resources, and strategies for mediating ICT-supported subject learning in their classrooms.

A socio-cultural basis for research

The main purpose of this paper is to present the nature and content of two teachers' reflections on their use of ICT tools in mathematics at grade 9. This will be addressed by a similar focus on teachers as Brown and McIntyre (1993) have:

> …any understanding of teaching will be severely limited unless it incorporates an understanding of how teachers make sense of what they do: …, and why, in their own understanding, they choose to act in particular ways in specific circumstances to achieve their successes. (Brown & McIntyre, 1993, p. 1)

The teachers in this research are studied and their actions analysed based on a socio-cultural view of learning, understanding, and knowledge and its connections to teaching. The socio-cultural theories explain cognition in interpersonal and contextual dimensions with great emphasis on the influential role played by culture on mind and thinking, as it is presented by Wood (1988): "interaction between children and more mature members of their culture" (Wood, 1988, p. 40). Vygotsky is a very central researcher in the early development of the socio-cultural theories. The main idea is that knowledge develops in social settings because human thought processes are socially linked. In the main study there will be more focus on how the school culture and its members (teachers, students, parents, school leaders, and government) and products (e.g. curriculum plans) influence teachers' practice and underlying thoughts, but there will be even greater emphasis on tools offered by the culture such as ICT tools. Bauersfeld (1995) offers an important contribution to classroom studies based on socio-cultural theories. He emphasises the importance of "the culture of the mathematics classroom" (p. 150), and describes that the "mathematical knowledge appear as social accomplishments of the specific culture" (p. 150). Consequently, a vital role of a didactician is to get to know the culture of both the classroom and the school, in order to understand a teacher's practice and her/his reflections and concerns.

Lave and Wenger (1991) argue that our thought is located in the functional system of which we are part. The notion 'functional system' is used almost synonymously with the word 'culture', but Lave and Wenger emphasise the surroundings including cultural tools, the activity, and its context in addition to other persons, for example students and teachers. ICT is in this setting interpreted as one such cultural tool besides for example language, symbols, diagrams, and more typical mathematical tools, for example ruler and compass. As a consequence they claim that we learn from the culture with its functional system and the members that are part of the culture. Therefore, if ICT is included as a cultural tool in the school culture, it can be an important mediator in teachers' work with mathematical tasks and for mathematical understanding. However, ICT is still quite a new tool and teachers' reluctance to change practice and include ICT as a cultural tool is understandable when seen in the light of Lave and Wenger's theory.

Thus, we locate our research in the area of use of ICT tools in mathematics, focusing on understanding of teachers' reflections and concerns with implementation of ICT in mathematics teaching. By using a perspective based on socio-cultural theory we set a further context of the research on understanding teachers by focusing on the surroundings including the cultural tools it offers, particularly ICT tools.

METHODOLOGY

This pilot study is designed as a case study using an ethnographic approach to the field. Our case is the teaching of two secondary mathematics teachers to a group of about 40 students at grade 9.

Data were collected[6] in the spring 2005 and consist of field notes and transcripts from lessons, meetings, and interviews. In order to increase our understanding of teachers' reflections, a focus group interview with the teachers together was conducted, with the hope that they would be able to sharpen each others' viewpoints. The outcome from the study of this case will be our interpretations of the teachers' reflections, where we aim to give important details relevant to the findings we present.

In this study we use a video recorded segment from one of their lessons as a resource in the focus group interview. In a number of lessons the teachers and students have used the calendar as a source for generalising patterns. The recording presented for the teachers shows three students working on this task using spreadsheet as a tool. Below, we have presented how the students have put numbers into a spreadsheet and marked the pattern on the screen as indicated below. When they generalise their pattern they name the largest number *e* (not the least number as in their earlier patterns). The students are supposed to present for the others their patterns (here a stair-pattern as indicated below) and the sum of their numbers. The other students are then supposed to be able to generalise the pattern and find their numbers:

$$e + (e-7) + (e-8) + (e-15) = 82 \Leftrightarrow 4e - 30 = 82 \Leftrightarrow e = 28.$$

	A	B	C	D	E	F	G
1	1	2	3	4	5	6	7
2	8	9	10	11	12	13	14
3	15	16	17	18	19	20	21
4	22	23	24	25	26	27	28

The three students try to generate the formula in a free spreadsheet cell below "the calendar" on the screen. The students are not using "spreadsheet-language" with cell references, neither for typing the numbers 1, 2, … , 28 (e.g. with the formula = A1+1 in A2, = A2+1 in A3 etc.) nor for generating the formula.

The selection of video segment for the focus group interview is worth attention. For what purposes are we using a particular segment? First of all we selected a segment that should be quite easy for the teachers to grasp. The students work together on a subtask on generalised arithmetic, from problem to a solution without teacher interruption. They are using ICT spreadsheet tool, which was a spontaneously choice by the students. Secondly, we wanted to focus on the students work and not the teachers' activity, partly since the latter may be uncomfortable for teachers. Further, this focus is appropriate since the aim of this paper is teachers' reflections on students work on ICT and more generally on ICT in mathematics learning.

[6] Data was gathered by the main author of this paper and another didactician at AUC, Espen Daland.

The approach to the interview is in many ways similar to the "Stimulated recall" approach used by for instance Jaworski (1994) and utilised in a similar setting as the case by Nardi, Jaworski, and Hegedus (2005). By referring to a previous lesson and letting the teachers think back and speak from experience, we hope to be able to get a better view of how teachers think and reflect on their own teaching and hope that teachers experience it as a sensible activity:

> This process of identifying significant, specific incidents of practice, and then engaging the participants in reflective discussion, as opposed to discussing practice at a general, dis-embedded level, has been identified…. as a "useful exercise". (Nardi et al., 2005, p. 290)

When we took part in classroom visits, preparation- and reflection meetings our main method was participant observation. The meetings give insights into what the teachers beforehand are focusing on, judgement they make, and what they afterwards see and reflect on.

We are well aware that we are not able to find out exactly how the teachers think and reflect and how it influences their further teaching. Each teacher's thinking is hidden in their brain but is in addition influenced by the setting, context, and the culture of which the teacher is part. With this in mind, we still believe that we will be able to indicate findings that will contribute to the aim of the research. We are also aware that, partly we are studying a developmental process as part of the ICTML research project. The teachers in our case, together with other teachers and didacticians including ourselves, take part in series of workshops at AUC, and get school visits from didacticians who also take part in meetings at schools with teachers. This gives us, as didacticians, a possible source to track changes and development from consecutive workshops, lessons, meetings, and interviews in addition to experience and reflections made by the teachers through each case in which they take part.

ANALYSIS

Our main approach to analysis of data is inductive and based on socio-cultural theory. We focus on what the teachers emphasise when they study a video recorded segment from their lesson and on their further reflections in regard to use of ICT in mathematics teaching. Based on the analysis of collected data[7] from lessons, meetings and an interview with two teachers, we have identified 16 questions that can be connected to teachers' reflections.

As a first dimension, teachers' comments and reflections can be classified in three broad areas according to what they address:

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[7] All the dialogues used in this paper are translated from Norwegian into English. T1 and T2 refer to the teachers in the focus group and D to the didactician.

- Mathematical tasks and use of ICT
- Group processes and commitment
- Organising ICT teaching in mathematics

Further, as a second dimension, teachers' reflections can be divided into two main levels in order of type of questions they deal with. Reflections on the first level are tightly linked to their lesson recalled by the video recording while reflection on the second level are on a more general level indicating pedagogical implications of how to use ICT in mathematics teaching:

- Questions linked to students mathematical activity on the task supported by ICT
- Questions in regard to teaching and learning of mathematics supported by ICT

Sorted into these two dimensions we have generated Table 1 indicating a classification of the 16 questions we identify, as either implicitly or explicitly addressed by the teachers. In this paper we restrict our analysis to teachers' reflections in regard to questions 1, 2, 5, and 7, which are all linked to the area "Mathematical tasks and use of ICT", covered in the first row of Table 1 below.

	Questions linked to students' mathematical activity on the task supported by ICT	**Questions in regard to teaching and learning of mathematics supported by ICT**
Mathematical tasks and use of ICT	1. Is the task suitable for use of ICT tool? 2. How do the students utilise ICT in their mathematical activity? 3. How do the students master the ICT tool? 4. What are the advantages of using ICT in this task?	5. What kinds of tasks/problems are suitable for use of different ICT tools? 6. Which competences in use of ICT tools do students need in order to utilise ICT in mathematics? 7. What are the advantages of using ICT in mathematics? 8. What kind of mathematical learning can take place when sitting by a computer screen?
Group processes and commitment	9. To what extent are the students participating and collaborating? 10. Do the students discuss problems? 11. Are there indications of motivation and engagement?	12. What kind(s) of learning may occur in groups working in this environment? 13. What is the teacher's role in this learning environment?

	Questions linked to students' mathematical activity on the task supported by ICT	**Questions in regard to teaching and learning of mathematics supported by ICT**
Organising ICT teaching in mathematics		14. Do we have access to proper rooms and equipment? 15. How are technical problems influencing? 16. How can use of ICT be organised in order to be a suitable tool in the teaching and learning of mathematics?

Table 1: Questions implicitly or explicitly addressed by the teachers

Is the task suitable for use of ICT tool? (Question 1 in Table 1)

Our data from the focus group interview indicate that this question is raised but not really discussed. Below, two separate sequences from the focus group interview are presented:

42 T2: Is it arranged for use of ICT tools?....
70 T2: But they did not use the spreadsheet to do what a spreadsheet is suitable for. They are not utilising (*pause*). I do not know if the task, really (*pause*). They use it as a blackboard as you say

In these sequences two main aspects in connection to the potential of the ICT tool are considered:

1. The students are not utilising the ICT tool in this task
2. The teacher consider whether this could have been a good task for utilising an ICT tool

The teacher's comment in paragraph 70, "I do not know if the task, really", followed by a short pause before a slightly different aspect is discussed, perhaps indicates that this question has not been critically considered by the teacher and they do not follow up this matter later in the interview. Paragraph 70 points to an important question to consider carefully in didacticians' contact with teachers in the main study: What kind of tools do students spontaneously use when facing a mathematical problem, and why do they choose this/these particular tool(s)?

What kinds of tasks/problems are suitable for use of different ICT tools? (Question 5)

This paper emphasises critical reflection offered by teachers and a continuation of question 1 is question 5 which was explicitly presented by one of the teachers in the interview (paragraph 153). However, none of the teachers give answers to the question and in that case why. In the sequence below, the teachers count on the project in this respect:

153	T2:	What kinds of tasks (*pause*, 3s) problems are suitable for use of different ICT tools? (*pause*, 5s). I believe that for the teachers who take part in ICT, these projects acquire some ideas. There is spreading of ideas going on so you do not have to be on your own
154	T1:	It is really just to get it started. It is a matter of getting students interested in this.

T2 formulates the question in paragraph 153 but T1 is not contributing because he is focusing on another aspect more tightly linked to organising and advantage of using ICT in mathematics. However, reflections covered by this question seems very important both with respect to teachers' own developmental process as part of their planning and carrying out of lessons and with respect to students' abilities to use tools reflectively. What kind of answers can we possibly reach? According to Dörfler (1993) the mathematical content is changed by introducing new tools such as ICT. This indicates that tasks, and perhaps the whole mathematical curriculum, have to be changed, at least new ways to approach tasks in mathematics.

How do the students utilise ICT in their mathematical activity? (Question 2)

This question is considered by the teachers in at least two broad ways: The first focus is on their students' lack of "using spreadsheet as a spreadsheet", more or less using it only as a typewriter. In the reflection meeting after the lesson, taking place before the interview, this issue is raised during the conversation, in which one of the teachers and a didactician take part:

5	D:	And when I saw that they (*the three students*) had drawn up (*interrupted*)
6	T1:	[Mm]
7	D:	the month-calendar in the spreadsheet
8	T1:	[Mm]
9	D:	And there are many elements that you can (*pause, 2s*). Because they did not use the spreadsheet (*interrupted*)
10	T1:	They used it as a typewriter

The sequence above presents some of the teacher's conception of his students' use of spreadsheet tool. The teacher introduces the word "typewriter" for the first time in the conversation, indicating awareness of his students' inability to utilise the possibilities of the spreadsheet tool in this task. The dialogue continues with a discussion on how the spreadsheet might be utilised in this task:

11	D:	Yes, because what they could have done, if you take A1 added that added that added that, then you could have made the formulas
12	T1:	[Mm]
13	D:	Then they could just have typed it and got it out.
14	T1:	Get it for free.
15	D:	Mm. So I sat there thinking (*interrupted*)
16	T1:	These are things that I hope I can do when I get them in and use it then (*The teacher refers to students handing in portfolios and his possibility to give them feedback on their ICT use in those*)
17	D:	Now, if you ask them whether it can be used in the spreadsheet
18	T1:	Exactly
19	D:	Yes they work with it and they can see how they possibly can use the variables (*interrupted*)
20	T1:	[Mm]
21	D:	The algebra in the spreadsheet (*interrupted*)
22	T1:	[Mm]
23	D:	They did not think of this, these students
24	T1:	No

These sequences are dominated by the didactician. The teacher mostly responds by short words indicating agreement and possibly uncertainty to the theme discussed. However, this conversation is a forum for exchange of ideas among teacher and didactician and may indicate that the teacher finds it satisfying to be given some ideas and the didactician experience expectations of giving feedback to the teacher as part of the developmental project?

In paragraph 13 the didactician points to potential benefits of using cell references and generating formulas in a spreadsheet as a helpful tool in the process of generalising patterns. The teacher's response in paragraph 14: "Get it for free" may indicate his awareness of this learning potential, but as the conversation goes on without any further comments from the teacher in this respect, his response may also reflect on the efficiency effect of using the spreadsheet with cell references in this task?

A second focus noticeable in our data in respect to how the students utilise ICT in their mathematical activity, is linked to their use of the ICT tool in connection with other tools. In the focus group interview this matter is raised in the sequence below:

91 T2: By the very fact that they are tied to one PC, none of them are writing anything beside. This they could have done. They could definitely have made a rough draft beside.

The teacher reflects on his students' lack of using different tools in combination, for instance paper and pencil in combination with ICT tool. These thoughts open reflections to the more general question: Which competence in use of ICT tools do students need in order to utilise ICT in mathematics? (Question 6)

What are the advantages of using ICT in mathematics? (Question 7)

Our data indicate that teachers primarily focus on two aspects with respect to this question: 1) ICT as an efficiency tool and 2) Its possible power to increase students' motivation to engage in mathematical activities. The paragraphs below present these focus:

128 T2: However, what is it suitable for? Within which areas may the use of ICT increase the interest? And in a way, for example use ICT as a time-saving tool instead of, as I have talked about x number of times, using more time. And also when we have…, yes then. Because then we really can learn and we have the testing out really quick instead of or in addition to throwing dice or whatever it is.

129 T1: It is not only that it is quicker. Quite a lot of people think that it is more interesting to work with ICT, too. When you work there you do not only get it better, but perhaps also feel that it is more fun to work when you have ICT.

130 T2: Talking of myself, I am starting from scratch in regard to how I can use this in my teaching

In some other sequences the teachers also focus on students' learning, not directly in connection with using ICT but more with respect to the work in the group sitting in front of the computer screen. In paragraph 130 T2 reveals his lack of confidence in his own ability and experience of using ICT in mathematics. According to Lave and Wenger (1991), a teacher's working functional system often obstructs a new cultural tool such as ICT. The teacher experiences ICT as something extra "out there" which is not part of his regular mathematical tools in the lessons.

Above we have presented an analysis which shows reflections made by teachers when they consider using ICT tools in their teaching in mathematics, when they plan their use of ICT, and when they carry out their teaching using ICT tools. This big issue will be further elaborated and extended in the main study, in which several teachers at a number of schools will be part of the data collection.

CONCLUSION

This pilot study has identified two teachers' reflections linked to a group of their students working with the ICT spreadsheet tool in a mathematics lesson. Teachers' reflections address questions that we divide into two levels: 1) Questions linked to students' mathematical activity on the task supported by ICT, and 2) Questions in regard to teaching and learning of mathematics supported by ICT**.** We present a table which relates teachers' comments on 16 questions which could be relevant for any teacher teaching mathematics. Teachers' reflections on 4 of these questions are analysed in this paper indicating their awareness of ways in which ICT spreadsheet tools can contribute to mathematics learning. The teachers are able to formulate critical questions about their teaching in mathematics and especially about the use of ICT as a tool, but they are to a small extent able to give any answers, thus indicating that ICT is not part of their working functional system (Lave & Wenger, 1991). As the main study progresses we hope to focus more deeply on teachers' ability to address these and other, related questions and possible development.

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STUDENTS' ATTITUDES, CHOICE OF TOOLS AND SOLUTIONS OF MATHEMATICAL TASKS IN AN ICT RICH ENVIRONMENT

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In a three year development and research project in lower secondary school the aim was to develop and study students' competence to use ICT-tools and to make reasonable choices of ICT-tools to solve mathematical problems. This paper will present some of the students' solutions in computer files, some observations and answers to the questionnaire that was given in close connection with their work. The students were asked some attitude questions and were asked what tools they chose for a specific task and why. Finally, the paper will outline a subsequent project which aims to build learning communities with teachers and didacticians[1] in order to develop ICT competence with mathematical tools and use of ICT in inquiry processes.

BACKGROUND AND RATIONALE FOR THE PROJECT

The students should "know about the use of IT and learn to judge which aids are most appropriate in a particular situation" according to the Norwegian curriculum plan for mathematics applied from 1997 (KUF, 1999). The new Norwegian curriculum plan (UFD, 2005) states ability to use digital tools as one of five basic proficiencies alongside with ability to read mathematics, express mathematics orally and in writing, and perform calculations. In higher levels this implies that the students should be able to use and assess digital tools for problem solving, simulations and modelling. We find similar recommendations in other documents about teaching and learning, for example the NCTM Principles and Standards (NCTM, 2000).

On this background the aim of the project was to develop the students' competence to use ICT-tools and to judge and choose suitable ICT-tools for mathematical problems. The research focussed on the students' appreciation of the ICT-tools and evaluates to what extent they could make reasonable choices of tools and give reasons for their choices.

The project was situated within a social constructivist view of learning, emphasising the students' collaboration and use of technological tools for experimenting,

[1] Didactician in this context is a researcher at the university college.

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exploring mathematics and thereby mediating students' development of mathematical knowledge and their competence with mathematics and the ICT-tools (Goos, Galbraith, Renshaw, & Geiger, 2005; Noss & Hoyles, 1996).

The view of the role of computers and technological tools in the classroom has implications for what tools to choose and how the tools are used in teaching. I see ICT-tools as reorganizers rather than as amplifiers (Dörfler, 1993; Pea, 1987). Computer software and calculators, as cognitive tools, have the potential to highly influence the development of mathematical concepts and connections. Amplifier implies doing the same as before, without changing the basic structure, methods and approaches. In contrast "re-organising occurs when learners' interaction with technology as a new semiotic system, qualitatively transforms their thinking" as expressed by Goos et al. (2005).

I think reorganising in this sense will stimulate students' learning, and make it possible to utilise the potential of ICT in mathematics, introducing new forms of representations and connections between different representations. In accordance with this view suitable tools are open and flexible, provide a learning environment for exploration and investigation, integrate different representations, give rich possibilities for mathematical representations, and stimulate reflection (Hershkowitz et al., 2002). I think of ICT-tools as open, flexible software that makes is possible for the user to decide what to do, not being directed by the software. In our project, a spreadsheet (Excel), graph plotter (Grafbox) and dynamic geometry programme (Cabri) were the main tools together with use of Internet for collecting data and information.

RESEARCH QUESTIONS AND METHODOLOGY

The research aimed to evaluate to what extent the students could make reasonable choices of ICT-tools for solving tasks. To illuminate and understand how far they appreciated and understood the tools, they were asked to give reasons for their choices of tools for a task and why they liked or disliked that task. Thus both questions about tools and about attitudes were part of the research.

The concept attitude in this context is as defined in McLeod (1992) dealing with if the students liked or did not like specific task and their reasons why.

The nature of the project and the research questions suggest mainly a qualitative research methodology with a developmental perspective and with the researcher as participant observer. Although a questionnaire was used, with some quantitative data, the purpose was mainly to collect qualitative data, in the form of students' written comments and students' work in computer files.

OUTLINE OF THE PROJECT

The first part of the project focussed on development of ICT use in the classroom with more emphasis the last year on observing classroom activities and collecting data for research (Fuglestad, 2004, 2005). The project intended to build an ICT rich environment, that is an environment with ample access to ICT-tools, and to create an open cooperative learning environment with opportunities to choose tools for given tasks, and to some extent for the students to develop their own tasks in a specific setting.

The teachers and project leaders met every term for project meetings to discuss experiences, teaching ideas and further development. The project leaders provided some ideas and material for use in the classes but the teachers were in charge of what they implemented in their classes. Later some of the teachers also developed their own material including computer files to work on in Excel or Cabri and made these available for the project group.

In the last part of the project the classes used 8–10 lessons over a couple of weeks for work on a booklet with a set of 12 tasks prepared for this period. The students could choose what tasks and tools to work on and cooperate in their work. The tasks were prepared to give a variation in difficulty, topics, and with options to use ICT-tools or other tools like paper and pencil, calculator or mental calculation. Some tasks had a set question whereas others gave only information and the students had to set their own tasks. The tasks were discussed with the teachers in a project meeting some months before the working period to secure that they were appropriate for all the classes. This was seen by the teachers as important to be fair to the students, since the classes had different levels of experience with ICT-tools.

Data collection

This working period was closely observed, partly with audio and video recordings of students' work, and their solutions to tasks on paper or in computer files were collected. About a week after this working period the students were given a questionnaire concerning their work with some attitude questions, questions about what tools they chose and why, and what tasks they liked or disliked to work on.

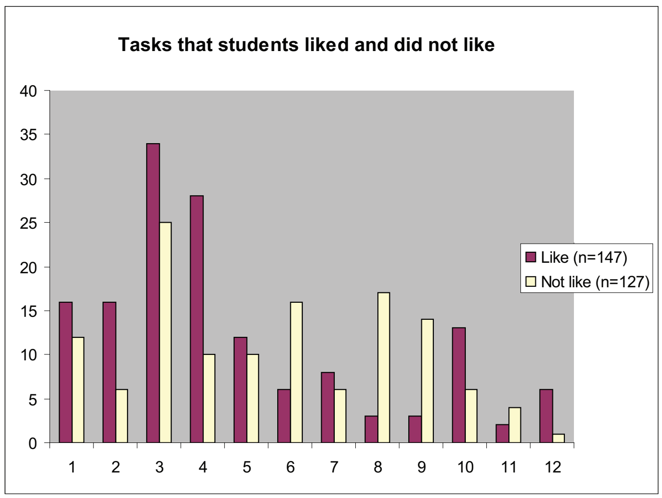
A variety of data was collected from the project: from observations, in questionnaire and computer files containing students' work copied either by the students themselves or by their teacher. In this paper I will restrict my presentation to the analysis of students' work as recorded in computer files and compare this with their answers to the questionnaire. In a few cases I will also report from some observations to complete the information.

In the questionnaire, administered via Internet, the students were asked to identify a task they liked to work on and why, and what tools they used for this task and give reasons for their choice. They ticked boxes for choice of one or more tools: paper and pencil, calculator, spreadsheet, Cabri or others, with the option to specify 'others'. Similar questions were asked for a task they did not like to work on. In the last part four tasks that they had not worked on were presented and the students were asked to read and judge what tools to use, but not to complete the task.

RESULTS

The aim of this paper is to look closer into some results from the questionnaire and relations between the reasons given by students in answers to the questionnaire and their solutions of tasks as shown in computer files.

The results from the questionnaire were transferred into Excel, and answers to why they liked a task and which tools they chose were coded according to the dominating content in their answers.



The results reveal that about 18 % of the answers to why they chose a particular tool gave good reasons with comments to features of the software, and thereby indicated good understanding of their choice. Examples of students' answers: I use this program often to calculate things that are mucky. It is easier to calculate sums in Excel. You can also use various formulae, so you don't have to write the same numbers several times (112)[2]. Another student also used a spreadsheet: Because then I don't have to write all the numbers, I can just copy all I need (219). The expressions "use various formulae", "do not have to write the same numbers", "don't write all numbers, just copy" gives references to commonly used features in Excel.

Varying from 48 % to 60 % on different questions the students gave acceptable but fairly neutral answers (Fuglestad, 2006). Answers like "it is the best choice" or "it is easy to set up on a spreadsheet" can be judged to be superficial. However, we can not expect extensive answers in a questionnaire with several questions in an

[2] Numbers in parentheses are labels for particular students

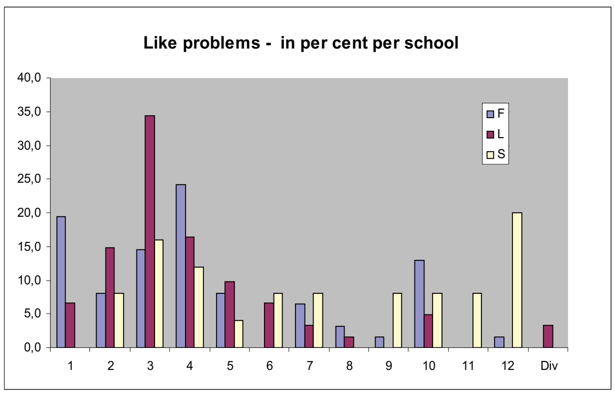
hour. For these students a look at their solution in computer files can provide more insight in their choice of tools and depth of understanding. In order to gain more insight examples of students' answers and solutions to the tasks will be presented and analysed in the following paragraphs.

The analysis of the students' solutions also revealed a vide variety of their competencies and ways of using the ICT-tools as will be shown in the following.

What tasks students liked and did not like

A simple counting of frequencies of tasks students liked to work on reveals that the first four tasks were the most liked. On the other hand two of these were also ranked highest as tasks not liked by many students.

The number of students varied between schools, with only 25 in school S and 68 and 70 from schools F and L. Not all students answered the questions, and a few gave answers other than 1–12. The response varied from 89 to 100 % for like and 76 to 86 % for not like a task.



With the distribution in per cent of class size for each class, it turned out that for school S task 12 had the highest score, followed by task 3 and 4. For school S task 1 had the highest score of not like. Task 3 had a high score for liking from school L, and a high score for dislike from school F. Task 4 had the highest score on liking from school F and as number two from school L.

In the following analysis of students' answers about choice of tools and solutions I will focus on the tasks 1, 3, 4 and 12 in the booklet and a couple of examples observed in the classes.

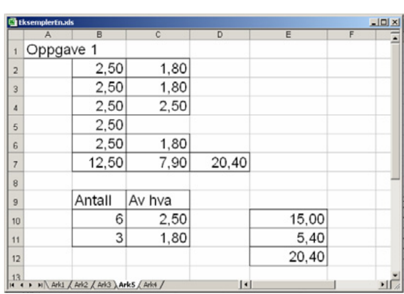
STUDENTS' SOLUTIONS AND COMMENTS

Task 1 was about combining the value of two kinds of stamps worth 2.50 NOK and 1.80 NOK to get the value 20.40 NOK. The task is easy to solve but requires some trial and error, not just pure routine. I found a wide variation of solutions with some using a spreadsheet, some a calculator, and some just mental calculation.

A solution was given like this when a student talked to the teacher: I looked at the point forty and found I had to use three times 1.80 to get that, so it was easy to find the solution just by trying with different numbers in my head. I think this is a good example of mental calculation and valuable for the discussion to highlight this method alongside with computer use. To be able to choose, the students have to see alternative solutions.

The solutions given in a spreadsheet varied. Some students mainly used the spreadsheet as a word processor, using none or few calculations and hardly any cell references. However, I also found a student with this kind of solution that solved other more difficult tasks very well. This may indicate either a misinterpretation of task 1 or that he learned to solve the tasks better. Others used the spreadsheet to experiment with different numbers of 1.80 and 2.50 in two columns and combining a sum (Example 1). Some students experimented with the number of stamps in two cells and calculated using a formula for the total. Variation of these methods and a table combining operations (Example 2) was also used. In one case that I observed that the students made a big table making all the combinations of the number of stamps, to be sure to get all the different solutions. It turned out that this was really not the students' solution but the teacher had found this for himself the evening before, and when students asked they were told how they could make this. When I asked the students about this, they could not really explain but said they had problems to understand it.

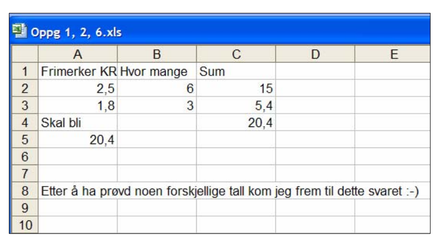
Figures 1 and 2 show two examples of students' solutions (202 and 218). In both cases they used formulae in the sums, product and the final result and experimented with different number of stamps. They had to plan a structure on the spreadsheet to make an efficient experiment. These two students, both boys, also gave acceptable or good solutions to a couple of other tasks, but had some they did not finish. There


Eksemplertn.xls

	A	B	C	D	E	F
1	Oppgave 1					
2		2,50	1,80			
3		2,50	1,80			
4		2,50	2,50			
5		2,50				
6		2,50	1,80			
7		12,50	7,90	20,40		
8						
9		Antall	Av hva			
10		6	2,50		15,00	
11		3	1,80		5,40	
12					20,40	

Ark1 Ark2 Ark3 Ark5 Ark4

Figure 1. Example 1, solution of task 1


Oppg 1, 2, 6.xls

	A	B	C	D	E
1	Frimerker KR	Hvor mange	Sum		
2	2,5	6	15		
3	1,8	3	5,4		
4	Skal bli		20,4		
5	20,4				
6					
7					
8	Etter å ha prøvd noen forskjellige tall kom jeg frem til dette svaret :-)				
9					
10					

Figure 2. Example 2, solution of task 1

seems to be reason to believe they mastered the task, and the comment that is was easy is in accordance with their solutions.

A students business

Task 3 was about running a students business. Their task was to plan for a kiosk at an outdoor sports arrangement a warm day in June. They could expect 300 participants of age 13–16 years and 200 adults. They had to plan the selection of items to sell, and judge how much to buy, deal with expenses and income and prepare an overview to show the result.

For this task an overview can very well be planned on a spreadsheet. That was also the choice made by many students. The students had to judge amounts and either collect prices or judge for themselves what were reasonable prices. The task was liked by 34 students and disliked by 25. A high proportion of students from school L (35 %) liked the task whereas many from F (25 %) disliked it.

The reasons students gave for liking the task were: Because we, in a way made the task ourselves. We could decide more make our own set up (108). Similar answers were given by many students. On the question why he (108) chose Excel for this: Because it was straightforward to make a set up in this. I addition we found that we could simulate the numbers by using RANDOM. The boy commented on what he learned: I felt that I became better and could manage the program. I revised the use of random (108).

He cooperated with another student and their solution in a table gave a good overview with texts explaining the use of rows and columns and had formulae in appropriate cells. They used random numbers to set number of items sold, and gave this within a range. Another student commented he liked task 3: Because we could use our fantasy and decide a lot ourselves. We chose for example how much to buy, purchase cost and retail price (221). He used Excel and gave this reason: Because it was simple and suits the task. This is a reasonable answer but do not provide any information about the depth of his answer. He solved the task correctly and confirms that Excel suits the task, and so his reason can be regarded as relevant. However, the way it is set up indicates that he had to write every single formula and could hardly utilise the feature of copying. Several students said that they liked the task because it was like having their own kiosk, and they were allowed to decide. One said that he knew the situation as he had had this task in practice.

The students that disliked task 3 used some of the same arguments but with opposite conclusion. They thought the task had little to do with mathematics, they did not like to check or invent prices, and judge how much to buy. A couple of quotes illustrate this: Because it was not interesting, really not a proper mathematics task. Find out what to buy for sports arrangement is really not the best mathematics, but fair enough to calculate prices and so on (238). This was a difficult task that did not have much to do with mathematics (251).

Calculating interest and status of account

Task 4 was about calculating interest and how the amount 9000 NOK increases over years with 4 % interest rate. Five questions related to this situation were given, including changing the interest rate to 5 %.

28 students liked the task and 10 disliked it. From the students that liked the task arguments were they liked to work on per cent and interest rates, they managed the task or they liked it for other reasons.

Mira (375) said she liked this task because it was simple. She chose only to use a spreadsheet, as did most of the students for this task, but she did not give any reason why. But she commented on what she learned: How I can calculate interest.

Mira presented the solution shown in Figure 3. Here she used multiplication by the formulae B7*104% in cell B8 and correspondingly further down. She did not utilise fixed reference to the cells B4 and B5 for the interest rates but the solution is correct. The first look at the solution indicates use of parameters, which she did not utilise but other students did. By using 104 % instead of a cell reference she avoided a possible problem with copying a cell for interest rate.

Other comments to why they chose a spreadsheet were: To use a spreadsheet is the best method, quick and simple. Just click and drag (337). And another: This was because when I had to set up for several years; it is easiest to use Excel (205). She gave a good solution and also made graphs for illustration. The comment about "setting up for several years" can indicate copying, as does also "click and drag".

Dora (374) disliked the task: I did not like this task well, because I am not good at such tasks and get frustrated when I can not manage. She commented that

	A	B	C	D	E	F	G
1	Renteberegning:						
2							
3	Kari har	90000	i banken per 01.01.2004.				
4	Hun får	4 %	som hun lar stå på kontoen.				
5		5 %					
6	ÅR	KAPITAL	Kapital med 5%				
7	2004	90000	90000				
8	2005	93600	94500				
9	2006	97344	99225				
10	2007	101237,76	104186,25				
11	**2008**	**105287,27**	109395,5625	a) Kari har	105287,27	kr i 2008.	
12	2009	109498,761	114865,3406				
13	2010	113878,712	120608,6077				
14	2011	118433,86	126639,038				
15	**2012**	**123171,215**	132970,9899	b)Det går 8 år før kari har minst 120 000 kr i banken.			
16	2013	128098,063	139619,5394				

Figure 3. Solution to task 4, using a table and indicating parameters

she used her peer Maria to explain and they worked together. And to the question what did you learn: I learned how to set up such tasks. The solution on files for Dora and Maria are the same and confirm their cooperation. The solution looks ok, and very similar to the one presented above, except with less written comment to the solution. They used a table in columns multiplication by 1.04 from one row to the next and so on. A similar solution was also presented by many other students.

Cost of mobile phone use

In task 12, titled "Theme: It costs to call", the last task in the booklet no questions were given, only prices from two different companies running mobile telephone service. The students had to set their own tasks related to the theme presented.

Six students in school S and one in F ticked they liked this task. Two students used Grafbox, and Internet to find prices. Three students used a spreadsheet for solution and two of them also used Grafbox. All gave good solutions.

A student (115) gave this reason why she liked the task: This was a task I could work on and it is possible to use other information than in the booklet. If it had been very difficult it would have been not so much fun, but this was appropriately difficult and not too easy. She commented why she used a spreadsheet: On a spreadsheet we can set up a table, and later find out the answers by using a simple formula, instead of using a calculator for every calculation. There are also many functions on a spreadsheet, and the one we used was to find the minimum. The point is to find what kind of subscription is the cheapest.

She solved the task together with another girl. Analysis of the solution given in Excel shows that the students made a very good setup with a list of different subscriptions and prices. The user can enter number of minutes in a cell and have the different prices calculated, and get a recommendation written below what subscription is best. The solution was correct and well documented by giving all formulae used in a textbox. Here I can see a good correspondence between the solution and the comments given. The comments do not give any details about the solution but give some main points of why a spreadsheet is suitable, and what possibilities it provides for similar tasks with its many functions and use of a table to make the calculations easy.

A boy that used Grafbox said he liked to make more out of it, finding prices for other companies on the Internet and he made graphs for several companies. The same student also gave a solution using Excel. The student from school F that liked the task (246) commented this task was interesting and useful for both adults and young people to calculate prices according to their own need, and make their own tasks.

A student that did not like task 12 thought it was a bad task and did not like that he had to find more information on the Internet and the task was without any sense. Perhaps he was just a little lazy or disliked tasks with no set question?

The comments given are all relevant to the choice of tools and solution. In most cases little is given concerning the details of the solutions.

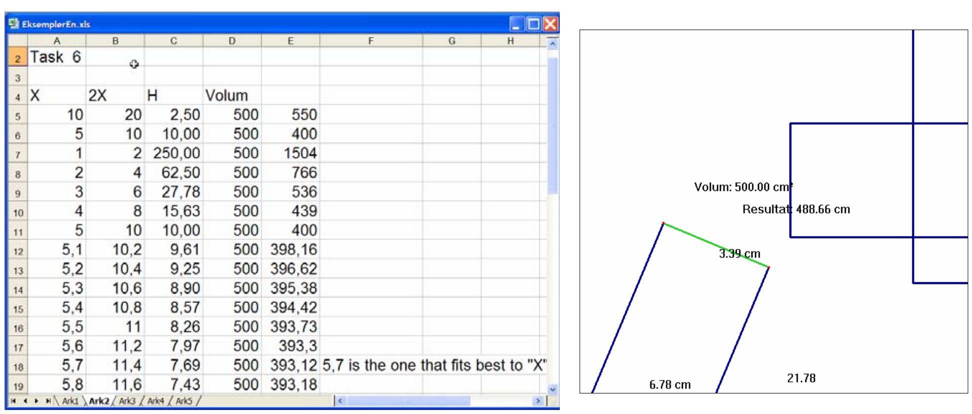
More than one tool

In order to develop good competence to choose tools, I think it is useful in some cases to solve the problem with more than one tool. Then the students' own experience can form a background for discussion of what is the best tool to use, or see connections between the different ways of solving the problem.

I observed two boys working alongside each other, on a task about a bus tour and judging different price models. One used Grafbox because he felt he managed this tool well, and the other chose to use a spreadsheet. They both expressed strongly their preference for the tool they chose and presented correct solutions. As they worked they started to compare their solutions (Fuglestad, 2005). One student liked the task and used Grafbox because he felt he got the grip on it and had very short comments to why he chose this tool: I also used Excel (107). The other one: In a spreadsheet we wrote in formulae and got the value in figures for 5 years. But in Grafbox we made curves, of formulae that we wrote in, and then we got a point where the cars had the same value (105).

Two girls used both Cabri and Excel to present different solutions to the same task. See Figure 4. This dealt with finding the smallest surface of a box with certain restrictions, having the width half the length and the volume 500 cm^3

The comments to why she chose the tools one of them wrote: In a spreadsheet it is easy to get an overview. In Cabri I can change the figures to look at different lengths (106). And the other one: Because it was a suitable way to calculate the result, and it became clearly set out (102). As we can see the answer for the first one refers to Cabri and the possibility of varying the lengths. The other one is fairly neutral, but relevant and their solution was very good.



	A	B	C	D	E	F
2	Task 6					
3						
4	X	2X	H	Volum		
5	10	20	2,50	500	550	
6	5	10	10,00	500	400	
7	1	2	250,00	500	1504	
8	2	4	62,50	500	766	
9	3	6	27,78	500	536	
10	4	8	15,63	500	439	
11	5	10	10,00	500	400	
12	5,1	10,2	9,61	500	398,16	
13	5,2	10,4	9,25	500	396,62	
14	5,3	10,6	8,90	500	395,38	
15	5,4	10,8	8,57	500	394,42	
16	5,5	11	8,26	500	393,73	
17	5,6	11,2	7,97	500	393,3	
18	5,7	11,4	7,69	500	393,12	5,7 is the one that fits best to "X"
19	5,8	11,6	7,43	500	393,18	

Figure 4. Solutions to task 6 in both Excel and Cabri

DISCUSSION

The computer files containing students' work is from an open working situation where cooperation was allowed and stimulated. We can therefore expect the solutions to be the result of a process going on in cooperation with other students and with some help from peers or teacher. For this reason the students' solution in data files are perhaps better than they would be on an individual test. In some cases the students also wrote in the questionnaire that they got help from their peers and learned from discussion. This is visible in comparing some solutions on files. However the students went through a process to get to the answer, and learned from it. It is possible to see in their files that their competence developed. This was clear from the example of students that did not manage task 1 but gave good solutions to later tasks. The social working environment with their peers stimulated further construction of knowledge, in line with the intention in the project.

Many answers were fairly neutral concerning why the students chose a particular ICT tool. Not all students' solutions were available. However, from analysing a fair portion of the students' files the general impression suggests a development over the working period with mostly correct and in some cases very good solutions.

In most of the cases I analysed, the tools were used correctly and in general gave acceptable solutions. The limitations revealed concerning competence with the tools were on preparing a systematic and efficient layout on the spreadsheet, not using fixed references to simplify experimenting and similar. Some students' layout did not easily allow copying of the formulae and so did not appear efficient. With Grafbox I observed problems to judge scales on the axes, giving problems of interpretation.

We might ask what stimulated the choice of a certain tool for a given task. A table will obviously be easy to transfer to Excel whereas a drawing could indicate use of Cabri. It is difficult to know to what extent this happened, but some answers indicate this could influence the reasoning. For example answers like "it suits the task" indicate some characteristics of tasks for specific programs. However, some students also solved a task with more than one tool and indicate competence to judge possible ways of solving the task beyond the first look at it.

Overall the students' answers to what tools they used and reasons for this were in correspondence with their solutions. Some solutions were fairly simple but correct whereas others were very good and included text explaining their methods. Although some students gave short and fairy neutral reasons for their choice, their solutions indicate there is a fair understanding of the tools.

At the outset we may wonder if it is necessary to have a deep understanding of the tools to make reasonable choices for a given mathematical task. The results reported here suggest that many students in these classes were able to make reasonable choices of tools, but with some limitations in the depth of their facility with the tools. They seemed to have grasped the main features and on that basis made their choices.

Limitations in experience

Although we found some promising results concerning students' choices, there is still a lot to develop concerning how to utilise ICT-tools in mathematics classrooms. This was revealed in the fairly simple use some students made of the ICT-tools. There is a need to develop further methods for use of the tools to provide situations for the students to experiment and investigate problems, and be stimulated to ask new questions themselves, inquiring into the mathematics and the use of ICT-tools.

I found limitations in teachers' competence concerning ICT and how to utilise the tools to provide an open and experimental teaching environment. Although they indicate they felt comfortable using ICT in the classroom, they revealed lack of insight into the software and they commented on their need for competence development both with ICT and to develop good teaching ideas. To some extent this limited the development in the project. For example, some teachers used mainly spreadsheet and had very little use of Cabri and Grafbox in their classes, because of their limited experience with the software. To some extent this choice was also influenced by their knowledge of the coming final examination where the students can use a spreadsheet but not the other ICT-tools.

I observed similar limitation in teachers' competence with computers and how to utilise tools in mathematics classrooms in contacts with teachers more broadly, for example in courses and on visits to schools. The development of ICT use in mathematics generally in Norway is a rather slow process as is also documented in other countries (Hennesey, Ruthven, & Brindley, 2005; Kilpatrick & Davis, 1993).

IMPLICATIONS FOR FURTHER RESEARCH – THE ICTML PROJECT

There is a clear need to develop teaching methods with ICT in mathematics classrooms. It is not sufficient just to introduce the tools and act as before. I think to fully utilise ICT in mathematics, we have to plan for a reorganising of the teaching approach and stimulate students' reorganising of their thinking. Experiences for the project reported here with an open cooperative learning environment using ICT-tools suggest this is a promising approach.

In a subsequent project, ICT and mathematics learning (ICTML), the aim is to build a learning community involving schools, teachers and a group of didacticians at Agder University College. The project has a close collaboration with the project Learning Communities in Mathematics (LCM), which aims to develop inquiry communities in mathematics (Jaworski, 2003; Jaworski, 2004). Both projects are funded by the Research Council of Norway[3] in the KUL programme, (kunnskap, utdanning og læring, i.e. knowledge, education and learning).

Inquiry as a focus in the work implies to ask questions, investigate and experiment and build an attitude to learning in line with this. This applies to all levels, both

[3] Norges Forskningsråd http://www.forskningsradet.no

for didacticians, teachers and students in schools. For ICTML a main focus is on how to utilise ICT-tools for inquiry to build students' and teachers' mathematical knowledge and competence with ICT-tools.

In the ICTML project we build on experiences and insight gained from the project reported here, for example in preparing material and ideas for teaching. In the ICTML and LCM projects we work closely with the teachers in workshops and in school teams. The teachers and didacticians meet regularly for workshops and most of the teachers in the ICTML project also take part in the LCM project. The workshops for LCM have a main focus on inquiry and explore an inquiry approach by doing mathematics, discuss experiences and implementation in schools. In the ICTML workshops the focus is particularly on use of ICT-tools and how to utilise these in an inquiry approach.

This cooperation will also provide further development of the teachers' competence to utilise ICT-tools as support for the learning environment in mathematics classrooms. In the ICTML project we will develop and investigate further students' use of ICT-tools for building learning communities with an inquiry approach.

Our experience so far from the project reported here and the ICTML project indicate that a long term effort is needed to develop good uses of ICT-tools and an inquiry attitude to teaching and problem solving in mathematics.

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SÁMI CULTURE AS BASIS FOR MATHEMATICS TEACHING

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A QRS-system (Barton, 1999) is a meaningful system that for a given group of people gives meaning to Quantity, Relations and Space. Each cultural group has their own QRS-system. This paper focuses on how the Sámi culture's elements lavvu, lasso, skis and duodji can function as basis for mathematics teaching. The elements are described and mathematised (Freudenthal, 1973); made accessible to mathematics treatment. The elements are further categorised into two groups, Relations and Space. There will be pointed out possibilities for how to use these elements in mathematics teaching.

INTRODUCTION

This paper aims to mathematise some elements of the Sámi culture and to build up a framework for analysing these elements. The research question is

> How can elements in the Sámi culture be mathematised with respect to Relations and Space in a QRS-system?

In another study (Fyhn, 2004) 10–12 years old students from Northern Norway made a two day winter trip to the mountains. The students went skiing to get to the camp area, they spent one night in a lavvu and they performed some activity with a lasso. This trip is referred to as the pilot study in this paper.

BACKGROUND

In my master thesis (Fyhn, 2000) I questioned if students who participate regularly in common leisure time activities succeed in some common mathematics tasks. There could be some connections. Girls who work with creative craft seemed to succeed best in tasks that concerned patterns. In addition students who take part in physical activities seemed to succeed in tasks that concerned understanding of space. Barton's (1999) *R* and *S* from the QRS-system was a suitable way of describing these findings. What I denoted as "patterns" was denoted as "relations" by Barton. Thus I wanted to explore the QRS-system and how it could work out as a tool in my further work.

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Barton (2005) points out that bilingual learners who are fluent in both languages have a deeper and more aware sort of knowing mathematics than those who only master one language. The Norwegian curriculum for the 10 year compulsory school (KUF, 1996) was implemented in 1997. This curriculum claims

> Sami culture and social life are important parts of the common cultural heritage, which all pupils in compulsory school should learn about. The Sami culture, language, history and social life comprise part of the common content of the different subjects. The education of Sami pupils shall promote their roots and security in their own culture and to develop the Sami language and Sami identity. At the same time it shall ensure that Sami pupils are able to participate actively in the community and obtaining education at all levels. (p. 65)

The mathematics teaching for Sámi students today make use of ordinary Norwegian textbooks translated into Sámi language. This is exactly what Barton (2005) warns against; just that one knowledge system has had enormously amounts of time and energy put into its development does not make it the only and correctly structured knowledge. Some mathematics teaching material is designed for Sámi students, for instance Nystad's booklets which are printed in both Sámi and Norwegian language (Nystad, 2002a, 2002b, 2003). Still there is a long way to go before the curriculum's (KUF, 1996) intentions are implemented in Norwegian schools.

MATHEMATICS

According to Lakoff and Núñez (2000) mathematics is a product of the human mind. They further claim (p. 6) "Metaphors are an essential part of mathematical thought, not just auxiliary mechanisms used for visualization or ease of understanding." Patterns in snow are examples of such metaphors. Some Sámi mittens have the pattern *grouse footprints* on their edges. These repeating patterns show a connection between embodied experiences and ornamentation. Ski trails as well as animal footprints perform patterns in the snow, patterns with different symmetry properties (Fyhn, 2002a).

Barton (2005, p. 100) points out that "Rather than thinking of the mathematics which is known world over through formal education, we need to expand our vision to include any form of quantitative, relational or spatial systems." Barton uses boats as a metaphor for mathematics. He claims that different boats can be used for different purposes, the fishing boat can go to rocky places where the ferry cannot navigate and the ferry can travel under conditions too hard for the fishing boat. "It is the same world, but it is a different understanding. Neither is the truth" (p. 100).

In this paper I intend to denote geometry and algebra like Lakoff and Núñez (2000) denote the Platonic mathematics – if it exists at all; "a disembodied mathematics which is transcending all bodies and minds and structuring both this universe and every possible universe" (p. 1). When it comes to Duodji I face the problem of whether some patterned ornamentation is geometry or algebra or both. As far as I can see this is an example of two different understandings of the same phenomenon, and neither is the truth.

Hans Freudenthal (1973) restricts geometry and does not include geometrical patterns:

> A pragmatic program for geometry could remain restricted to a small treasure of theorems like the Pythagorean, a few obvious theorems on similar figures and a few formulae for perimeters, areas and volumes. (p. 406)

Freudenthal strongly problematises the distinction between geometry and algebra: "Geometrical algebra, … was the disease which killed Greek mathematics" (p. 5). He continues: "The first to cut free from the Greek tradition was Descartes, the challenger of all tradition. He put the chart before the horse: rather than geometrizing algebra, he algebraized geometry" (p. 6). To me these statements are related to the question about whether "The one and only Mathematics" exists. Barton (2005, p. 98) asks "...why should we think that mathematics is the single universal created from human experience?"

S AND R IN A QRS-SYSTEM

According to Freudenthal (1973) geometry is grasping space.

> And since it is about the education of children, it is grasping that space in which the child lives, breathes and moves. The space that the child must learn to know, explore, conquer, in order to live, breathe and move better in it. (p. 403)

Lakoff and Núñez (2000) support this by claiming that prior to the mid-nineteenth century space was conceptualized as most people normally think of it – namely naturally continuous. "It arises because we have a body and a brain and we function in the everyday world" (p. 265).

This work focuses on *Space* like the students experience it in their daily lives, geometry as grasping space; space as the living arena where geometry activities take place.

When a three dimensional figure from the real world is scaled down and transformed into a two dimensional figure on paper, it is far from as concrete as its origin. School mathematics has traditionally made use of such drawn images instead of making use of the original figures themselves. Berthelot and Salin (1998) divide the space into three different ones: The micro space where school geometry usually takes place, the meso space where the students' physical and social interactions usually take place and finally the big macro space which contains the whole mountain/forest/city.

According to Berthelot and Salin (1998) one of the main sources of learning difficulties in geometry among students 12 years and older, is probably the previous treatment of geometrical figures on paper during elementary school. In the teaching process, many students think of geometrical figures as if they were objects, while teachers refer to the same figures talking about geometrical concepts. In addition, Berthelot and Salin claim they have good reason to expect that geometrical knowledge is not spontaneously transferred to solve space problems. Berthelot and Salin are here interpreted to support geometry teaching based on the mathematising of space activities.

Different more or less advanced plaited wool ribbons are elements of Duodji. If you plait your hair you split it into three equal parts. This plaiting procedure can be described by numerous repetitions of "take the right part and cross it over the mid-part. Then take the left part and cross it over the mid part". The right part, whichever it is, can refer to all of the parts of the hair, and so is for the mid part and the left part as well. This is what we understand with conceptual metonymy (Lakoff & Núñez, 2000) and it exists outside mathematics.

This everyday conceptual metonymy …plays a major role in mathematical thinking: It allows us to go from concrete (case by case) arithmetic to general algebraic thinking… This everyday cognitive mechanism allows us to state general laws like "x + y = y + x", which says that adding a number y to another number x yields the same result as adding x to y. It is this metonymic mechanism that makes the discipline of algebra possible, by allowing us to reason about numbers or other entities without knowing which particular entities we are talking about (p. 74–75).

An analogy for conceptual metonymy in music is the pitch when two or more persons would sing a song together. The structure of the song is given on beforehand; independent of what particular pitch to use. Lakoff and Núñez (2000) further claim that "Algebra is the study of mathematical form or 'structure'" (p. 110). According to The Penguin English Dictionary (Allen, 2002) a synonym for "structure" is "interrelation of elements". In other words – how elements are related to each other.

I interpret the structured process of working with concrete patterned ornamentation as conceptual metonymy. In this paper I focus on the *Relations* that concern the structure of concrete and visual patterns; the metonymy in Duodji ornamentation.

SOME PARTICULAR ELEMENTS IN THE SÁMI CULTURE

The Sámi tent lavvu looks like the Native Americans' tipi. Inside it has a central fireplace and its stable structure bears the strong winds that occur in the Scandinavian mountain plateaus. The lasso is a tool for catching running reindeers. In the Sámi skiing championship there is a biathlon discipline; a lasso-ski competition. For youths of twelve and thirteen the distance they throw the lasso is about half a lasso length. The target is a reindeer's horns, fastened to a stake that is stuck into the snow. The lasso is used by more cultures than the Sámi. But neither the lasso nor the lavvu traditionally belongs to other Scandinavian cultures than the Sámi.

Skiing activities is an element in the Sámi culture (Birkely, 1994). Skis are traditionally used in the rest of Scandinavia, too, as the ground is covered with snow about half of the year in many Sámi areas. According to Birkely the Bysantic historian Prokopios wrote about Thule (Scandinavia) about 1500 years ago. Among the people of Thule Prokopios describes “Skrithifinoi” which means skridfinnar; skiing Sámi people. Birkely claims there are reasons to believe that Sondre Norheim hundreds of years later could have learned his telemark technique from the Sámi people at the Hardangervidda in southern Norway.

The lavvu, the lasso and the skis were all natural belonging elements of the mountain trip in the pilot study. These three elements were naturally connected to activities which took place in *Space*. Because the trip took place in *Space* there was no focus on *Relations*. However, based on Fyhn’s (2000) findings there are reasons to believe that *Relations* would come to surface by mathematising the patterned ornamentations of duodji.

> “Duodji” is a collective term for various activities such as homecrafts, artware (handicrafts), woodwork and trades, and for many Saamis it is a way of life… “Duodji” mirrors the Saami mode of living and cultural traditions. (Aagård, 1994, p. 2)

According to Dunfjeld (2001) duodji includes the process going from idea until the complete product and it is a common concept for Sámi art and handicraft in Norway, Sweden and Finland.

METHOD

The intention of this paper is to build up a framework for analysing mathematics content of some elements in the Sámi culture. In the pilot study the students were videotaped performing activities and some of their discussions were recorded on cassette as well. These videos were initially meant as basis for further tasks in mathematics; tasks that focused on concept understanding. The analyses of this paper concerned the natural use of lavvu, lasso and skis in the tapes and videotapes: What – if any – is the mathematics in these activities which take place in “the naturally continuous space” (Lakoff & Núñez, 2000). Patterned ornamentations do not necessarily take place in the meso space (Berthelot & Salin, 1999), but many people claim this is geometry. Thus I wanted to investigate some patterned ornamentation from Duodji in order to bring in an element that differed from lavvu, lasso and skis.

According to Freudenthal (1991) mathematising includes both moving from the world of life to the world of symbols and deeper diving into the world of symbols. In mathematising the tapes and the videotapes I categorised the mathematics into *R* and *S* and related this mathematics to what I believed was geometry and algebra. Here I was influenced by my Norwegian mother tongue. This paper does not focus on the Sámi language, but Barton (2005) warns that ideas of quantity, space and relations are not necessarily expressed the same way in different languages. Further work based on this paper could be to let teachers and students with Sámi language as mother tongue mathematise these four elements. One question will then be how their mathematising differs from mine.

Because Duodji is a genuine Sámi tradition and because the ornamental Duodji patterns are structured patterns, I open for the possibility that Duodji ornamentation can function as a gate to understanding *Relations*. This will be discussed because ornamentations often are denoted as geometry and thus do not concern algebra.

MATHEMATISING OF SOME ELEMENTS IN THE SÁMI CULTURE

The Lavvu

The lavvu is recognised by its conical shape, it represents a geometrical figure. The central fireplace with people around it represents the centre of a circle and the people around it can represent points on the periphery. I claim that the lavvu's mathematics mainly belongs to S. A description of the lavvu will include usage of some geometrical concepts; circumference, point, area, periphery, triangle, cone and sector.

In the pilot project some questions were asked in order to let the students focus on why the lavvu shape is useful compared to a tent. This showed to be a much too difficult discussion for the students. This was an example of the researcher's own joy caused by her mathematising of the lavvu. I was trapped just the way many teachers are trapped: "When you first have done mathematising like this, the mathematics seem so obvious that it is a great risk to forget that other people do not see it at all." Sitting inside the lavvu the students were capable to watch the floor and make a good estimate about how many people who could sleep in there.

A lavvu that is pitched up can be used as a basis for a discussion about the relation between radius and circumference in a circle (Fyhn, 2004). It is not obvious to the students that this can be investigated by checking if there is any relation between the number of steps from the centre of the lavvu's fireplace to its doorway and the number of steps in the lavvu's circumference. This investigation takes place in meso space (Berthelot & Salin, 1998).

Freudenthal (1991) claims that reflection on one's own activities is an important aspect of mathematising. During my reflections after the pilot research, I found many possibilities for how to use the lavvu in mathematics teaching. In raising the rods of a lavvu the students make use of the knowledge that a right cone has its top straight above its centre and that all the rods then are of equal length. The lavvu is centred round its fireplace. On the ground the distance from the fireplace and to the lavvu's edge is the same all over the lavvu.

One possibility for a further project is to let the students raise a lavvu – that is an activity which takes place in the meso space. Sitting inside the lavvu afterwards the students can then discuss and describe as many as possible of the lavvu's properties and connect these properties to mathematics.

The Lasso

For the lasso to catch a target, there are two conditions that need to be fulfilled. First the distance need to be correct and second the direction must be appropriate. Given the correct distance, the locus of the possible catching points is a circle with the lassoer in its centre. Given a fixed direction, the possible catching points constitute a ray from the lassoer and in the fixed direction. The intersection of the circle and the ray represents that both conditions are fulfilled; the lasso catches its target.

Based on this I will claim that the mathematics of the lasso as element in the Sámi culture mainly belongs to *S*. In other words: some geometry teaching can be based on the use of lasso. More precisely the geometry teaching that concerns the concepts; circle, angle, point and ray.

In the pilot study the students were divided into groups which took part in some outdoor activities. One of the activities started with letting the group find the middle of a lasso. One group member stood still and held on to this mid point. Another group member held on to the lasso's ends and walked around person number one with the lasso straightened. The walking had to be repeated until the footprints shaped a circle's periphery. The complete lasso was straightened out again and the students were asked how many lasso lengths were around the complete circle periphery. They guessed and checked and all the groups concluded the answer should be somewhere between 3 and 3,5. Some of the students even discussed if the correct answer was 3,2. Afterwards one of the group members stood in the circle's centre and the other ones on the periphery. The one in the middle then tried to catch the others by the lasso.

Because 3,14 or π not was suggested as answer to the lasso task, I took for granted that the students were not introduced to π yet. Next day at school I presented the students for some tasks in their book that concerned relation between diameter and circumference in a circle. Some of the girls quickly replied that they had done these tasks last week and they could not find any reason to do the same tasks once more. This showed quite clearly that to these students, as for many other Norwegian students as well, mathematics is about finishing tasks in a book, not necessarily to understand what these tasks are all about. Some of the girls that more than once proved to be able to do smart reasoning in mathematics did not see any connections between the lasso task and the similar tasks in the book. This supports Berthelot and Salin's (1998) claim that geometrical knowledge is not spontaneously transferred to solve space problems.

One possibility for a further project is to let the students use a lasso. Afterwards they can list up properties of a lasso and describe how the lasso is used. Then they may connect the listed properties with what they believe can be mathematics.

The Skis

One of the parents transported the lavvus and most of the other equipment by snow scooter to get it to the camp area. The students used skis to get there. I did not focus on mathematising the students' skiing. The following day at school the students could have described how they got to and from the camp. They used several different skiing techniques which all resulted in symmetric patterns in the snow.

The students performed some tasks by the use of skis, tasks that I had prepared beforehand. These were not tasks that concerned mathematising; the students just concretised some mathematics by use of their skis. The intention was to do some mathematising but I realised afterwards that I got caught by traditional "school reasoning". I started to focus on the mathematics concepts instead of focusing on the skis to be mathematised. This reminds me how difficult it is, to me as well as to other teachers, to change the way you teach; to put away some of one's own old teaching experiences in preparing and implementing teaching.

One skiing task started this way: The students should stand on their skis and turn around while the back ends of their skis were located on the same place all of the time. This activity is a bit difficult to perform and it is an activity with focus on mastering the skis. The result of this task is that you make a nice circle with radius equal to one ski's length. The centre of this circle is the place where the back ends of the skis were located and the circumference of the circle consists of the places touched by the skis' tips. The trampled area represents the area of this circle. The PISA test (Kjærnsli, Lie, Olsen, Roe, & Turmo, 2004) showed that space and shape is the area where Norwegian students perform lowest and that the Norwegian results in this field even have decreased from 2000 until 2003. Thus there are reasons to believe that area is a difficult concept for Norwegian students to understand. However, the PISA test was not translated into Sámi language thus Norwegian students with Sámi as mother tongue did not take part in this study.

One possibility for a further project is to let the students use skis in meso space (Berthelot & Salin, 1998) in a way they choose themselves. Afterwards they can list up properties of skis, how skis are used and then describe what all these have to do with mathematics.

Duodji

Regarding Duodji, this paper's focus is limited to patterned ornamentations. Such structured ornamentations have lots of inter-*Relations* between their elements; thus they are categorised into *R*. The structured work with these patterns includes metonymy and thus gives possibilities for the work with algebra in school.

In Dunfjeld's (2001) descriptions of ornaments she, to a large extent, refers to how geometrical figures and other ornaments are structured and why they are structured the way they are. A superficial reading of this text can lead to claiming it is about geometry but I will claim it mainly concerns structure – *Relations*. An interesting approach to research on this area could be a more profound analysis of the mathematics in this text.

Patterns usually have names, for instance the pattern *fish bone* is well known. This pattern has got its name from the backbone and ribs of a fish. A similar pattern is made when a skier goes uphill, and both the skiing technique for this ascending and the skiing pattern is named fish bone. This fish bone metaphor is what Lakoff and Núñez (2000) denotes as a conceptual metaphor: "Conceptual metaphor is a cognitive mechanism for allowing us to reason about one kind of thing as if it were another" (p. 6). Dunfjeld (2001) chose to interpret the ornaments by the metaphor and the embodied understanding she had made from the ornament. She states: "To watch ornaments just as decoration was not usual" (p. 118, my translation). Dunfjeld further claims that "the ornamentation is ambiguous dependent on which context it is interpreted or analysed into" (p. 39, my translation). This tells me that I must be careful in mathematising Duodji. I cannot avoid doing it because I usually mathematise much of what I experience, but I have to be aware of these close connections between Duodji and context.

Duodji patterns often have names related to nature. This indicates an embodied way of using metaphors. The pattern for sewing a coffee bag consists of different parts and each part has its own name like for instance "njálbmi", the part where the opening is (Johansen, 1983). The Sámi word "njálbmi" means "mouth" in English. However, this pattern is an ordinary geometrical figure and not a patterned ornamentation.

Mathematising (Freudenthal, 1973, 1991) some particular Duodji ornamentation could start with pointing out how an embodied metaphor is used as name for this ornamentation. For instance the grouse footprints refer to the similarity to the footprints grouses make in the snow while they are walking around and eating. It is a Sámi tradition to catch grouses in snares during the winter time and the birds' footprints tell where to put up the snares. These repeated footprints shape an ornamented pattern. Knitting of this pattern can be described by use of the conceptual metonymy; first turn: three times bottom colour, second turn: one bottom colour, one different colour and one bottom colour. The third turn equals to the first one, the second turn equals to the second one and so on. According to Lakoff and Núñez (2000) this is algebra.

As for the QRS-system the description of this knitted pattern to a large extent concern *Relations*.

A FRAMEWORK FOR ANALYSES OF CULTURAL ELEMENTS

Freudenthal (1973) claims that geometry is grasping space. Space is an important part of mathematics. Both Barton (1999) and Lakoff and Núñez (2000) make use of the term space in their descriptions of mathematics. So do I (Fyhn, 2000).

Lakoff and Núñez (2000) use the terms essence and structure in their approach to algebra,

> Algebra is about essence. It makes use of the same metaphor for essence that Plato did – namely, Essence is form.
> Algebra is the study of mathematical form or "structure". Since form (as the Greek philosophers assumed) is taken to be abstract, algebra is about abstract structure. (p. 110)

Fyhn (2000) used the metaphor pattern quite similar to Lakoff and Núñez' structure. Barton's (1999) term *Relations* points to some extent at algebra, but it seems to be a wider concept than structure.

Descartes introduced the world to the metaphor "space-is-a-set-of-points" (Lakoff, & Núñez, 2000). Barton's (1999) QRS-system can be visualized by the following: "Let (q, r, s) be an ordered triple of non-negative real numbers. Then the QRS-system is the space which is generated by (q, r, s)." Each of q, r and s can be described as basic gates of mathematics understanding. Most people have their understanding and interpretations of different kinds of mathematics somewhere in this space. These close connections between the Q, R and S make it difficult to categorise some particular mathematics as just Q or R or S.

Tables 1-2 show a framework for analyses of lasso and lavvu with respect to R and S. Just some aspects of these analysed objects are shown; the intention is to present the framework, not to fulfil these analyses.

	relations	space
geometry	The connection between diameter and circumference	Concept building: Properties of the lasso, the circle, the segment. The concepts distance and direction
algebra	πr^2, $2 \pi r$ and other formulas	

Table 1. A framework for analyses of lasso

	relations	space
geometry	The location of the fireplace and its relation to the length of the rods and the location of the top point. Different locations inside the lavvu	Concept building: Properties of the lavvu, the rods, the lavvu cloth, the cone, the circle, the sector The concepts area, circumference, angle and height
algebra	$V = \pi r^2 h$, $r^2 + h^2 = (\text{rod})^2$ and other formulas	

Table 2. A framework for analyses of lavvu

SUMMARY AND CONCLUSION

The pilot study in this research here points at an example of what Berthelot and Salin (1998) claim; geometrical knowledge is not spontaneously transferred to solve space problems. Work with the lasso gave reasons to believe that students do not intuitively find connections between their experiences from activities outside the mathematics classroom and tasks in their books. Thus one focus in a possible further research on this issue must be to focus on the process that goes on while the students try to connect their practical experiences to the written tasks in their books. There is a risk that when the teacher once has mathematised an element, its mathematics content becomes so obvious to the teacher that she or he forgets the state they were in before this mathematising took place.

The natural use of the lavvu, the lasso and skis take place in "the naturally continuous space" (Lakoff & Núñez, 2000). Mathematising of these elements mainly categorise them into *S*. The framework shown in the tables 1-2 points at the difference between understanding of a concept and of a formula as the latter is an algebraic expression for some geometry.

There seems to be no clear distinction between *Space* and *Relations* on one side and geometry and algebra on the other. Still I will claim that the use of the distinction points to where the mathematics comes from; *Space* and *Relations* focus on the real world while geometry and algebra traditionally focus on the mathematics.

During this work I realised that focusing on concepts of mathematics could prevent me from being open minded towards the outcomes of the mathematising. That is because my view at the Sámi culture is from the outside. I believe this is the greatest challenge to me in a future work in this field.

The structures of the Duodji ornamentations are concrete. Mathematising of the structures in plaiting and weaving leads to permutations which is part of algebra. Based on this a new question arises: Will mathematising of structures in Duodji ornamentations lead to abstract algebra? If so, then mathematising of Duodji could function as a tool to enlighten algebra. It could even be possible that algebra could function as a tool to describe Duodji. This could be an issue for future research.

During this work I have got into more than one discussion about whether ornamented patterns belong to geometry or algebra. My master's thesis (Fyhn, 2000) gave reasons to believe that as for *Space* and *Relations*, students who enjoy taking part in different kind of activities succeed with different kinds of mathematics. By now I know about two ways of mathematising ornamented patterns. My mother tongue is Norwegian which is totally different from Sámi. What I do not know is how people with Sámi mother tongue will mathematise Duodji. Maybe their mathematising will result in one or more new species in the flora of mathematising Duodji ornamentations.

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ARTEFACT MEDIATED CLASSROOM PRACTICE

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This paper reports on a classroom study of meaning being made by students working in small groups in a mathematics classroom at an upper secondary school. A naturalistic study, it observes student activities within the classroom from a socio-cultural-historical perspective. It addresses the artefact mediated classroom practice, wherein student participation and meaning making is guided and apprenticed by the teachers. Mediated by artefacts within intentional practices, the mathematics classroom constitutes a culture which is also the medium for meaning making.

INTRODUCTION

In this paper I discuss classroom participation, classroom practice and classroom culture evidenced in a study conducted by me with students aged 16 to 17 years in a mathematics classroom at an upper secondary school in Norway. My two research questions are: How do artefacts mediated in the mathematics classroom contribute to meaning making? How do artefacts mediated within the mathematics classroom contribute to problem solving?

I base my classification of artefacts on Wartofsky (1979) as primary, secondary and tertiary and give examples of these in the mathematics classroom. Primary artefacts have a representation, which I sub-classify into physical, for example blackboard, notebook, calculator and intellectual, for example language, writing, algebraic notations, graphs. I consider secondary artefacts as ways of working with primary artefacts, for example ways of speaking, ways of solving equations, ways of plotting graphs. Tertiary artefacts are constructions derived and abstracted with primary and secondary artefacts, for example properties of proportional quantities, relationships between consecutive numbers.

In this paper I zoom out from the study of specific artefact mediation and address the practice of the classroom, within which all mathematical activity and problem solving has meaning (Lerman, 2001). I elaborate below how in classroom practice various artefacts and human beings act as mediators of the meaning being made.

The interplay of mathematical artefacts, participants and social practices enables the mathematics classroom to be viewed as a culture. Within this culture, meaning is negotiated and renegotiated by the participants. Interwoven in a discursive prac-

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tice I see classroom culture not only as a pool of artefacts but also as the medium in which meaning is being made (Cole, 1995).

I first outline aspects of socio-cultural-historical theory that I draw upon, throw light on appropriate methodology and specific methods, lead a focus on classroom practice and culture and follow it by data and discussion concluding finally upon analysis.

SOCIO-CULTURAL-HISTORICAL PERSPECTIVE

My study is grounded in the socio-cultural-historical tradition of Vygotsky, Luria and Leont'ev and their followers. It focuses on higher psychological functions which are mediated and socio-cultural in origin. Unlike elementary processes which are biological, higher processes involve auxiliary means; which I call artefacts and which form the medium and are instrumental in mediating psychological activity.

I take the term artefact to refer to artificial aspects of culture which are modified into goal-directed human action. Also referred to as cultural tools, artefacts are both ideal and material. Accumulated in the course of historical experience they provide the medium of human development and history in the present (Cole, 1996). Wartofsky (1979) suggests the crucial feature of human cognitive practice to be the production of artefacts. They present meanings, intentions and relations which are understood in their use and production. Daniels (2001) describes artefacts as imbued with meaning and value through their existence within a field of human activity. As mediators, they serve as the means by which individuals act upon and are acted upon by social, cultural and historical factors.

Vygotsky explains how human beings master their behaviour with auxiliary means in a manner analogous to tools of labour. Whereas tools serve to influence human activity externally, auxiliary means or artefacts influence human behaviour internally. Vygotsky subsumes the mediating function of artefacts under a concept of mediated activity. He identifies 'higher' psychological functions as mediated human behaviour, with language the tool of tools as having a two-sided mediation, directed externally and internally (Cole, 1996; Luria, 1978; Vygotsky, 1978).

I see mathematical meaning being made through the active mediation of artefacts between the teachers and the students corresponding with the classic Vygotskian triad of subject, mediating artefact and object. I follow Kozulin's (1998) classification of mediators and take human beings in addition to physical and intellectual artefacts as mediators of meaning.

I follow Bruner (1990) who sees culture and the quest for meaning within that culture as the cause of human action. Within classroom culture I see the teacher changing an interactive situation from an incidental one to a purposeful one. In my study I observe mathematical artefacts introduced by the teacher gain significance upon use and mediating action within cultural, historical and institutional settings (Wertsch, 1991, 1994). Wertsch, Rio, and Alvarez (1995) describe the task of socio-cultural analysis as explicating the relationships between human action, on one hand,

and the cultural, institutional and historical contexts in which this action occurs, on the other.

Wells (1999) sees artefacts and their mediational means form an integral part of the activity of knowing: with social practices, human participants and intentional inquiry. I follow knowledge building in the formation of what he calls knowledge artefacts. Created in one cycle they mediate and begin theoretical knowing in the next.

CLASSROOM PRACTICE

I take teachers and students as practitioners in the cultural practice of the classroom with their participation in it as the principle of learning. I follow Lave and Wenger (1991) who emphasise cognition, learning, thinking and knowing as relations among people in activity, arising within and from a socially and culturally structured world. They describe participation as situated negotiation and renegotiation of meaning and imply understanding and experience as mutually constitutive. Such participation implies transformation and is generative of artefacts and social structures.

Säljö (1998) explains thinking as an integral part of human practice and learning as appropriating tools, conceptual resources and products of culture. He argues for the analysis of thinking pursued in conjunction with and through artefacts in a situated activity, shared between actors and mediational means. This he says leads to an understanding of how individuals expand their intellectual repertoires and practical skills through participation in collective activities.

Wertsch and Tulviste (1998) caution that though mediated action inherently involves artefacts which fundamentally shape it, this does not mean that action can be reduced to or mechanistically determined by them. They reason that semantic mediation involves an inherent tension between the mediational means and the individuals using them. I thus see the importance of artefacts *becoming* mediational means and find it important to study student participation and apprenticeship in classroom practices.

In guiding student participation, Rogoff (1990) describes the role of language as a cultural form of bridging. By providing more than a labelling function language structures the user's attention to and the understanding of events. She explains and I follow, that participation in a variety of structured strategies is evidence of participating in meaning. Bruner (1990) says that it is by virtue of participation in culture, that meaning is rendered public and shared.

Nickson (1992) describes shared meanings accepted in any practice, the content of that classroom culture, and that teacher-pupil and pupil-pupil interactions determine individual mathematical thinking and meaning making within it. Doyle (1988) explains how teachers' task of orchestrating classroom events is complex and affects student learning. He outlines the problem of meaning in classroom work as serious and that any attempt to reform teaching of mathematics must come to grip with the situational forces that shape curriculum and hold it in place as a classroom event.

Bartolini Bussi (1998) explains that verbal discourse does not exhaust the whole of classroom activity but gives a relevant perspective on the teaching-learning process due to the importance of verbal communication in school settings. Seeger (2001) calls for including the concept of mediated activity in the analysis of discourse. Again Wertsch (1991) explains the analysis of utterances as consistent with a focus on mediated activity. He quotes Bakhtin (1986) who insists that meaning comes into existence when two or more 'voices' come in contact in a socio-cultural context.

METHODOLOGY, METHODS: ETHNOGRAPHIC PRACTICE

The mathematics class I observed used both Norwegian (L_1) and English (L_2), making it possible for an English speaking non-Norwegian to make a study. Permission for data collection was obtained from the Norwegian Social Science Data Services (Project No: 11168 dated 31st Aug 2004 of the NSD). The class was taught by two teachers, sometimes singly and sometimes together, who combined whole class teaching with activities specially designed for group work. The students were seated in small groups at all times. I conducted my study in the culture of the classroom (Geertz, 1973) in an ethnographic tradition (Eisenhart, 1988).

I adopted a naturalist approach (Lincoln & Guba, 1985) which recognizes that realities are multiple and constructed, and interactions are simultaneous and context sensitive. Towards design and analysis emerging from data in a grounded approach (Strauss & Corbin, 1998) I chose a combination of methods so as to yield multiple levels of triangulation to describe the reality of the classroom towards answering my research questions. I collected data in three ways; as a participant observer, a survey element, and audio-recording of a problem solving session.

As a participant observer I made field notes recording utterances, gestures and events *in situ*. With this I began the analysis of meaning making, seeking to understand the mathematics class in terms of its actors. I sectioned my data to coincide with topics of the mathematics curriculum that culminated in a school test. My field notes are made up analytically of two parts: the didactics of the topics by the teachers with the entire class and those that I make by sitting with one particular group (group-in-focus) of the many groups of students. Observation of a group-in-focus helped me record the specific mediation of mathematical artefacts. During year long observation I observed a new group-in-focus for every new topic.

In addition to the textbook, students worked at activities, specifically designed by the teachers. Responses of all students to these activities handed out as worksheets formed part of my survey element which confirmed or questioned the interpretations of my field notes providing insight on majority performance and individual strategy.

I designed and conducted a problem solving task with my group-in-focus looking closely at the mathematical artefacts used within the topic I observed them work on. I audio-recorded and transcribed the task, where for analysis of intentionality and meaning of the speakers I drew upon analysis of my field and survey data.

For this paper I draw chiefly from field notes of my observations in the classroom. Towards data reduction I followed Evertson and Green (1986) and sectioned chunks of classroom discourse into episodes reflecting action directed towards a particular goal. I analysed discussions held and utterances made in English and obtained data triangulation in two ways: among the three or four student participants within a group-in-focus and between the teachers, student participants in the classroom and me while following whole class discussion.

ANALYSIS OF DATA

In this paper I discuss data relating to the first two topics at the commencement of the academic year. The class was taught during the first topic by teacher T1, as teacher T2 was on leave. T2 joined the class from the beginning of the second topic. T1 taught the class in the first topic with students seated in groups around their respective cluster of tables. I had the opportunity to observe how the teaching in the first topic differed from the teaching in the second topic, when T1 knew he would be and was joined by T2.

In my analysis of classroom practice, I have taken classroom activity as my unit of analysis. I have recorded and numbered the utterances and actions within episodes as events. I share extracts from episodes using the following conventions, indicating myself as researcher by RES and students by STD when not sure of their name.

Convention	Explanation	Convention	Explanation
((…))	Researcher comments	………..	Writing on the board
…….	Writing in student notebooks	………..	Excerpt from textbook

I first discuss data belonging to the topic Number and Number Understanding, taught over three and a half weeks and concluding in a school test. I analysed twelve episodes within this topic and found T1 manage many tasks simultaneously. His actions revealed to me that he was conscious that the students were new: to the school, a new level of learning, him as a teacher and his practices. I quote below T1's utterances from classroom practice on different occasions:

1. Turn to page 14 … there are some rules in the box.
2. If you have a problem, box first, partners next, then me.
3. Unfortunately the calculator can reduce any fraction.
4. It is nice to see you using different strategies.
5. There is a simpler way … what is the common denominator.

6. Does it make any sense … two multiplied by itself negative three times?
7. Anything you'd like me to explain or do before we go on.
8. Are these brackets necessary … can we remove them and write it this way?
9. 3^5. How do you do that on your calculator?
10. What do we square here?
11. If the index is zero the answer is one always.
12. Can you remember that … can you be my first class in twenty five years to do so?

T1's utterances evidence a variety of ways in which he encourages participation. I see him guide, direct, explain and raise questions that he thinks the students may have. I draw attention to the various physical artefacts of use: the textbook (1, 2), blackboard (5, 6, 7, 8, 9, 10) and calculator (9). I observe simultaneously the use of many intellectual artefacts: rules in a box (1, 2) and mathematical notation (5, 6, 8, 9, 11). Enculturation of students by T1 into ways of using primary artefacts included: ways of participating in the classroom (2, 4, 7, 12), ways of working with the calculator (3, 9) and ways of working with notations (5, 6, 8, 9, 10, 11).

I now offer an extract from an episode on the first day of teaching. It records the only two utterances made by the students, as against twenty by T1, out of a total of thirty eight events in the episode.

Event	Person	Utterance	Action
11	T1	How do we multiply this	
12	T1		$\frac{16}{15} \times \frac{5}{8}$
13	STD1	We multiply the numerators and the denominators	
14	T1		$\frac{16}{15} \times \frac{5}{8} = \frac{80}{120} = \frac{2}{3}$
15	T1	How else can we do this	
16	STD2	Reduce first	
17	T1	A good idea	

The above extract again evidences the use of primary artefacts: physical (blackboard), intellectual (fractional notation) and secondary artefacts (ways of working with fractional notation). There is also evidence of encouragement that I see T1 provide the students in their participation. However in comparison to the practice that is to follow in the teaching of the second topic, extracts of which I discuss below, I find classroom participation to be limited and teacher driven.

The day T2 joined T1 the students worked at an introductory group activity. The second topic Formulae and Equations was covered over four and a half weeks and inluded a week long autumn break. With this topic I started observing a group-in-focus work at classroom tasks and began to view the class from their point of view. I observed that in the time I observed my group-in-focus; other student groups were having an independent and parallel experience of classroom tasks. The practice of group work in the classroom meant that more mathematics was being discussed at various group tables. In this connection I observed and took note of an important practice of the teachers. To gain access to student working I saw them visit student tables, watching over and guiding both individual and group participation.

I now relate data from the second topic which is representative of how students, from then on, collaborated in groups within classroom teaching. Three students Egil, Lea and Stine are working at a problem from the textbook with hardly any utterances.

Event	**Person**	**Utterance**	**Action**
5	RES		Q2.224: Per weighs 5 kg more than Hans and 20 kg more than Grete. Together they weigh 200 kg. How much does each weigh?
6	RES		Notes group-in-focus students Stine, Lea and Egil use *x* from the beginning.
7	Stine		P = x + 20 + 5 H = x + 20 G = x
8	Lea		$200 = x + x - 5 + x + 20$
9	Egil		((Egil who has worked at the question explains to the others as below)) x | 75 x - 5 | 70 x - 20 | $\underline{55}$ 200
10	Lea and Stine		((Listen to Egil's explanation and take down in their notebooks)) $200 = x + x - 5 + x - 20$ $200 = 3x - 25$ $225 = 3x$ $\frac{225}{3} = x$ $75 = x$

In the above extract I evidence the use of textbook, student notebooks and student 'solutions' through which the three students mediate their understanding of mathematics. Egil, Lea and Stine make independent attempts (7–9) before Egil's attempt convinces Lea and Stine. The above episode is also indicative of the way of working together in groups by the students at classroom tasks.

The teachers who were at different tables had by now observed different individual and group solutions. In my year long observations I found this practice of T1 and T2 very crucial. Through this practice they gathered students' understanding evidenced their actions in their respective groups and the various group solutions that emerged in relation to their teaching. I also observed T1 and T2 exchange notes between themselves. They summarised the above episode as follows.

Event	**Person**	**Utterance**	**Action**
12	T1	There are many ways of doing the question	
13	T1	The question doesn't say use an equation.	((Looking at T2 and speaking))
14	T2		((Nods in agreement))
15	T1	We can use any one way as long as we can 'guess and check'	((Norwegian: Gjette og sjekke))

I see the use of many secondary artefacts in the data discussed above: ways of working by students in a group, ways of observing individual and group solutions by teachers, ways of working together as teachers. I see another important primary artefact being introduced. Attention to 'guess and check' as an intellectual artefact was used to invoke the cultural practice of 'trial and error' in solving problems.

The teachers then set the students to work at the next question from their textbook. They offered guidance with the word 'consecutive' which was not mentioned explicitly in Norwegian in the text book: Find three whole numbers which follow one another so that the sum of the numbers is 123. After allowing time for group work I noted the teachers call attention at the board.

Event	Person	Utterance	Action
20	T1	Someone said the question could be done as x + y + z = 123.	
21	T1		$x + y + z = 123$
22	T1	Is it a good idea?	
23	STD1	No	
24	T1	Why is it not?	
25	STD2	Because you won't find the answer	
26	T1	((Explains in Norwegian while writing))	17 18 19 17 + 1 17 + 2
27	T1	Thus …	
28	T1		$x + (x+1) + (x+2) = 123$ $x + x + 1 + x + 2 = 123$ $3x + 3 = 123$ $3x = 120$ $x = 40$
29	T1	Someone suggested the middle be x	
30	T1		$x \quad + \quad y \quad + \quad z \quad = 123$ $(x - 1) + \quad x \quad + (x + 1)$
31	T1	What will x be now?	
32	STD	Forty one	

In the above extract T1 uses a hypothetical equation for discussion (21). Whether a student's actual attempt or as a rhetorical device, T1 used it as an intellectual artefact. Further I see T1's numerical example (26), equation (28) and reformulation (30) as mediating knowing about consecutive numbers, leading to a tertiary artefact.

After this episode I record T1 use the sequence (20–32) as an example or another intellectual artefact to set as task the next question: Find five even numbers which follow one another so that the sum of the numbers is 240. After another round of working T1 brought teaching to conclusion as below.

Event	Person	Utterance	Action
37	T1	Perhaps we can take your problem now	((Addressing a particular group))
38	RES		Notes Ulrik come forward and explain while writing at the board
39	Ulrik	They weigh 200 together. So --	
40	Ulrik		$200 = P + (P - 5) + (P - 20)$
41	Ulrik	This is because --	
42	Ulrik		((At another location)) $P = \text{Per}$ $\text{Hans} = P - 5$ $\text{Grete} = P - 20$
43	Ulrik		$200 = 3P - 25$
44	Ulrik	Move this over to the other side	((Pointing to 25))
45	Ulrik		$\frac{225}{3} = P$ $75 = P$
46	RES		Notes Ulrik explain as he proceeds
47	RES		T1 acknowledges and accepts Ulrik's solution and goes up to the board.
48	T1	You could have written this the other way	
49	T1		$P + (P - 5) + (P - 20) = 200$

In the extracts from the episode relating to the second topic I see an increase in student participation. I recorded fourteen student utterances against twenty one by the teachers, out of a total of fifty three events. However it is the change in the kind of participation by the students and teachers that is important. Ulrik spoke on behalf of his group mates (39–46) and explained their group solution. This enhanced the use of the blackboard in classroom practice. It now became a forum for classroom discussion, on alternate strategies and solutions to problems.

DISCUSSION

From a teacher driven practice in the teaching of the first topic I observe a discursive practice set up in the teaching of the second topic. There were a number of ‘ways of working’ or secondary artefacts that constituted classroom practice. This allowed for primary: physical and intellectual artefacts, like blackboard and ‘guess and check’ to become mediational means (Wertsch & Tulviste, 1998).

While discussing the first question on consecutive numbers T1 built a tertiary and knowledge artefact and used it to mediate knowing in the next question (Wells, 1999). In bringing teaching to conclusion with the display of Ulrik’s group solution T1 and T2 established an intentional practice and apprenticed students in the praxis of mathematics (Rogoff, 1990). I see the discussion of problem solutions as instances of students appropriating conceptual resources and thinking, pursued in conjunction with and through artefacts in a situated activity as pointed out by Säljö (1998).

The situated practice established by the teachers (Lave & Wenger, 1991) provided students the opportunity to voice their mathematical thinking. They could observe, share, conjecture, reason, convince and be convinced. Various artefacts primary, secondary and tertiary were part of the culture nurtured and constituted within the mathematics classroom. Participation within this provided opportunities by which artefacts *became* mediational means. Several instances allowed students to control their behaviour through artefacts *as* mediational means (Vygotsky, 1978).

Human mediators (T1, T2, students) were part of the negotiation and renegotiation of mathematical meaning. Students ‘voiced’ their thinking (Bakhtin, 1986) with and through their actions and teachers spoke with and through their actions and practices. Meaning made through collaborative inquiry was constantly made public and shared (Bruner, 1990) and became part of classroom events (Doyle, 1988).

CONCLUSION

Constituting classroom culture as a discursive practice allowed for many primary artefacts within it to become mediational. Various secondary artefacts allowed classroom practice to become participatory. This allowed opportunities for students to voice their understanding in mathematics. Depending on the context primary, secondary and tertiary artefacts mediated different meanings. The thoughtful and intentional praxis established by the teachers enabled a medium of meaning making and classroom culture to be mutually constituted.

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PRESERVICE TEACHERS' CONCEPTIONS OF THE FUNCTION CONCEPT AND ITS SIGNIFICANCE IN MATHEMATICS

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The purpose of this paper is to examine preservice teachers' conceptions of the function concept as well as their conceptions of the significance of functions in mathematics. A further purpose is to study the effects of an intervention regarding the function concept. Two groups of preservice teachers who are specializing in mathematics and science are participating in the study. The findings indicate changes, firstly in the preservice teachers' view on the function concept related to the intervention, and secondly with respect to different themes of relevance in their reasoning on the significance and presence of functions in various contexts.

INTRODUCTION

The concept of function is one of the central concepts underlying mathematics. It is important that preservice mathematics teachers acquire a well-developed understanding of the function concept, partly to become successful in their studies in mathematics but also critical in their planning and teaching of mathematics (e.g., Even, 1993; Even & Tirosh, 2002; Lloyd & Wilson, 1998; Vollrath, 1994). Having a deeper understanding of the function concept can help teachers to make knowledgeable decisions about the place of functions in the curriculum and create settings for their students to become aware of the function concept as a powerful mathematical idea. However, previous studies indicate that preservice teachers' conceptions of function might possibly not be as well-developed as one may expect of proficient mathematics teachers (e.g., Even, 1993; Hansson, 2004; Wilson, 1994). The aim of the current study is to examine preservice teachers' conceptions of the function concept as well their conceptions of the significance of functions in mathematics. A further aim is to study the effects of an intervention concerning the concept of function.

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RESEARCH QUESTIONS

Based on the aims of the study the research questions are: What conceptions do preservice teachers have of the function concept? How do preservice teachers view the significance of functions in mathematics? How does the intervention make a difference to the preservice teachers' conceptions of the function concept?

THEORETICAL FRAMEWORK

The framework relates to principles of constructivist learning where knowledge is an individual construction built gradually (seen as e.g., Novak, 1993; Steffe & Gale, 1995). Understanding is described in terms of building mental structures where previous built structures affect subsequent constructions. To promote understanding includes important dimensions in mental activities of the learner such as constructing relationships, articulating what one knows, extending and applying mathematical knowledge, reflecting about experiences etc. A central part of the theoretical framework is the notion of concept image (Tall & Vinner, 1981; Vinner, 1983, 1992; Vinner & Dreyfus, 1989) as well as theory related to the notions of learning with understanding (Carpenter & Lehrer, 1999; Hiebert & Carpenter, 1992) and meaningful learning (Ausubel, 2000; Ausubel, Novak, & Hanesian, 1978).

Ausubel (2000; Ausubel et al., 1978) describes learning in terms of meaningful learning as opposed to rote learning, where learning becomes meaningful when the learners are able to relate new knowledge to prior knowledge. In order to obtain successful learning students must acquire knowledge actively and establish relationships between concepts to be learned and the ones the students know. Hiebert and Carpenter (1992) refer to Ausubel's theories as a "bottom-up approach" in the way knowledge develops building upon prior knowledge, and describe a view of learning with understanding similar to meaningful learning. That is, understanding grows as an individual's cognitive structures become larger, more organized and richly connected networks of knowledge. Understanding can be rather limited if only some of the mental representations of potentially related ideas are connected. These ideas are consistent with a more concrete approach by Carpenter and Lehrer (1999) considering how students construct meaning for mathematical concepts and processes and how classrooms support that kind of learning.

An individual's understanding of the function concept and its correspondence to the definition of function is by Vinner (1983, 1992; Vinner & Dreyfus, 1989) described by utilizing the notion of concept image, which is applicable to formal concepts in general (Vinner, 1991). A concept image is in the current framework considered to consist of all parts of cognitive structure associated with a concept in the mind of an individual, "including all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152). In thinking, different parts of the concept image are evoked at different times. During an individual's reasoning, the concept image will almost always be evoked, whereas the concept definition will remain inactive or even be forgotten. When students meet an old concept in a new context, it is the concept image, with all the implicit assumptions abstracted from earlier contexts, that responds to the task.

The framework is further discussed below in relation to the intervention in the study.

METHOD AND PROCEDURE

The current study is part of a larger study related to the views of preservice teachers on the concept of function. It is conducted during two consecutive spring terms with two groups of preservice teachers, one group for each term, in relation to a course in calculus in which the function concept is a central concept. The calculus course is part of the concluding courses in mathematics during the sixth term of a four and a half year teacher preparation program. Both groups of preservice teachers are specializing in mathematics and science for grades four to nine. They comprise twenty-five and sixteen students[1], respectively. The study is conducted at a smaller[2] university in Sweden and the two groups of preservice teachers make up all the third-year students specializing in mathematics and science on the education programme.

A questionnaire was completed on two occasions, before and after the calculus course. Twenty-two students in the first group and fifteen students in the second group answered the questionnaire on both occasions. The questionnaire comprises eight questions that the students typically spent half an hour or more to answer. Shortly after the second questionnaire was distributed volunteer preservice teachers were interviewed individually. The interviews were based on the questionnaire, and the preservice teachers were asked to study their responses from before and after the calculus course, for each question, and to comment on them. Follow-up questions were asked to clarify the preservice teachers' reasoning. The time for each interview varied but often continued for an hour or more. Each interview was recorded on tape and transcribed. Twenty students in the first group and nine students in the second group were interviewed. The interviewed students represent, in each group, students on a variety of levels in their studies of mathematics.

Two of the questions in the questionnaire are considered in this paper. The first question is "Describe in your own words your interpretation of the concept of function" and the second is "Give your opinion about the extent to which functions are of significance in mathematics. Provide reasons for your opinion". The questions are open ended, and they frequently became a starting point for further reasoning

[1] In this paper the preservice teachers of the study are sometimes called students, since they are students on the teacher preparation program.

[2] There are approximately 10000 students enrolled at the university with about half of the students in full-time study.

about the function concept in mathematics and school mathematics by the preservice teachers during the interviews.

The second group of preservice teachers also participated in an intervention that was partly designed with respect to data from the first group in the study. The intervention is further described in a section below. (Because the study described in this paper is part of a larger study, more data was collected from both groups of preservice teachers during the period the study was conducted. For instance, concept maps, preservice teachers' solutions to mathematical problems, video recordings of preservice teachers working in groups etc. These data are however not used in the current paper.)

Categorizations used in the study

The students' conceptions of the function concept were categorized following a categorization scheme presented by Vinner and Dreyfus (1989). Furthermore, a categorization of the preservice teachers' accounts of the significance of functions in mathematics was derived from a number of distinctive themes that appear in their answers to the questionnaire. The categorization of the significance of functions is further described in the section of results from the study.

Different groups of college students, in addition to inservice teachers in mathematics, participated in the study conducted by Vinner and Dreyfus (1989). A categorization based on their conceptions of the function concept was developed. The categorization consists of the following seven categories:

I. A function is any correspondence between two sets that for each element in the first set assigns exactly one element in the second set (the Dirichlet-Bourbaki definition).
II. A function is a dependence relation between two variables.
III. A function is a rule, which is expected to have some regularity (whereas a correspondence may be "arbitrary").
IV. A function is an operation or a manipulation (one acts, e.g., on a given number, in order to get its image).
V. A function is a formula, an algebraic expression, or an equation.
VI. A function is identified with a representation, possibly in a meaningless graphical or symbolic form.
VII. Others.

The categories are derived from the answers to the questionnaire where the participants were asked to give their opinion on what a function is. The current study applies the categorization[3] in the same manner as Vinner and Dreyfus, where an answer only is allotted to one of the seven categories, and thus makes it possible

[3] During the preparation of the current paper, Vinner and Dreyfus confirmed in a correspondence that they have not made any changes to the categorization or later developed it further.

to compare results from the two studies. The categories are, however, not disjoint, and from the samples of answers that are presented by Vinner and Dreyfus one may conclude that among the four categories II, III, IV, V a higher numbered category is generally given higher precedence in deciding to which category an answer will be assigned. Also from the given sample of answers, the four categories can often be seen as a sequence of decreasing subsets; this was also observed in the current study. Moreover, if an answer contains a description of function equivalent to the definition, then it is always allotted to category I. The categories VI and VII will be considered if an answer is not applicable to any of the categories I–V.

THE STUDY BY VINNER AND DREYFUS

The participants in the study by Vinner and Dreyfus (1989) were drawn from several groups of first-year college students as well as a group of junior high school mathematics teachers in inservice training. The study makes it possible to compare a range of different groups of students and the preservice teachers in the current study. The college students majored in different areas and were divided into four groups by the level of mathematics courses required for their majors: Low level, 33 students majoring in industrial design; intermediate level, 67 students majoring in economics or agriculture; high level, 113 students majoring in chemistry, biology, or technological education; and mathematics level, 58 students majoring in mathematics or physics. Moreover, a fifth group consisted of 36 junior high school mathematics teachers.

Category		I	II	III	IV	V	VI	VII
Participants	Low	2 (6)	12 (36)	4 (12)	2 (6)	6 (18)	4 (12)	3 (9)
	Intermediate	12 (18)	18 (27)	9 (13)	1 (1)	13 (19)	6 (9)	8 (12)
	High	17 (15)	36 (32)	9 (8)	7 (6)	8 (7)	11 (10)	25 (22)
	Mathematics	26 (45)	12 (21)	7 (12)	3 (5)	3 (5)	3 (5)	4 (7)
	Teachers	25 (69)	3 (8)	3 (8)	1 (3)	0	1 (3)	3 (8)

Table 1: Distribution of conceptions of the function concept for the participants in the study by Vinner and Dreyfus (1989), number and percentage x (y).

The study is conducted before the concept of function was introduced in the courses and focus on participants who gave an account of function that is consistent with the definition of function, that is category I. Due to limited space, the reader is directed to Vinner and Dreyfus (1989) for further details.

THE INTERVENTION

The intervention was conducted during the first two weeks of a five-week calculus course. I implemented the intervention, and the same teacher who had supervised the first group in the study supervised the remaining part of the course. The preservice teachers were informed about the intervention but not about the research questions of the project. The intervention also became apparent for the students for instance as a result of handouts that complemented or replaced the course literature during the execution of the intervention. Some ideas and purposes of the intervention that are related to the research questions in this paper are described below.

The concept of function was introduced as a special case of the concept of relation[4]. One reason for this approach was to give the students a concept to relate to with consequences for meaningful learning (Ausubel, 2000; Ausubel et al., 1978), and learning with understanding (Carpenter & Lehrer, 1999; Hiebert & Carpenter, 1992), and appropriate language in which they could express themselves in situations of a more general context related to the concept of function. Based on experiences from the first group in the study (where the students' conceptions of function frequently were limited to a formula, an algebraic expression, or an equation), the inclusive notion of relation was intended as an anchoring idea for the concept of function (Ausubel, 2000; Ausubel et al., 1978). The concept of relation was considered as an association between objects, in everyday situations as well as in mathematics, and as such thought to create opportunities for meaningful learning and render possible for the students to further develop their view on the concept of function.

An individual's concept image of the function concept is constructed during a longer period of time and is based on experiences of all kinds of the concept. Different portions of the concept image, that is evoked concept images, are used at different times in an individuals reasoning (Tall & Vinner, 1981; Vinner, 1983, 1992; Vinner & Dreyfus, 1989). An individual's concept image may be based on experiences that do not necessarily form a concept image that is consistent with the definition of the concept. One intention of the intervention was to stimulate feedback from evoked concept images to the concept definition. This would give the preservice teachers an opportunity to further develop their concept image of function in different contexts, and construct a concept image that is more consistent with the Dirichlet-Bourbaki definition of function. For this reason, the concept of function was given in a range of examples and problems with varying domain, codomain and representation, in applied as well as in non-applied contexts (which

[4] A (binary) relation was given as an association of elements between two non-empty sets, rather than a subset of a Cartesian product. A function was thereafter given as a special case of relation; which for each element in the first set (the domain) gives exactly one corresponding element in the second set (the codomain), and thus satisfying the criterion of uniqueness.

is coherent with results by, e.g., Vinner & Dreyfus, 1989; Schwarz & Dreyfus, 1995).

Feedback from an evoked concept image to the concept definition is usually not established with standard types of problems, according to Vinner (1992). This justifies the inclusion of examples and problems that traditionally are not given in a calculus course. For instance, functions between finite sets, the number of such functions, and the use of sets other than sets of numbers (this was mainly part of the first week of the intervention). The second[5] week of the intervention was generally dedicated to real valued functions in one real variable and their different properties. Furthermore, the preservice teachers were given opportunities to reflect upon the concept of function and the presence of functions in mathematics and school mathematics, for instance in assignments with the task to formulate problems where the function concept was considered to be present to a varying degree. These are examples of activities consistent with principles of constructivist learning were preservice teachers are invited to reflect upon the concept of function and articulate what one knows about functions and their relations to other concepts etc. (Carpenter & Lehrer, 1999).

RESULTS

The concept of function

Before the calculus course, category V is the dominant category in the first group, while the students' conceptions of the function concept are somewhat more evenly distributed among the categories in the second group with V and VI as the two largest categories, see Table 2. In both groups of students, the two categories V and VI together comprise half of the students' answers before the course, that is answers reflecting a view of the function concept as a formula, an algebraic expression, an equation, or a representation possibly in a meaningless graphical or symbolic form. None of the students gives an account of a view that is consistent with the definition of function. The students' conceptions of function are distributed over five categories in each group, where no answers are allotted to category III in the first group or category IV in the second group, that is answers describing a function as a rule or an operation, respectively. (However, this does not exclude the case where an answer allotted to a higher numbered category includes conceptions related to a lower numbered category, for the categories II–V). In comparison with the study by Vinner and Dreyfus (1989), the preservice teachers' conceptions of function are in both groups, before the calculus course, more similar to students in the lower level groups than to students majoring in mathematics and physics or the inservice teachers.

[5] Further elaboration on the subject of preservice students' view on properties of different classes of functions is planned to be presented in a future paper.

After the calculus course, conceptions related to category V are yet more dominant in the first group than before the course, that is answers that give account of a function as a formula, an algebraic expression, or an equation. (However, these answers may also include parts related to categories II–IV, for example a dependency relation between two variables). There are still five categories represented in the first group, where category V now is more than twice as large as any other category. Moreover, answers related to category VI are no longer represented, that is conceptions of a meaningless graphical or symbolic representation of function. The students' answers now also represent category III with conceptions of function as a rule. As before the course, none of the students' answers in the first group describe the function concept in a way that is consistent with the contemporary definition of function.

While category V is the dominant category in the first group, both before and after the calculus course, there are more significant changes in the students' conceptions of the function concept in the second group. The previously two largest categories V and VI are no longer part of the students' answers after the course, while category IV now is the largest category in the second group, reflecting a view of function as an operation. (This is sometimes a less dramatic change in a student's conception of function, where parts of a previous answer in category V may include conceptions related to category IV). After the course, four categories are covered by the students' answers in the second group where categories V, VI, and VII are no longer represented. Only one student's answer is consistent with the contemporary definition of function, that is a view of the concept which includes a domain, a codomain, and a correspondence that satisfies the uniqueness criterion. However, answers in category IV in the second group often contain conceptions of the function concept which involve an operation where the uniqueness criterion is made explicit but the answers lack the notion of domain and codomain, as well as the notion of set. The lack of domain and codomain is a significant reason for the large differences in the distribution of answers after the calculus course when the second group is compared to the group of inservice teachers in the study by Vinner and Dreyfus (1989).

Table 2 below displays the distribution of students' answers for the 21 preservice teachers in the first group and 15 preservice teachers in the second group who gave their interpretation of function both before and after the calculus course. The table displays number and percentage x (y), respectively. The number of students who gave their opinion on the concept of function both before and after the calculus course was reduced. This was due to students' absence from lectures at the time the survey was carried out and due students who before the calculus course did not answer the question in the survey.

Category	I	II	III	IV	V	VI	VII	
The first group	Before	0	5 (24)	0	3 (14)	8 (38)	4 (19)	1 (5)
	After	0	4 (19)	4 (19)	1 (5)	11 (52)	0	1 (5)
The second group	Before	0	2 (13)	3 (20)	0	4 (27)	4 (27)	2 (13)
	After	1 (7)	2 (13)	5 (33)	7 (47)	0	0	0

Table 2: Distribution of students' answers.

During the interviews it was common that the preservice teachers express an uncertainty in their interpretation of the function concept, and to a higher degree in the first group. It was not unusual with responses like "… I am still uncertain about functions. After five weeks of calculus, I am still uncertain about functions. It feels very silly, but that's the way it is." (F10[6], in the first group). The interviewed students in the second group could usually give a description of function that was more consistent with the contemporary definition when given an opportunity to reflect upon their two answers. However, this was not a conception of function that seems to be spontaneously evoked. Students in the first group gave the impression to be less aware of a definition of function, and during the interviews they were more likely to mix terminology related to the concept of function with concepts associated with category V, like the concept of equation, formula, or a mathematical expression. In their reasoning about functions, students in the first group also more frequently seem to think of functions in terms of symbolic manipulations and procedural techniques.

Few students included the notions of domain and codomain in their description of the function concept, in the questionnaire or during the interviews, but usually related to numbers in the context of function. It was also typical for students from the first group to limit their view on different forms of representation of function to a symbolic or visual representation. Students in the second group may during the interviews however refer to examples of functions from the intervention with domain and codomain other than sets of numbers as well as a larger variety of representations. The notion of set was almost absent in the students' answers to the questionnaire and seems to be a notion that forms an obstacle in the preservice teachers' interpretation of domain and codomain as two components of the function concept. The students would during the interviews sometimes use terms like "bubbles" or "groupings" and so on, in the context of set, which made it obvious that they were not really accustomed to the concept of set.

During the interviews, the students, primarily in the second group, referred to the language used in mathematics – this was also a theme brought up by the students in relation to lectures during the intervention. The preservice teachers noticed that a chosen name for a concept did not always seem to have any meaning for them and did not help them to clarify the meaning of the concept. In this context the

[6] Code (F-female) used during data analysis

preservice students in the second group referred to the concept of relation as a concept they believed to be more true to their everyday experiences of the notion, than the notion of function. This was for instance the case for the student (M2[7]) in the second group, who in response to the questionnaire after the calculus course described the notion of function in a way that is consistent with current characterizations of function. For him the concept of relation felt natural and like "a word that you recognize", and he applied it in a variety of contexts during the interview, and then looked upon the concept of function as a special type of relation.

The significance of functions in mathematics

A number of distinctive themes appears in the preservice teachers' answers to the question of the significance of functions in mathematics, which gives rise to the following categorization together with samples of answers:

1. Functions in different applications of mathematics: "Functions are 'real' math where relations can be described that are taken from reality", "To theoretically be able to describe a practical example", "Good examples are everyday situations as fuel consumption (the cost of this) for a certain distance [for a vehicle]".
2. Presence of functions in mathematics: "Functions are part of quite a lot of mathematics, but you really don't think about it", "Most of mathematics is built by functions, so that's why they are of importance", "I guess that harder mathematics uses functions".
3. Use of functions in mathematics: "Functions are useful to 'see' math!", "They are used to solve problems in mathematics which can be solved both graphically and by calculations", "To describe relations".
4. Functions in the context of teaching and learning mathematics: "Pupils can learn to illustrate relations graphically", "If the functions are related to the pupils' interests, then I think its good", "In ninth grade you (the pupils) don't think it is relevant, I suppose".
5. No opinion: "I don't know!", "?", "I have forgotten how, when and why you use functions".
6. Other: "… I think it is fun … (Maybe more [people] do so, then it is significant!!)", "Important to know what a function is to be able to consider yourself mathematical", "You should be able to calculate with unknown numbers, variables".

More than a third of the preservice teachers' answers in both groups is allotted to category 5 before the calculus course. They are students who did not know how to answer a question on the significance of functions in mathematics. After the course

[7] Code (M-male) used during data analysis

all students gave an opinion on functions' significance in mathematics, see Table 3 below. The table displays the distribution of students' answers on functions' significance in mathematics, number and percentage x (y). Participants are the 22 preservice students in the first group and 15 preservice students in the second group, who took part in the survey at both occasions before and after the calculus course. In the survey, 6 students in the first group and 4 students in the second group did not give any answer to the question on the first questionnaire. They are assumed to have no opinion and have been allotted to category 5.

Category		1	2	3	4	5	6
The first group	Before	10 (45)	3 (14)	3 (14)	2 (9)	8 (36)	5 (23)
	After	12 (55)	2 (9)	9 (41)	5 (23)	0	2 (9)
The second group	Before	3 (20)	0	3 (20)	1 (7)	6 (40)	2 (13)
	After	8 (53)	6 (40)	3 (20)	2 (13)	0	2 (13)

Table 3: Distribution of students' answers on the significance of functions in mathematics.

In the first group, a major theme in the students' answers to functions' significance in mathematics is related to the use of functions in applications of mathematics, represented by category 1. Almost half of the students' answers relate to category 1 before the calculus course; that is more than two-thirds of the students who actually expressed an opinion on the significance of functions in mathematics. The significance of functions in applications of mathematics is still a major theme after the calculus course. It is a theme that is part of more than half of the students' answers in the first group; this represents however a smaller part of those students who actually gave an opinion on the significance of functions than before the course. The applications of functions that the students give are frequently related to the areas of science, usually physics. Furthermore, there is a threefold increase of answers related to the use of functions in mathematics, that is category 3, which often is connected to the use of functions in standard problems from the calculus course, and this is the second largest category after the course. Moreover, answers that relate to teaching mathematics, category 4, have more than doubled after the course and represent almost one-quarter of the students' answers, while answers in category 6 – answers that not always seem to make sense – have decreased.

In the second group, before the course, answers with no opinion on the significance of functions in mathematics, that is category 5, are twice as frequent as for any other category. After the course all students gave an opinion on the significance of functions in mathematics, where the two major themes are: functions in applications of mathematics, and the presence of functions within mathematics, that is categories 1 and 2. Half of the students' answers relate to category 1 after the course, which is almost a threefold increase. Furthermore, more than one-third of the students' answers relate to category 2, which is a category that did not exist before the course. Answers in category 2 are basically of two types of reasoning: the first type of reasoning is based on a quantitative view from the students' experience of functions in different contexts, and the second type of reasoning is to a higher degree based on the ability to perceive the concept of function in different situations. The numbers of answers that concerns categories 3 and 6 have not changed in the second group, while answers related to teaching in mathematics, that is category 4, have increased although this is still a small category.

The preservice teachers perceived the significance of functions in mathematics motivated from their importance in different applications of mathematics and often in the area of science. Applications in the area of physics are more dominant in the first group both before and after the course. Whereas in the second group, applications of functions are less frequent before the course while the majority uses it as an argument in their answers after the course, but with fewer references to science than in the first group. Moreover, connections to teaching were only made by a few students in the two groups in relation to the significance of functions in mathematics.

During the interviews, students in the first group did to, a higher degree, relate to their experiences of the function concept in physics, as a formula etc., in the context of the significance of functions in mathematics. Moreover, students in the first group usually restricted their view of functions to real valued functions in one real variable. In the interviews with students from the second group, the use of functions in science was also a central part in their reasoning of the significance of functions in mathematics, but they did, in addition, relate to functions with domain and codomain other than sets of real numbers. From the interviews with students it became obvious, in both groups, that applications of functions in real world situations were an important factor in their reasoning about the significance of functions in mathematics.

In the context of applications of functions during the interviews students often seemed to relate to functions as a dependency relation between two quantities. After further consideration students frequently came to the understanding that such relations are "all around us" and believed that this is something you usually do not talk about. Many students thought the question of functions' significance in mathematics was a hard question to answer, and some students felt that they did not really have the knowledge to answer the question.

CONCLUDING DISCUSSION

The study implies that a more traditional calculus course may strengthen preservice teachers' conceptions of the function concept that are inconsistent with current characterizations of function. While their conceptions of function are more similar before the course there are clear indications of different conceptions of the function concept between the two groups of preservice teachers after the calculus course. One significant difference between the groups is that a function is, after the calculus course, to a higher degree considered as an operation in the second group. This is in contrast to the first group, where the previously most frequent conceptions of function are strengthened after the course, that is conceptions of function as a formula, an algebraic expression, or an equation. Students in the first group are also less aware of a definition of function and more frequently tend to mix terminology related to other concepts in the context of functions. It was however observed during the interviews that the preservice teachers could vary their reasoning to some extent and focus on different aspects of functions in different contexts. For example a dependency relation seemed to be of higher relevance for the students in the context of applications of functions.

Few students, including students in the second group, gave an account of the notions of domain and codomain as two components of the function concept. The notion of set seemed in this context to be an obstacle in the preservice teachers' interpretations of the definition of function. Students in the first group almost exclusively associated numbers with functions, that is real valued functions in one real variable, usually in a symbolic or graphical form. This is not the case in the second group in which students during the interviews were usually able to reason about functions with different representations and in contexts not always related to numbers.

Preservice teachers in the second group said they "recognized" the notion of relation from their everyday experiences and that the use of relation in mathematics felt in accord with their view of the concept as an association between objects. The students' previous conceptions of the notion of relation seem to be important for their ability to reason about functions with domain and codomain other than sets of numbers and about functions with a variety of representations, in the case a function is known as a special type of relation. Students' preexisting ideas of the notion of relation appear to be an essential conceptual infrastructure to which the notion of function can be anchored – in the sense of Ausubel (2000), analogous to the notion of cognitive root by Tall (1992) – and seem to have the potential to play an important role in students' conceptual development of the function concept. The study indicates that with the notion of relation as an anchoring idea, the function concept is less likely to be learned by rote or limited to real valued functions or linked to a specific representation.

The significance of functions in mathematics is by the preservice teachers frequently justified by the importance of functions in different applications of mathematics. This is a major theme in the first group both before and after the calculus course, with reference to science and especially physics which is most likely a consequence of their experiences of functions in physics. In the second group, applications of functions also became a major theme after the calculus course, with reference to a variety of applications. The significance of functions in mathematics was for the students to a high extent connected to the ability to apply functions in different real world contexts. The students' conceptions of functions' significance in mathematics indicate that applications of functions are relevant in the development of their concept image of function, and a factor of importance to create opportunities for meaningful learning (Ausubel, 2000; Ausubel et al., 1978) and learning with understanding (Carpenter & Lehrer, 1999; Hiebert & Carpenter, 1992).

The idea that teachers' conceptions of function will influence the quality of the understanding developed by their students has received support from research findings (e.g., Even & Tirosh, 2002; Lloyd & Wilson, 1998). A teacher's view on functions (for instance, as a formula, an algebraic expression, or an equation) will most likely have consequences for the students' conceptions of the function concept. One might expect both lesson goals and structures to be consistent with teachers' views of functions, and thus expressed by examples they put forward, activities they design, questions they ask, and ideas they consider of value and so forth.

The current study illustrates that preservice teachers need support to develop their view of the function concept. In comparison with the study by Vinner and Dreyfus (1989), the preservice teachers' views on the concept of function tend to be similar to students majoring in less mathematics intense fields. This might further indicate that preservice teachers need opportunities to further develop their conceptions of function and meet functions in a variety of contexts. Contexts that provide opportunities for preservice teachers to realize that the concept of function is a central concept in mathematics and an important concept to introduce to their future students.

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PROBLEM SOLVING – WHAT AND HOW DO PUPILS LEARN?

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This study forms a part of a project, RIMA, where pupils aged 13–16 years and their teachers were working with ten so-called rich problems during mathematics lessons. In this paper I discuss the work of two groups which were video and audio tape recorded during their work. They were also interviewed individually after the problem solving sessions, and their written solutions were collected. In this way I tried to find out what opportunities for learning arose during the problem solving and how the pupils made use of these opportunities. I found that most of the pupils, though bewildered in the beginning, could find procedures and strategies to solve the problem and in that way got a better understanding of the concepts involved.

INTRODUCTION

This study forms a part of a project, RIMA, carried out by two of my colleagues and me, where four classes in years 7–9 (pupils aged 13–16 years) and their teachers were working with ten so called rich problems during lessons in mathematics. Together we have made up seven criteria for a mathematical task to be called a rich problem. These criteria are:

1. The problem should lead into important mathematical ideas or certain problem solving strategies.
2. The problem should be easy to understand, and every pupil should have the capability to work with it.
3. The problem should be experienced as a challenge, demand an effort, and be allowed to take time.
4. The problem should be solvable in many different ways, with different strategies and representations.
5. The problem should initiate a mathematical discussion from the pupils' different solutions, a discussion that points to different strategies, representations and mathematical ideas.
6. The problem should act as a bridge between different mathematical fields.
7. The problem should influence teachers and pupils to formulate new interesting problems.

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The teachers and the three researchers jointly decided on the problems but the teachers designed the problem solving sessions on their own.

In this paper I discuss one of the problem solving sessions in two classes with respect to where opportunities for learning came up and how the pupils made use of these possibilities or could not make use of them. We found that this problem turned out to be a rich one in the actual classes. However, I will not discuss this issue here owing to space limitations.

BACKGROUND

As a point of departure for the study I have used social constructivism. I have been especially interested in the link between the individual and collective aspects of mathematical development, as it is discussed by others, for example Cobb, Boufi, McClain, and Whitenack (1997), and which, in my opinion, is clearly put forward in the following statement

> In this formulation, the link between discourse and psychological processes like reflective abstraction is indirect. This perspective acknowledges that both the process of mathematical learning and its products, increasingly sophisticated mathematical ways of knowing, are social through and through. However, it also emphasizes that children actively construct their mathematical understandings as they participate in classroom social processes. (p. 264)

This would mean that the individual will, to a great extent, be helped by the common discussion either in class or in a smaller group, to reflect upon mathematical concepts, procedures, and strategies. But at the end, it will be the pupils themselves that construct their own knowledge.

This position is also taken by Hershkowitz (1999). She refers to the amount of research that has been done on interactions in groups of pupils and she considers the shared construction of knowledge important. But, at the same time, she sees the individual as the “smallest autonomist 'unit' that can carry his/her constructed knowledge to different communities even simultaneously (the pair, the small group, the whole classroom community), and share it with other individuals in various communities during a lifetime” (p. 12).

Goos, Galbraith, and Renshaw (2002) discuss the same matter in connection with peer groups and ZPD (Vygotsky's zone of proximal development, (e. g. Vygotsky, 1978). They refer to Vygotsky's observation that children playing together are able to play above both their own and their partner's normal level of development (see Minick, 1987) and add:

> Applied to educational settings, this view of ZPD suggests there is learning potential in peer groups where students have incomplete but relatively equal expertise – each partner possessing some knowledge and skill but requiring the others' contribution in order to make progress. (Goos et al., p. 195)

On the other hand, Kieran (2001) has shown that it is not always certain that a dialogue between two co-operating pupils will be equally productive to both of them.

It seems important that both partners are active in the discussion, that each one of them has an opportunity to take part in the other's thinking. If once one of the partners has found the key to the solution the interaction between the pupils decreases, which will have a limiting influence on the other pupil's knowledge construction.

Another pair of notions that will come into play in the analyses is concept and procedure and how one is dependent on the other. Many researchers have stressed the link between these notions, for instance, Hiebert and Lefevre (1986) say: "Linking conceptual and procedural knowledge benefits conceptual knowledge as well as procedural knowledge. The benefits for conceptual knowledge are cited less often but are equally significant" (p. 14). The connection between these notions and problem solving is stressed by Silver (1986): "A major thesis underlying this paper is that mathematical problems are useful vehicles for the study of conceptual and procedural knowledge, since problem solving typically requires the application of both kinds of knowledge" (p. 191).

More recently the term "procept" has been used to stress the connection between the two notions. For instance, Gray and Tall (2001) write: "When the symbols act freely as cues to switch between mental concepts to think about and processes to carry out operations, they are called *procepts*" (Gray & Tall, pp. 67–68, original emphasis). These authors distinguish between procedure as step-by-step algorithms and process as "procedures (having the same overall effects) as seen as a whole" (p. 67).

Sfard (e. g. 1991) uses the terms "operational concepts" and "structural concepts" to denote procedures and concepts as mentioned above. By doing this, she stresses the importance of the interplay between these two notions. She says: "... the terms 'operational' and 'structural' refer to inseparable, though dramatically different, facets of the same thing. Thus we are dealing here with *duality* rather than *dichotomy*" (p. 9, original emphasis).

Although considering Sfard's ideas about the notions, I will stay with the terms "conceptual knowledge" and "procedural knowledge" in this paper. I see them as different kinds of knowledge, separated but with a strong link between them.

AIM AND RESEARCH QUESTIONS

The aim of this study was to find out what pupils in lower secondary school learn by working with rich problems. However, I realised that this is quite impossible to decide in a study like this and therefore I concentrated on the opportunities for learning that could be observed during the course of problem solving and how the pupils used these opportunities. Thus, I hope to be able to answer the following research questions:

1. What opportunities for learning might arise when pupils are working with a rich problem?
2. How do the pupils make use of these opportunities?

METHOD

Design and data collection

The problem used in this study was called The Collector's Pictures:

Five pupils have a number of collector's pictures each. The one who has the majority has 40. The mean is 22, the median is 20 and the mode is 20. How many pictures does each pupil have? What possible solutions of the problem are there?

The problem was also extended to include nine pupils and to let the pupils pose a similar problem themselves.

A group of two girls in one class and a group of three boys in another class were video and audiotape recorded during the problem solving sessions. These girls and boys were also interviewed individually one or two days after these sessions. These interviews were audiotape recorded. The two teachers' actions were video recorded during the problem solving and all her or his utterances were audiotape recorded. Interviews with the teachers were held before and after the lessons and these interviews were also audiotape recorded. In addition, I also collected the written solutions from the chosen groups. All audiotape recordings were transcribed.

Method of analysis

1. I carefully trawled the interview transcripts and the transcripts from the problem solving sessions for concepts, procedures, and solution strategies that appeared during the problem solving or during the interviews. I checked the audiotape recordings from the problem solving sessions with the interviews with the pupils and also with their written solutions.
2. In the transcripts I found out how the pupils treated these concepts; for example if they used them in the problem solving process, if they neglected them, if they discussed them with their peers and/or with their teacher and so on.
3. I then checked with the video recordings and with the pupils' written solutions to see if I could find more concepts that appeared during the sessions or if I

could find that the pupils treated the ideas more thoroughly than I had managed to identify in the transcripts.

RESULT AND ANALYSIS

Working with the concept mean

The girls tried different values and took a chance as far as the mean was concerned. For instance, they tried the values 10, 20, 20, 30, and 40. They then calculated the mean with a calculator and found that it was 24. They had some difficulties to find out if they should raise or lower the number of pictures to get the correct mean. By an oversight they tried the values 9, 20, 20, 39, and 40 the next time and got confused because the mean was higher than before, although they had lowered the value 10 to 9. After a long time of trial and error, they finally found the values 2, 20, 20, 28, and 40 and started discussing if there could be other solutions.

Cecilia: But it is not possible with more cases (inaudible), it doesn't work, does it? If all that should be the same.
Linda: Yes, yes, it works, you know, it works with three and twenty seven.
Cecilia: (Inaudible). Does it work?
Linda: It has to work, if you take that away, you add it there.
Cecilia: But then you can do as many as you like, you know. It's only to take one away ...

They wrote down the values 3, 20, 20, 27, and 40. Linda thus found a strategy, if you subtract one from one value and add one to another, you will get the same mean. The girls went on to use that strategy twice more, thus getting the values 4, 20, 20, 26, and 40 respectively 5, 20, 20, 25, and 40.

The boys solved the problem a little differently. One of them got the idea: “Five times twenty two (the mean), then you will get what they all had together.” They used the mode by maintaining that two of the pupils ought to have twenty pictures each. They then subtracted: 110 minus 40 (the maximum) minus 2 times 20 (the mode) leaving 30. Then one of the boys suggested that the two remaining pupils could have 15 each. After this was rejected (see below) they found that three pupils could have 20 pictures and, as the maximum value was 40, the fifth one had to have 10. Then, they came to the question: “What possible solutions are there?” They went on

Bertil: If he has twenty five, then, he has five.
Adam: Yes
Bertil: Will the mean be the same?
Adam: Mm.
Bertil: Check if the median is still twenty.
Adam: Mm. No, it probably isn't, is it.

Bertil: Yes (silently)
Adam: Yes, it is.

After that they very soon found a general strategy, to add one to one value and subtract one from another, and wrote the following table

The 4th	21	22	23	24	26	27	28	29
The 5th	9	8	7	6	4	3	2	1

The 4th	19	18	17	16	14	13	12	11	10
The 5th	11	12	13	14	16	17	18	19	20

They also wrote: "There are more solutions but it will be too tough a job to count them all out. Thus, that's what we have written, 'cause you can't take away any of the upper twenties."

The boys omitted one solution (25, 5) that they had already found and which is mentioned in the excerpt above. We can see that what the boys called the 4th and the 5th in the upper table are in reality the 4th and the 1st, if you start with the smallest value. The first four values in the lower table are the 2nd and the 1st. The next four values can be omitted; they actually make up a repetition of the first four. The last one has been mentioned above. In reality, the boys found all possible solutions without being aware of that and without noting that they repeated some of them.

Later, when solving the extension of the given problem, the girls also realised that the number of values times the mean is equal to the total and used this fact in the same way as I have already described for the boys. In the subsequent interview, Linda told me that next time she would do so from the very beginning.

When working with a problem of their own, dealing with stamps and where the mean was 88, one of the girls stated: "But if they all had the same number of stamps then they would all get eighty eight, but now they have like this. But then they would have to share and share equally. Do you see?" This is obviously another way to express the meaning of the concept mean. They wrote an interesting table

1	***2***	***3***	***4***	***5***	***6***
88	88	88	88	88	88
87	87	92	86	93	80
89	87	92	87	93	80

Under the table they wrote: 80 87 87 89 92 93. Their problem ran like this:

Sune, Sally, Sara, Sandra, Sigurd and Sigfrid collect stamps. The mean is 88, the median is also 88. The one who has most has 93 stamps.

- How many stamps does each person have?
- If all have the same number of stamps, how many will they have then?

The values in the third line are not a solution to the problem, but the values in the fourth line are. They wrote the values in order under the table to get hold of the median, which in this case is not one of the given values. Their teacher suggested they have an even number of values in their problem to make it a little more difficult to find the median.

The boys also worked with the extension of the given problem. They however, got a wrong solution without being aware of it and I had to tell them this fact during the interviews.

Working with the concepts median and mode

As mentioned above, the boys' first suggestion was: 15, 15, 20, 20 and 40. However, the following discussion arose:

Adam:	Yes, but that's it. We have two of twenty, then we subtract another forty.
Bertil:	Then, there will be thirty left.
Adam:	Then, we have thirty left, then, then the others have fifteen. Then we have solved it.
Bertil:	(Unaudible.)
Adam:	What did you say?
Bertil:	Then there will be two who have fifteen and two who have twenty.
Adam:	Yes, mm.

What Bertil tried to tell his peers was that in such a case there would be two modes, which was not in accordance with the problem text. Bertil had a clear understanding of the concept mode, which he tried to convey to the others.

In the interview after the problem solving session, the two boys Adam and Bertil showed that they understood the concept mode. The third one, David, who was more passive during the session, did not. Adam and Bertil told me that they learnt the concept when solving the problem.

According to the transcript from the problem solving session, the girls did not explicitly work with the concepts median and mode when dealing with the original problem. However, they saw to it that there would be two pupils with 20 pictures each, and they checked that the median was 20 in every solution. The girls mentioned in the interview afterwards that they learnt the concept of mode during the lesson and they both knew the concepts median and mean beforehand.

The boys only mentioned the concept median incidentally. Bertil said: "the median it is the one that's in the middle, you know". However we saw in the preceding paragraph that the boys took care to keep one of the values 20 in the middle. It also turned out in the interview that all the boys had a clear understanding of the concept.

The third boy, David

David did not take part very much in the solution. This was confirmed by Adam, who told me in the interview: “David was sitting looking most of the time.” When I interviewed David after the problem solving he did not know the concepts mean and mode, although he had some grasp of the concept median: “Yes, you arrange all of them and then you take the one in the middle.” On the other hand, he was not present when their teacher went over the concepts.

I also asked him why Bertil multiplied 5 times 22, the given mean times the number of values. Int is the interviewer, utterances in [] are uncertain.

Int:	... Well, I will put more concrete questions, here it's written twenty two times five.
David:	I don't know where they got that from.
Int:	You have no idea.
David:	Noo, yes, the mean.
Int:	Yes, but why do you think they multiply the mean by five?
David:	Er, 'cause there are five.
Int:	Yes [there it starts getting better]. Yes, and what do you get then, if you take the mean and multiply by five?
David:	What kind of answer, or?
Int:	Yes, I can see that myself, hundred ten, but what does it stand for, what does the answer tell?
David:	Er, what they all have together.

We can see here that David understood quite a lot of the solution strategy, but he needed my help to express it clearly. It was the same thing when I asked him about the boys' strategy to add one to one value and subtract one from another to keep the mean unaltered.

David said himself that he could not contribute to the solution of the problem. He felt that he did not know anything. These statements are in accordance with the transcript from the problem solving session and with the opinions of the other two boys.

I asked David how he felt when working with the problem. He found it difficult and meaningless. He thought that it would have been more fun if they had got an easier problem. At the same time, he said that it might have been easier for him if he had been present when his teacher went over the concepts.

Summary

We can see how the girls, though bewildered in the beginning, found a strategy to get more than one solution, subtract one from one value and add one to another. They also found that they could multiply the mean times the number of values to get the total. Finally they even come up with a different and interesting definition of the mean; what each one would have if they all had the same number of stamps.

The boys started a little faster with the idea that the mean times the number of values equals the total, and they could even, without their knowledge, get all the solutions of the original problem.

The mode was not discussed very much, one exception being when one of the boys rejected a solution with two modes. However, both the girls and the boys took care always to include at least two twenties. Anyhow, as well as the girls two of the boys told me that they learnt the concept mode during the problem solving.

They also saw to it that the middle value was always twenty, without discussing this a lot. The only case, in which the median was a matter of deeper consideration, was when the girls posed a problem with an even number of values and purposely saw to it that the median was not included among the values. They got this idea from their teacher.

The boy David did not take part in the solution of the problem. He thought that the problem was too tough and the problem solving session meaningless.

DISCUSSION

It is interesting to see how the girls could, after some time, overcome their bewilderment then find first a possible solution and then even a strategy to get more and in fact find all the solutions, even though they did not follow that line throughout. It seems that by working with this strategy as a procedure, they got a good grasp of the concept mean, which could be seen in their use of the mean in the extension of the problem and also in their new "definition" of the concept mean. The procedure helped them get to know the concept better as for example Hiebert and Lefevre (1986) and Silver (1986) indicate.

They discussed their solution strategy together. Though Linda seemed to be the leader of the conversation, she let Cecilia enter the solution process. One of the girls came up with the idea for the solutions but both girls seemed to profit from their joint work (Hershkowitz, 1999; Kieran, 2001).

In the case of the median it is interesting above all to see how they made up their own problem. Their teacher gave them the idea to have an even number of values, and the girls made use of it and saw to it that the median was not one of the values. Here, it was the teacher's contribution that helped the girls get going, but they themselves made use of the idea and got an interesting and instructive problem out of it. Here again the procedure to arrange a suitable problem and find a solution of it helped the girls to understand the concept median better. The discussion among peers and the discussion with their teacher were both an important factor for the girls' progress.

In about the same way, the two boys who were active helped each other progress towards all possible solutions. During the solving procedure, they also found that two modes were not in accordance with the problem text, thus strengthening their conception of this concept (Hiebert & Lefevre, 1986; Silver, 1986).

The two active boys had to take the median into consideration as well, although they did not talk about it expressly. I ascertain that the problem still helped them understand the concept better.

Very often, Bertil was the one that came up with new ideas that caused a breakthrough in the solution process. In spite of that, he did not dominate the discussion but let Adam take part in it. My conclusion is that at least these two boys took advantage of the common work in the solution process as far as their construction and strengthening of knowledge is concerned (Hershkowitz, 1999; Kieran, 2001).

It is interesting to compare the work of these two groups trying to solve the problem given, with a lesson where the pupils are told to find the mean, median and mode of a set of values. In the latter case the pupils will probably soon learn a procedure, an algorithm, to find the results, without being forced to consider what the three concepts really involve.

The third boy, David, was a matter of concern. Did he learn anything from the problem solving session? On the one hand, his motivation was not very high, as he found the problem too difficult and the problem solving difficult and meaningless. However, my conclusion is that David did not try to solve the problem, not because he did not manage, but because his two peers had clear ideas about the solution before David had a possibility to act on it, as Kieran (2001) elaborates on in her article.

Was the problem really too difficult for him, and was the problem therefore not a rich problem? The post interview with David shows that he could express some knowledge about the mean in connection with the total, when he got some help and encouragement. I think that if his peers had not taken over the work on the problem, David had also been able to contribute to the solution, and thus, his action is not contradictory to the problem being rich.

In this connection I want to add that it seems to be very important that the teacher encourages all the groups to see to it that every member has been able to understand and follow all procedures in the group's solution of the problem.

The conclusions that can be drawn from the data are, in my opinion, that the girls and the two active boys could take advantage of the problem solving as far as their construction of knowledge is concerned. They all had the opportunities to look upon the included concepts mean, median, and mode in new ways that could help them get a deepened understanding of the concepts. It is more questionable if the third boy, David, could benefit from the lesson. I argue that he would have had greater advantage from his peers' discussing the solution more with him, encouraging him to be more active during the solution process.

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TEACHERS AND RESEARCHERS INQUIRING INTO MATHEMATICS TEACHING AND LEARNING: A CASE OF LINEAR FUNCTIONS

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The teachers in this study are participants in the LCM[1] - (Learning Communities in Mathematics) project run at Agder University College. The project emphasises the development of communities of teachers and researchers focused on inquiry in mathematics and teaching. This paper deals with a well-planned lesson in an eleventh grade class in a Norwegian upper secondary school. We will present data that illustrate how the teachers intervened to reach certain goals that had been identified through the planning process. These interventions identified discourses, which showed that there was a discrepancy between how the teachers interpreted and used a certain word for a mathematical concept and how the students interpreted the same word. Our findings illuminate common challenges faced when trying to build a community of inquiry in the classroom.

INTRODUCTION

This paper deals with a well-planned lesson in an eleventh grade class in a Norwegian upper secondary school. We[2] followed three teachers during their planning, implementing and reflection process. The teachers in this study are three out of forty participants in the LCM- (Learning Communities in Mathematics) project run at Agder University College (AUC).

The project has two principal aims: first, the development of communities of inquiry that include both teachers and didacticians[3]; second to study the development of these communities. This concerns the different levels in school, from first

[1] LCM is supported by the Research Council of Norway within its KUL programme 'Kunnskap, Utdanning og Læring' (Knowledge, Education and Learning: Norges Forskningsråd, project no. 157949/S20)
[2] Unless otherwise indicated the personal pronouns 'we' and 'our' refer to the representatives from Agder University
College that followed the inquiry process in this school, in this paper also referred to as 'didacticians' .
[3] Didacticians are people who have responsibility to theorise learning and teaching and consider relationships between theory and practice. In the project we refer to the university college educators as 'didacticians' in order to recognise that both teachers and didacticians are educators and both groups can engage in research.

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to eleventh grade. The teachers in our study are brought together at AUC engaged in workshops that emphasise inquiry in mathematics and teaching, and collaboration processes. We want collaboration to take place on different levels; both in group work in workshops between teachers participating in the project and by collaborative activities in the school teams. The didacticians participate regularly in both forums.

THEORETICAL BACKGROUND

In the project, we have emphasised the premise that people learn through responsible participation in a collaborative enterprise. We draw on a rich tradition of inquiry in the learning and teaching of mathematics developing from problem solving in mathematics itself, through inquiry approaches to learning mathematics in classroom to teacher inquiry in which teachers take on the mantle of researcher to explore questions about effective means of stimulating and sustaining their students' mathematical growth (Jaworski, 1994, 1998; Lambdin, 1993; Mason, 2000; Mason, Burton, & Stacey, 1982; Polya, 1945; Schoenfeld, 1992).

In the project, *community building* means that we work with the teachers on the same level. We value their experience and knowledge. We are not trying to impose our thinking on schools but we want to facilitate learning and teaching through sharing ideas, practices and goals. In building the community the intention is to establish ways of thinking and ways of working that become shared among us. In the work the concept of *inquiry* is crucial. The conviction is that through questioning and investigation we will together be able to address questions in and about learning and teaching mathematics. Inquiry in the mathematics classroom will enable the students to work with the mathematics together with their peers and learn together with them. In our joint work with the teachers, inquiry into mathematics has mainly been done through investigation of open ended problems. We have been concerned with activities such as explorations of patterns, generalisations, efforts to find general structures, extend problems, and find new approaches for investigation. We see inquiry into mathematics as quite different from learning specific procedures and work on routine tasks.

Implicit in our thinking about inquiry is a cognitive perspective. We believe that mathematical knowledge is constructed by individuals. We are influenced by von Glasersfeld's consequences for teaching where he among other things emphasises the importance of what can be inferred to be going on inside the student's head, rather than on overt "responses". Further we do not believe in transfer of knowledge but that each learner has to construct their own knowledge, and that "the function of cognition is adaptive and serves the organization of the experiential world, not the discovery of ontological reality" (von Glasersfeld, 1995, p. 18). We are exploring together what inquiry can mean in our developmental work in the community we are working to establish. Wenger (1998) elaborates widely on building communities and what practice means. We want to build a community of inquiry, where inquiry becomes a way of being. The meaning of inquiry will be explored in the community we are establishing. Inquiry can help students to discover other perspectives of the mathematical concepts. They get the opportunity to learn what questions can be asked in mathematics and what kind of answers one can expect to get. This corresponds to the competences listed by the Danish KOM (Competencies and the learning of mathematics) project. In this project eight competences important in mathematics were identified. Four of the competences are related to "The ability to ask and answer questions in and with mathematics" (Niss 2004, p. 184, our translation).

We see mathematics to be firmly related to language as a tool for communication, but also as a language in its own right (Bergsten & Grevholm, 2004; Pimm, 1987). We realise that the communication between students and between students and teachers takes place through language of different kinds. Both parts (students and teachers) speak the same language (Norwegian), but we can see their language as two different languages where the common elements are in their words and grammatical structure. According to Cestari (1997) and Forman and Ansell (2001), the teacher and students together develop a classroom mathematical discourse.

IMPLEMENTING THE PROJECT

During phase 1 of the project (from August 2004 to June 2005) we emphasised the importance of trust building between the different participants in the project and development of a common consciousness about inquiry. To this end we worked together with open-ended tasks in the workshops, to give us necessary and shared experiences of inquiry. We also spent time together with the teachers discussing didactical issues about how and why we should develop inquiry communities. In this context, the teachers in an upper secondary school developed lessons based on inquiry perspectives around the issues equations, straight lines and linear functions. Their preparation process lasted for several weeks. The idea was first mentioned to the didacticians in one of the workshop sessions during autumn 2004. The teachers signalled that the motivation for their choice of topics was related to their experience. They thought that the topic suffered from a too traditional presentation in the textbook. They also characterised their own instruction patterns as traditional, using tasks which to a small extent invited the students to work in an inquiry-based style. This motivation corresponded with different aspects of our theoretical framework for the project (didactician's field notes from meeting 27th October, 2004).

An important aspect of the LCM-project is the teachers' engagement in inquiry into processes of learning and teaching. They interpreted the curriculum in collaboration with their colleagues, didacticians and other teachers linked to the project. It was therefore important that the teachers later presented what they had done and their reflection on this to the whole project community. This process gave the teachers a new opportunity to reflect about the process and the outcome of it. The presentation and the discussion that followed gave the community the opportunity to gain from their experience.

RESEARCH QUESTIONS

Here we focus on a particular example, which illustrates different aspects of the teachers' approach to teaching. We have data from our common planning session that identify some of the goals that were outlined by the teachers at this stage in the process. The goals were linked to specific skills and understanding of relationships. By studying the instruction material we can identify these goals. By observing video recordings we recognised the goals by noticing how the teachers intervened in the students' work.

From a broad perspective we study how the teachers execute a teaching sequence, which emphasises inquiry and a community of learning. From a more narrow perspective, we are interested in how the teachers intervened in the students' learning process and the rationales for these interventions. Thus, breaking down these broader interests of study, we want to answer these questions:

- Does the teacher give clear answers when students ask questions? Does the teacher encourage the students to explore mathematics or is she leading them to the correct answers?
- To what extent does the teacher direct the students towards the presupposed goals? To what extent does the teacher's responsibility to the official syllabus interfere with the students' free inquiry process?
- Do the teacher and the students have shared meanings of the words used in the classroom?

METHODS

In our efforts to answer the questions above, we used a variety of different research approaches. In the first part of the process, in the period 7–4 weeks before the lessons, our role can be interpreted as participant observers. The didacticians participated in two planning meetings. In these meetings we were all engaged in the brainstorming processes. We discussed mathematics and drew close links to the curriculum goals and the textbook. In both planning meetings, the teachers had identified learning goals that they saw as important to reach. Our role as participants in this phase must be seen in relation to our broader role in the project. In the first phase we placed ourselves in a promoter role, where the important issue was to start the process of inquiry development. Our more detailed research questions have therefore come into being later.

After our joint meetings, the teachers went back to their school and finished the preparation of the lessons. They had to be clear about the choice of tasks, how they would like the students to work, their own role as teacher and various organizational matters. The didacticians were not involved in this work. Therefore we see the final product as heavily influenced by the teachers' own thinking and philosophy. We videotaped the lessons in three different classes. The three teachers experienced the implementation of the same lesson plan. Later we held informal interviews with the teachers. On another occasion two of the teachers gave a presentation to the whole community during a regular workshop and this was discussed in the community. During the preparation period, we observed that the teachers had specific aims for the lessons. These were expressed in a workshop where Osvald (one of the teachers) said to one of the didacticians that they wanted to use inquiry methods in teaching linear functions and straight lines.

The textbooks (the school used two different books) played an essential role in the teachers' rationale for the topic. They often referred to the content of the chapter in the textbook, which deals with linear functions and equations. The teachers had definite opinions about what competence the students should achieve. They had experienced that students had trouble with the relationship between algebraic and graphical solutions of two equations with two unknowns. They wanted students to be able to understand and answer questions like "Why is the point of intersection of the lines the solution to the system of equations?"

The teachers presented their complete teaching plan the day before the lesson. They had after our last joint meeting created four cards ('card 1', 'card 2', 'card 3' and 'card 4') with questions or tasks. Each card contained some questions for students to work with. The teachers explained that the students should be working systematically through the cards. In this paper, we give an account of a specific episode during the work with the first card in one of the groups. 'Card 1' focused on the equation $x + y = 7$. The four tasks in 'card 1' are presented with our translation here:

1. Each student writes two integers, which added together equals seven.
2. Are there more such number pairs?
3. How many such pairs exist?
4. Draw these number pairs on millimetre-paper.

The term 'number pairs' refer to the Norwegian term 'tallpar', which may be interpreted as an ordered pair of numbers, for instance$_{(3,4)}$. As didacticians, we interpreted their intentions to be a compromise between several needs. The teachers felt a deep responsibility for 'getting the students through' and preparing them for the exam. There seemed to be a common understanding among the teachers that the textbook covered the syllabus quite well. By not following the usual approach, and instead introducing inquiry tasks, it became a challenge to direct the students towards certain syllabus aims. They did that by "closing" the inquiry process. The tasks seemed to represent a careful and conscious progression. First they focused only on positive integers, and then during the work with the next cards, the students had to handle also negative numbers, fractions and decimals. The tasks were rich in different perspectives from adding two unknown numbers to pairs of numbers. Then there was a focus on equations and solutions. Everything should finally be illustrated with plotting points on ruled millimetre-papers. It seems that the teachers made an effort to break the learning aims into small learning bits, which they could control (Gagné 1977; Orton, 1992).

We also notice that some aspects were hidden and not explicitly presented. The lesson was the first of a sequence of lessons that followed a chapter in their textbook. This chapter was about linear functions, graphs and about solving sets of linear equations, by calculation and graphs. Nevertheless, the teachers never use the terms 'equation', 'solutions' or 'function', and they did not present the word 'graph' in their written material. It may be that the teachers want the students to get an intuitive and informal understanding of these issues before the rest of the topic is presented.

The teachers explained their aims in a meeting with the didactician the day before the lesson. Mari said, "I am very curious whether this method results in more students understanding what they are doing when they solve equations graphically". They all agreed with this aim. Kristin confirmed our interpretation of this approach as 'guided discovery'. She said that she did not believe that the students would achieve this understanding without any form of guidance. Osvald added that they in a way had made a sort of algorithm that the students could follow. Later in the conversation, we asked them "What is it exactly you want the student to learn?" The discussion revealed some disagreement concerning which aims they tried to achieve by adopting this approach. Osvald claimed that the primary goal was to learn to solve equations; Kristin gave most attention to the importance of interpreting the function written in explicit form. Nevertheless, they seemed to agree about the content and design of this lesson.

None of the teachers found it necessary to introduce the tasks. In the planning meeting they said that they did not want to say anything before the students started to work, except taking care of practical matters. At our suggestion they considered to discuss inquiry with the students before the lesson, but they found this unnecessary. This may be interpreted as an evidence of the fact that the teachers felt that they had formulated the questions in a clear and concise way to the students. Nevertheless, it seemed that the teachers were conscious that they should not 'teach' or 'explain'. They wanted their students to think mathematically with as little intervention as possible. They seemed to be comfortable in motivating and helping students in their own construction of knowledge. This view is consistent with the philosophy behind the LCM project, which the teachers are participating in.

KRISTIN'S LESSON: DESCRIPTION AND COMMENTS

We have in this section chosen both to present data and give comments. This is because our analysis is closely related to the specific dialogues that are presented in this paper. In Kristin's class, we followed one student group closely by using video recording. In this paper, we draw special attention to one episode where the teacher intervened when the group was working with 'card 1'. We relate what happened in this episode to the teacher's aims for this lesson expressed prior to the lesson, and their evaluation afterwards.

The students quickly ran into trouble meeting task 4 ("Draw these number pairs on the millimetre-paper"). In Norwegian the word 'draw' (tegne) does not have a defined mathematical meaning. During our conversations with the teachers and from their summing up process afterwards, we realised that the teachers by using this expression meant to ask the students to plot the number pairs (x,y) produced into a coordinate system. The teachers expressed afterwards an expectation that the students should have been familiar with this task from the mathematics lessons in lower secondary school.

Below, we quote from an episode that illustrates how the teacher influenced the students' thinking when they struggled with understanding the concept of 'draw'. The students start to list the different solutions "6+1", "5+2" and so on, on the graph paper, apparently without thinking of any geometric representation.

46 Student 1	Draw these number pairs on graph paper.	Tegn disse tallparene på millimeter-papir.
47 Student 2	We have done that.	Det har vi gjort.
48 Student 1	[We have done that] Yes, actually we do 1a and 4 at the same time. Okey, was that all?	[Det har vi jo gjort] Ja, vi gjør egentlig 1a og 4 sammen da. Okey, var det alt?
50 Student 3	It seems like that.	Det virker slik.

We have not found indications that imply uncertainty about whether the students believed that they had solved the task correctly. They seemed to be satisfied with their own achievement asking Kristin for card 2. It may be surprising that none of them question why they had received millimetre-paper.

58 Student 2	Oh, Kristin, do you have more tasks?	Øj Kristin, har du flere oppgaver?
59 Teacher	Hm? [she did not hear what the student said]	Hm? [Hun hørte ikke hva eleven sa]
60 Student 2	Do you have more tasks?	Har du mer oppgaver?
61 Teacher	Eh yes, but now, now you have set up a lot... (Teacher comes and places herself behind student 1 and student 2)	Eh ja, men nå nå har dere satt opp masse… (Lærer kommer bort og stiller seg bak elev 1 og elev 2)
63 Student 2	Yes?	Ja?
64 Teacher	And then, you are going to try to draw on the graph paper these number pairs.	Og så skal dere prøve å tegne på millimeterpapir de tallparene.

65 Student 1	Shall we draw like this	Skal vi sånn tegne?
66 Student 2	Draw?	Tegne?
67 Teacher	Yes, how is it possible to draw it?	Ja, hvordan er det mulig å tegne det inn..?
68 Student 1	Can't we just say that one cell has the value one, or is it like that?	Kan vi ikke bare si at liksom en rute er verd en liksom eller er det sånn?
69 Teacher	Yes, you may perhaps…	Ja sånn noe kan du kanskje…
70 Student 2	Yes	Ja
71 Teacher	Try, and try to think how it is possible to *represent* it in one way or another. How you can do it with the other number pairs. [Our italics]	Prøv og prøv og tenk åssen det er mulig å *representere* det på en måte. Hvordan dere kan gjøre det på de forskjellige tall-parene. [Vår kursivering]

The above dialogue reveals that the students understood the task differently than the teacher expected it should be understood. The word 'draw' created problems. First, Kristin probably realised that the students needed a short reminder of what 'draw' means in this context. Instead of just telling them, she asked questions and hoped that the students would manage to make progress. Later, she introduced a new word, 'represent'. She was careful not to mention 'graph' or 'coordinate system'.

In the dialogue, there are signs indicating that the teacher looked upon the word 'draw' as a well-defined mathematical concept. It seems that in her opinion the students have just forgotten the knowledge linked to this concept. She therefore continued to give hints that led towards how she wanted the students to react to the task. Since the lesson plan presupposed that the students should work in an inquiry mode, she was conscious not to explain, but she rather tried to provoke the students to think and discuss with each other. The question here is whether she was aware of the difference between the meanings of the word 'draw' as a linguistic convention and 'draw' interpreted as a mathematical concept that may be used instead of for instance 'graph'. The above and later dialogues indicate that she has implemented the word 'draw' as a mathematical concept, which means to plot points in the coordinate system or to sketch a graph. Our data seems to show that this is the situation for her two colleagues too. In the teachers' summing up process afterwards there are new findings that support this view: "I believe this film illustrates very well how incredibly important it is that concepts are understood. For example, what does it mean to 'draw number pairs'? They have to understand what to do, here it was typical that they did not understand what the concept 'draw' means, At least not in a mathematical sense" (Osvald, plenary, 2nd March 05). The teachers seem to think that such a unique and objective interpretation of 'draw' really existed despite the fact that the word 'draw' was not used in any of their textbooks. We see this episode as an example that illustrates how different interpretations of language may influence the learning process. Kristin said in the plenary session that "I am still uncertain about exactly what word we should have used" (Kristin, 2nd March 05). This maybe illustrates the importance of being aware that working with mathematics is a constructive process and not a discovery activity where the concepts are objective and an a priori given, as in the Platonic view of mathematics (Ernest, 1991). In Hundeland, Grevholm, and Breiteig (2004) we demonstrated that this view of mathematics is probably quite common among teachers.

Later, one of the students suggested an alternative way to draw these number pairs. The suggestion included to use the cells on the millimetre-paper, and divide them into suitable dimensions consistent with the number pairs. The teacher seemed not to like the idea and gave new hints:

78 Teacher	But, if you have x eh, it it is actually x plus y that equals seven!	Men hvis dere har x eh, det det er egentlig x pluss y som er lik syv!
79 Student 2	Yes	Ja
80 Teacher	You could in a way have chosen something as x and something as y and so	Du kunne på en måte ha valgt at at noe var x og noe var y og.
81-85 Student 2	But if we for example divide the sheet and then we could divide it in two for example so it is x and so it is y. And then we must draw so we can for example draw seven seven, we shall have seven plus zero so seven cells there or something like that, is that what you mean? *(The student uses the pencil and shows in the air)*	Men hvis vi for eksempel deler arket inn og så kan vi dele det i to for eksempel så er det x og så er det y. Og så må vi jo tegne så kan for eksempel vi tegne syv syv, vi skal ha syv pluss null så syv ruter der og så blir det null ruter der eller noe sånn, er det det du mener? (*Elev 2 bruker blyanten og viser i lufta*)
86 Teacher	Yes, How do you usually draw number pairs?	Ja. Åssen er det du pleier å tegne tallpar
87 Student 2	I have no idea	Aner ikke jeg
88 Student 3	Eh…	Eh…
89 Student 1	[Draw?]	[Tegne?]
90 Teacher	[x and y values] How do you draw them?	[x og y verdier] åssen tegner dere det?
91 Student 1	As coordinate system or something like that?	Som koordinatsystem eller sånn?
92 Teacher	Yes, maybe. How would you have done it in a coordinate system?	Ja, kanskje det. Åssen ville du gjort det i et koordinatsystem?
93 Student 1	With this (…) ruler?	Med denne (…) linjalen?
94 Teacher	(Gives student 1 a ruler)	

The student used the ruler to connect the points they had marked in the coordinate system. The student seemed to understand that the line can be used to identify the different pairs of integers that add up to seven. Later, the teacher returned and gave new hints to the group. These hints resulted in an expansion of the solutions to negative numbers when the line was extended in both directions. Later, the student met questions such as "What happens if $x = 2$?" , "What if $y = 4.7$?" and "How many pairs of numbers are possible now?" In co-operation with the teacher, they realised that an infinite number of solutions exists. The dialogue illustrates the teachers' wish to establish an inquiry milieu in the group. The teacher is conscious to ask questions and give small hints. We especially noticed two things. Firstly, the questions directed the students towards specific learning goals. Her references to the x and y values illustrate her wish to lead the students towards the coordinate system. Secondly, her response to the students' questions became in some occasions too rigid. For example, she did not want to clarify her interpretation of the word 'draw' and to the student's correct response in line 92 "As coordinate system or something like that?" she did not confirm the student's correctness. The teacher was unexperienced in this specific teaching situation. It seems reasonable that it will be difficult to consider how to respond to different questions from students. That is because the teacher is aware that the students in these learning situations should preferably construct their own knowledge and not simply be told the result or answer.

We received feedback from the teachers immediately after the lessons, and they also summarised what they learnt two months later in a workshop for the whole LCM community. We found that there was a considerable agreement between their spontaneous reactions after the lessons and in their later reflection. After the lesson, Kristin expressed enthusiasm about how she experienced the lesson. She believed that the implementation of the lesson had opened possibilities for all the students. She was concerned especially about low attaining students. She observed that they also profited by this lesson, since indeed they *did* some work and were not passively following the other students. Normally, during her lessons, only a few students engaged in the discussion. Both the teacher and the didacticians observed that the students talked more, and probably also reflected more on the mathematics.

In the workshop presentation two months after, the teachers agreed that the teaching had been successful. They concluded that they had not lost time, and that they experienced that the students understood the subsequent parts of the topic more easily after this first introduction. Osvald described the situation in this way: "When we talked more about this… progress in the syllabus, I felt that it was a great transfer value because it went so fast" (Osvald, plenary, 2nd March 05). Our interpretation of this statement is that he meant that the subsequent learning process went easier because the teachers could draw on the students' experience in the lesson that followed.

CONCLUSIONS

In this particular study, we have observed that the word ‘draw’ was interpreted differently by students and teachers. We have observed the teachers’ surprised reaction to the students’ difficulties of understanding what the teacher meant. It seems to be fairly clear that the teachers had a common and particular interpretation of the meaning of ‘tegne tallpar’ (‘draw number pairs’) but they were not aware that their students did not share this interpretation. Related to our question whether the teachers and students had a shared meaning of words used in the classroom, this incident seems to illustrate the existence of words where the teacher and the students did not share a meaning.

The study also illustrates challenges and problems that may occur when teachers try to establish a community of inquiry. For example, our teachers decided not to introduce the tasks before they started working. The incident illustrates that the students were exploring both the mathematics and the meaning of the concept “draw”. It seems problematic that the inquiry process should include problems concerning how the teacher interprets specific words. It may be a learning outcome from this case that before starting such inquiry processes, teachers have to analyse the use of language in their presentations according to what meaning they give to central concepts. Such events may be more likely to happen in an inquiry based lesson, than in a teacher-dominated lecture where he or she defines all central concepts. Future investigations of the three teachers will tell us more about this hypothesis.

The teacher’s responsibility to achieve the official aims seems to have influenced the interference in the students’ inquiry process. Despite this fact, the teacher did not give straightforward answers, but inspired the students to investigate and construct their own knowledge. Concerning our research questions, we have observed that the teacher gave hints, not answers. The teacher encouraged the students to inquire, not to look for a specific answer. It also seemed clear that the teacher was concerned about her responsibility to give the students possibilities to achieve certain syllabus aims. It may be that this situation “closed” the inquiry process to deal with a limited mathematical issue. We identified the word ‘draw’ as one word/concept where there was no shared meaning between the teacher and the students. We also observed that there was no negotiation according to the meaning of this word. The teacher directed through hints and tips the meaning of the word in a way that was consistent with her own meaning of the word.

Finally, our data, which show students actively engaged in inquiry discussions, have convinced us that it is possible for teachers to design and implement inquiry based lessons in mathematics that may result in effective learning and that this way of working may result in a situation where students find it easier to engage in the subsequent issues, as stated above. This finding can be related to Per-Eskil Persson’s discussion about time for learning (Persson, 2005). One of his points is that concepts develop through higher and higher levels. This progress from one level to a higher level takes place after a certain period of work with the mathematical issue.

Our teachers probably experienced that the students had more time to develop their concepts of linear functions in this teaching context.

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META-LEVEL MATHEMATICS DISCUSSIONS IN PRACTICE TEACHING: AN INVESTIGATIVE APPROACH

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This paper refers to a project where the preconditions for a subject based and reflective approach in the context of practice teaching in teacher education are investigated. The project is linked to a compulsory 30 ect credits study of mathematics education. Five second-year student teachers and three tutors were invited to participate in the investigation. This paper focuses on how the preliminary analysis provides a basis to inquire further into the didactical conditions for including a subject based discussion within the conversation in practice teaching. More specifically it discusses contradictions implied by the findings that practice teaching communication is imprinted by an evaluative approach that can restrain the development of a subject based, reflective approach.

BACKGROUND

In this paper the notion of practice refers to the study within teacher training where students[1] are teaching in school. Under supervision, the students are allowed to teach and organize children's learning activities. The students are intended to consider, structure, try out, and discuss the mathematical and didactical[2] content they have been studying at the university college. The experience gained in practice teaching is intended to give direction to their study of mathematics education and other subjects. The importance of practice teaching can be verified from different angels:

Practice teaching has a strong tradition and can be seen as a subculture within teacher education. The corps of tutors (special trained primary and/or secondary school teachers) has been recruited, which has a high degree of autonomy in respect of tutoring the students. This tradition has a well-articulated and clearly defined position.

[1] In this paper the term *students* refers to student teachers and *pupils* to primary school pupils.
[2] The term didactical refers to didactics as it often is used in a Norwegian tradition. In this paper, the notion more explicitly refers to an area constituted by "how (mathematical) knowledge is developed, used and communicated». Didactics implies (theoretical) considerations relevant for educational practices (inside and outside schools) and deals with conditions for learning, using and communicating knowledge.

2007. In C. Bergsten, B. Grevholm, H. S. Måsøval, & F. Rønning (Eds.),
Relating Practice and Research in Mathematics Education. Proceedings of Norma 05, Fourth Nordic Conference on Mathematics Education (pp. 311–323). Trondheim: Tapir Academic Press

The student evaluations indicate that they consider their practice to be one of the most important aspects of their education (Hove, 2004).

The teacher training curriculum focuses on practice teaching as an important part of the teacher education. The development of a stronger bond between practice teaching and other parts of the study programme is emphasised. More than in the past, the curriculum now focuses on the responsibilities of the mathematics educators (as well as those in other subjects).[3]

THE FIELD OF INVESTIGATION

Most often, one tutor (T) is responsible for a group of 2–4 students (S) for a set period of time (for instance 4 weeks). The students and their tutors work closely during this period. The learning environment is affected by both the tight-knit natured group and the professionalism of the tutors. This is evident in the way the tutors' knowledge of their pupils, the curricula and the teaching tradition is handled in interaction with the students.

In considering the relationship between the students and their tutor(s) we found it useful to employ the notions of communities of practice and master-learning, as elaborated by Wenger (1998), Lave (1999), and Kvale and Nielsen (1999). The students are engaging in a practice that is already established; their interactions promote their own learning, but also a renewal of the practice. We also find our investigations related to an ongoing project where Nilssen studies the interaction between students and their tutor and discusses the role of the tutor as a mentor and a model in context of the students' learning. In Nilssen (2003) she elaborates on how a concept of "imitation" may challenge the discussions on the processes of student development in the context of established practice.

By taking part in the didactical discussions in practice, teacher educators are able to study and join in the interaction with the students and their tutors. As such we wanted to develop our role as practicing teacher educators and as researchers. The participants were invited to cooperate analytically. In this process, insight from the LCM-project (Learning Communities in Mathematics) at Agder University College (Fuglestad & Jaworski, 2005) is relevant for our work. The Agder-project introduces mathematical tasks that teachers and researchers explore together. This provides a common basis for the implementation of projects in classrooms and for further investigative cooperation. Our ongoing inquiry approach has links to the LCM-project, although our initial focus is on the students' teaching practice (Jaworski, 2004).

The field that we are entering bears the imprint of the tutors' two foci: the children's learning and the students' learning. The students share this dual focus which is a challenge. The tutors and the students observe each other when they are teaching. They discuss the class sessions both beforehand and afterwards.

[3] http://www.dep.no/filarkiv/235560/Rammeplan_laerer_eng.pdf

The tutors are responsible for establishing the conversation, and the post-teaching conversations are seen as important parts of the didactical[4] conversations that take place regularly. The discourse generally is initiated without the intervention of the mathematics educators (the Ms). When present, the Ms are visitors, observing the teaching/learning sessions and taking part in the post-teaching conversations. As Ms we often feel as if we are entering a group where the participants know each other fairly well. We are entering or are perhaps confronted by a discourse that is already established when we join in the didactical post-teaching conversation (here called SMT-conversation). The discussion has started earlier.[5]

PRELIMINARY APPROACH – RESEARCH FIELD

In order to establish a stronger emphasis on cooperation in the field of practice teaching, we have chosen to focus on the SMT-conversation as a forum for discussions between S, T and M. In this project we want to explore the didactical potentials of SMT-conversations.

We invited two groups of students and their tutors to cooperate in an investigation.[6] We all observed teaching-learning situations where students were actively engaged with the pupils. These observations served as a common reference point for SMT-conversations. As Ms, we brought a meta-perspective into these discussions. In order to gain insight into the nature of such didactical discussions and to explore how suitable preconditions for such conversations can be established, the position of mathematics[7] in practice teaching were investigated and also the role of teachers from the university college, especially the mathematics educators. The students and their tutors were invited to join in a discussion which would foster our joint understanding of the qualities and the conditions for interaction.

METHODOLOGICAL APPROACH

The project is initiated and implemented by the mathematics educators (M1 and M2)[8]. It is based on data collected during observations and (SMT-) conversations in the practice field. During the research period, the two groups were teaching in

[4] Didactical as referred to in footnote 2

[5] The fact that we enter a discussion that we interpret (also by the participants telling us) that started earlier, makes an analytic perspective based on M.M. Bakhtin relevant. In a perspective of continuity we do not see a discussion as started and ended. A discussion is to be seen as a complexity where a variety of discussions are brought to the fore (Johnsen Høines, 2002, 2004). These perspectives are important in the project, but are not developed further in this paper.

[6] Five students, three tutors and two mathematics educators (researchers) participated in the project.

[7] Mathematics and mathematics education.

[8] M1 and M2 are the researchers and authors of this paper.

different schools; one group in first grade and the other in second. On an initial visit, the tutors showed to be open and cooperative. It was this positive attitude that constituted the only, but nonetheless important, criteria for the groups to be invited to join the project. The eight persons that were invited were all eager to participate in the project.

The project was organized in these phases:

1 Observation of sessions in which S, M and T watched the students' teaching. These observations generated references for 2. During the session notes were taken, some of which were detailed and well-developed text.
2 SMT-conversations based on these observations. The discussions were intended to address the issues mentioned above: How a subject based discussion can develop and can be understood as part of the didactical conversations. In addition, it was an issue to investigate how the role of the mathematics educator can be viewed. Notes from phase 1 provided the basis and notes from phase 2 were written up.
3 Individual interviews with the students and the tutors based on the theme in 2 were recorded.

The emphasis was put on the development of subject-based approaches as part of the SMT-discussion, and the development of a conversation about the SMT-discussion itself (2). In order to gain insight into the positioning of subject-based perspectives within the didactical discussions in practice, we tried to frame the interviews as investigative dialogues (3). By implication therefore it became important not to "pose questions for them to answer" – or at best to minimize this. We invited the participants to join us in developing and sharing insights; to expound our perspectives; to turn and twist issues and search for possibilities. We build upon a dialogical approach developed by Alrø and Skovsmose (2002) where they underline dialogue as a conversation of inquiry:

> Entering an inquiry means to take control of the activity in terms of ownership. The inquiry participants own their activity and they are responsible for the way it develops and what they can learn form it. The elements of shared ownership distinguish a dialogue as an inquiry from many other forms of inquiry where, for instance, an authority sets the agenda for the investigation and the conversation. (p.119)

This chosen methodological approach had implications for the analysis. The SMT-conversations and the interviews were analytical processes in the sense that we discussed how we understood and developed relevant concepts. All the participants made serious contributions to the analytical processes. Consequently analysis on some levels has been included in the empirical data. There appeared to be a complex interaction between the processes of gathering, interpreting and analysing the data. The analytical processes may be characterised as inquiry so far as they are investigative and questioning processes, and they developed on different levels:

L1: The didactical post-teaching discussion was to some extent analytical by nature. The discussion was intended to reflect on practice.[9] The teaching/learning situations basically serve as a basis for the discussions, but previous experience and issues from other phases of the study are made relevant. Participants: SMT.
L2: We initiated a discussion about the discussion itself in order to investigate how it can be described and how it can be developed. We developed criteria for identifying and describing different aspects or types of didactical discussions. The empirical material was related mostly to L2, where our discussions of L1 were discussed. Participants: SMT.
L3: Analysis of L2 (and L1) discussions was undertaken. Participants: M.

The levels described here are not to be interpreted as being sequentially organized. For instance the analyses in L3 would have consequences for further communication with S and T, either in later interviews or in informal conversations.

PRELIMINARY FINDINGS AND RESEARCH FOCUS

Already in the preliminary discussions effort was made to create an awareness of the kind of discussions we were participating in. We identified the most common approach as the evaluative. All the participants expressed a familiarity to the SMT-conversations as "evaluative conversations" where the focus was on the learning/teaching session interpreted in terms of what had or had not worked well, what could or could not have been done, and why specific choices had been made. This kind of conversation represented a "what-happened" or retrospective perspective. Quite often the discussion focused on organisational problems, such as "good, you asked them to take their caps off", "you should have waited until they were quiet before you …".

A conversation in this evaluative mode might, as was explicitly stated on several occasions, enlighten students regarding the consequences of their later teaching and learning sessions. However, this approach relies to a dominant subordinate relationship and is to a high degree directed towards the past – an evaluation of what has been done.

At an early stage it was made clear that, as Ms, we intended to challenge the discourse developed within the SMT-conversation by having beyond evaluation to initiate or stimulate an educative[10] discussion about the mathematical content, teaching and learning. We felt that discussions should be based on reflections about what happened in the learning sessions without the common "what-I-did-well-or-wrong-focus". We applied the term *subject-based discussions* to the kind of didac-

[9] Even though we have focused mostly on post-teaching discussions, didactical discussions before and during practice teaching are included.
[10] We use *educative* as the English word that we find closest to the Norwegian *dannende (danning)*

tical communication we sought to develop. We recognized the importance of such communication even before we were able to explain what a subject-based discussion should be. Our efforts to foster an alternative kind of discussion implied a challenge to the established discourse. We interpreted this as indicating that our interference made the characteristics of the established discourse more visible to all the participants.

We wanted to learn about the established discourse and the discursive possibilities and offered a common focus: *How to describe a subject-based discussion? And how do we envisage the didactical conditions for including a subject-based discussion within the didactical conversation in practice teaching?*

EXCERPTS FROM SMT-CONVERSATIONS

Excerpt 1: "The back table did not follow"

As mathematics educators we sought to establish our background role in the beginning of the discussion by telling the others to: "Just carry on as you normally would." T1[11] was in charge and asked S1 to describe what had happened in the lessons in relation to the preparations and previous supervision. The other students were asked to add their comments, as were the mathematics educators. T1 made the point that the group had emphasized on a supportive communication, in which alternatives and possibilities were sought for. She wanted to foster a climate where it was possible to articulate constructive criticism.

Phase 1

S1 had been working with 28 six-years-olds on Lurvelegg[12] and number-concepts. The children had made models, they had drawn and they had sung and danced about Lurvelegg. We had enjoyed ourselves as we observed how S1 managed to handle various aspects of the number concept in a flexible and creative classroom dialogue. The children helped to chose the symbols to be used (drawings); they negotiated numbers, letters and positions and they discussed and made changes on the blackboard.

Phase 2

S1 started the SMT-conversation by stating that "it went pretty well as planned": "Most of the pupils followed what I said and grasped it, I think." He seemed satisfied. "We noticed that the back table did not follow," T2 commented.[13] The conversation turned to issues of behaviour, norms and limits. After a while, M2 shifted the focus

[11] There are two tutors in this group.

[12] *Lurvelegg* is a cultural fantasy-figure. He has one eye, two noses, three ears and four legs…

[13] The pupils were organized in groups. *The back table* refers to one of the groups seated back in the classroom.

to number concepts and how communication could be used to foster linguistic-mathematical creativity. S1 seemed to resist entering the discussion. Several times he responded with utterances like " I muddled the order..." "I should have done…" "No, I did not…" "I tried to…". M2 tried to reassure him that she did not think this was a case of importance. It was obvious however, that S1 did not change his focus to join the discussion. He returned to evaluate what he might not have done well enough.

M1 intervened, emphasizing that S1 had taught a very good lesson. Her description was detailed and had a strong evaluative component. She described his muddling with numbers, signs and order as a brilliant didactical example. She gave grounds for why the lesson had been very good: "You invited the children to play and argue, and they joined in. You handled a variety of number-concept features brilliantly," she said. M1 stressed that we were interested in going beyond a discussion of what was good and was not; that we wanted to establish a basis for a subject-based discussion independent of whether he had succeeded or not. This appeared to help; S1 seemed more relaxed and satisfied. Nevertheless he soon returned to comments like: "Yes, I should have…"

Phase 3

During the phase 3 interview we told S1 that we interpreted the way he interacted as resistance and he commented: "Yes, that is how I handled it, because that is what we are used to doing. In practice teaching last year we always focused on what we did well enough and what we did badly. The only interest we were supposed to have was about what we did badly. The post-teaching conversations focused on *what should have been done differently*. That was the point. As if … well, the evaluation was introduced by: *yes … but ... you should have done so and so*. The discussions we have had this year has been fantastic in comparison." He was referring positively to the sessions led by his tutors this year."[14]

S1 proceeded to tell us about how the comment "we recognised that the back table did not follow" had influenced him. T2 had referred to "we" implying himself and M2: "I understood that the two of you had been talking about this, and I took this as an indication that you considered it as important." S1 went on to say that he had felt relaxed initially since he felt that the children had been active and well focused. The comment about *the back table* had changed his attitude. T2's words had been impossible to erase from his mind and he could not concentrate on the discussion. Even at the time of the interview the back table was what he remembered most clearly from the lesson and from the post-teaching discussion. He referred to it as something that disturbed him: *they did not follow, and he had not recognised this*. He claimed that even when we discussed the function of such utterances and tried to see how it might restrain or stimulate the didactical discussions, the bad feeling in his stomach constantly repeated: *the back table did not follow*.

[14] The students have two periods of practice teaching each year. The present period is their fourth.

Excerpt 2: "Is it … strictly speaking … concrete?"

Phase 1

S4 had started the class by singing, talking about which day it was and what the date was. It was the tenth. S4 took out a small rack with five wooden sticks in it. One child puts nine white balls on the sticks to indicate 9 – yesterday. Then ten white balls were shown. Communicating with the children, S1 changed the ten balls into one red ball, put it on the next stick in order to symbolize 10 – today.

Phase 2

In the SMT-conversation S4 received supportive comments from everybody for the interactive way she had started the day. In this context T3 emphasized how well S4 had used the balls on the sticks in order to concretize the numbers.

M1 followed up with exploratory questions: Is it, strictly speaking, concrete? To use a red ball in order to symbolize ten; to put it on a stick you have decided for the tenths? … Is it abstraction, or concretizing? … What is symbolizing?

The conversation shifted. A new thoughtfulness prevailed. The conversation was characterized by slowness, hesitation and pauses; unfinished sentences were followed by new unfinished sentences, the tentativeness was underlined by the questioning tone of voice.

> M1: This is not meant to imply that you should have done anything differently. I just consider it interesting. (This remark from M1 passed without comments. We could not observe that this comment had any disturbing effects.)
>
> S4: I have never thought about that.
>
> T3: This might be the case … most often … when we talk about concretizing. Especially when it is concerning … the positions … the number-system?
>
> S4: What about when we are playing "to buy in a shop"? I have seen that lots of children mix 10-kroner, 5-kroner and 20 kroner?
>
> S5: Is it abstraction or concretization when we use money as teaching material in the context of playing shopping?
>
> S4: When we say "fifty-two" and make it "concrete" by using piles of kroner and changing them into tens… and into a 50-kroner-note…?

These two excerpts were chosen to illustrate the nature of the kind of subject-based discussion that we were trying to implement, as well as the struggle to introduce and position a subject-based discussion within the established discourse.

Excerpts like these were used as examples and tools in the SMT discussions and interviews.

CONFLICTING APPROACHES

In the beginning of the SMT-conversation in excerpt 1, M2 encouraged the participants to start in the way they were used to. T1 took the lead in this phase. M1 and M2 participated as visitors, and were invited to join in. Drawing upon Foucault, we view this discourse as being generated institutionally (Mellin-Olsen, 1991; Popkewitz & Brennan, 1998). The participants knew which questions were relevant (and which were not). They knew how questions should be posed and responded to. When we visit different groups, we find that although the communication pattern differs from group to group, there are marked similarities. The post-teaching conversations have established a discursive identity. The discourse is based in the culture of tutors established through their practice, training and seminars. The tutors are given the responsibility; each is in charge of establishing a constructive basis for the discussions. However, they do their tutoring in a context that sets discursive limitations and possibilities. When analyzing the data, the power of this discourse became evident.

S1 initiated the discussion by commenting that “it developed all right” and that “most of the pupils followed what I said and grasped it, I think.” This comment can be viewed in connection with a comment from another part of the interview where he was “defending” a fellow-student (S2) and explained the following: “Of course the focus of S2 is on whether the pupils really did what he intended them to do. What he had planned that they do. That was his focus!”

S1 felt satisfied, therefore, the pupils had done what he wanted them to do; they joined in. Therefore he had concluded that they had “grasped it”. His interpretation was questioned by T2 in the comment about the back table that did not follow. This led to a change in his attitude. His body-language, his silence, his tone of voice when he was talking and the brevity of his utterances all indicated to us that he felt confused and insecure. This was highlighted by the evaluative approach: “I muddled the order...” “I should have done…” “No, I did not…” “I tried to…”.

We tried to challenge him by leaving the evaluative mode and starting a subject-based discussion. However, even when M1 started by praising S1’s teaching and then explicitly explained the intended discussion concerning children’s learning of number concept without discussing what was done well or badly, he did not fully participate. S1 never regained the relaxed and well confident attitude that he had had in the beginning.

In the interview, S1 later commented on what had happened by referring to what they “were used to”. Even though this interview took place at the end of second year, he still referred to the first year in his statement that “the only interest we should have (…) was what we did badly (…) the evaluation afterwards was introduced by: Yes… but ..you should have done so and so.” S1 described the changes in approach from the previous year. By so doing, he also strengthens that the foundation was laid early in his studies. Later in the interview, he commented on the discussion concerning a fellow-student: “That is how it is – we easily retreat to the perspective of evaluation (…) it is difficult to leave one’s focus and look for other aspects (…) The focus is on *getting the pupils to do what we have planned* ”.

S1's comments strengthen our view that the dominant approach is evaluative. He actually moves the focus still further. Rather than evaluating what was good, or could have been better in a broader sense, he simply questioned whether he had succeeded in getting things done as planned.

S1's inability to shift focus from "the back table that did not follow" was also unexpected to us. According to S1, T2's comment (which included M2) played a critical role for him. He returned to it over and over again. He did so even after M2 said that she had considered saying that she did not quite agree with T2. She had felt a bit uncomfortable. However she had not interpreted the impact the comment had on S2's impression of his teaching. Later in the interview S1 exclaimed: "You should have said so! You should just have said explicitly that you did not see it quite as strongly and that it should not have such great importance." M1: "Do you think that M2 had the power to do so?" S1: "Yes … well I see … I see … that T1 and T2 are in charge and that you want to include all the participants in the discussion, but … yes, I think…you should have. Because I could not let it go. But I see it is not easy." M1 had praised his work. She had said how highly she evaluated his communication of number-concepts. However, neither she nor M2 had commented on the back table.

We interpret this as an indication that S1 is describing characteristics of the discourse of practice teaching by describing what is considered relevant to talk about (and in which ways) and what is not. He describes a discourse imprinted by evaluative connotations. He also describes how the discourse is implied as a frame for interpretation. The comment about the back table is interpreted by the student in the light of how such conversations have been practiced.

Analyses of the transcripts[15] from the SMT-conversations and interviews highlight the aspects referred to above. It is possible to identify strong evaluative tendencies characteristic of the discourse as it is articulated by all the participants (S, T and M).

By trying to foster a didactical discussion without a strong evaluative focus, we challenged an established discourse. The established discourse became more clearly visible when viewed in the light of an approach that the participants described as unfamiliar. The fact that we tried to develop an alternative discussion and to question what kind of discussion we were trying to develop influenced the way we questioned the established discourse.

We found the students inquisitive and cooperative when it came to analyzing the interactions. The paragraph in which we discuss "the back table did not follow" sequence shows traces of how a cooperative inquiry has developed (Jaworski, 2005). The following three statements by students illustrate this development:

[15] The limitation of the paper restricts us from presenting broader analyses here.

> As S2 was on his way out of the interview, he turned back and looked at us: "If I were to define a subject-based discussion, well … I think … we are moving forward … from the teaching-learning situations … *it is about winding up a discussion for later use … or for relevant situations later on"*.
>
> Earlier in the interview, S2 commented: "I am ready for this. We went through the "What? How? Why? Did it work as planned? Last year … I need to move on … not just to circle around in a what-did-happen perspective … and we have learnt to cope with the "what you did wrong"-comments. Just forget about it. Go ahead. Expel the whole situation. So the comments are for no use. No … I am ready to move on."
>
> When interviewing S3, we used excerpt 2 – "Is it … strictly … speaking … concrete?" as one of the examples. She commented: "That is the benefit of having you there (at the SMT-conversations). We really discussed things in another way than we usually did. We have an excellent tutor, caring and supportive. But these questions we pose when you are present are different on nature. I would never have thought in that direction. It's interesting." She explained that she did recognise that she managed to change from an evaluative approach, but that she thought that her fellow student S5 would not manage. "He is showing resistance; I see that. It is not easy for him and it is not easy to push him; he simply does not want to. I think I can understand his resistance."

The students talked about "leaving an approach", about "resisting", about "moving on", about "later use". They talked about being ready for a change, and they understand the resistance they observe. In studying the kind of expressions that were used, we get evidence for describing the discourses established and the interactions between them. Additionally we have to be aware that these expressions have been given relevance within the discourse we have established in our group.[16]

We have identified two different approaches:

An evaluative approach that aims to focus on a discussion of the learning/teaching session in terms of what was considered to work well and what did not, what could have been done (and was not), why choices had been made. It represents a '*what happened'* perspective, a retrospective perspective. A conversation within this mode might discuss the consequences one sees for later teaching and learning sessions. However, the perspective is to a large degree directed towards the past through the evaluation of what has been done.

A subject based, reflective discussion would be educative approach that aims to explore how the situation might generate discussions for further development;

[16] We have actively questioned these issues. As mathematics educators we have stimulated and influenced on how they understand the didactical conversation in practice and how they express their understanding. As this is the case for all the participants; we have influenced each other within a group where the power is not shared equally. The questions about symmetry – asymmetry are important for this project, but it is not elaborated explicitly in this paper.

a future-oriented perspective. It is generated in the practice teaching situation, released from the evaluative aspects and developed as a subject-based interest; as a foundation for subject-based reflectiveness. This perspective implies an investigative and dialogical approach. (Alrø & Skovsmose, 2002; Johnsen Høines, 2002)

These two approaches are very different and may appear to be conflicting approaches. They do not occur as equal. The evaluative discussion has a strong tradition in the didactical conversations in practice teaching and in the SMT- conversations. It is constructed in exchanges between the tutors and the students[17]; it is embedded in the discourse shared with other tutors. The dominant position of the evaluative discussion can be attributed due to the discourse that has developed in context of practice teaching (Mellin-Olsen, 1991). Early findings in our project indicate that a strongly evaluative approach might restrain the development of a subject-based, reflective discussion approach.

The two approaches can be represented by individuals in the sense that each participant brings her/his personal approach into interaction with others. It can also be identified as intrapersonal, in the sense that each person moves between different approaches in the process of gaining understanding. S1 described how he retreated to an evaluative argumentation. S4 claims that she understands the resistance of S5, and the difficulty he has in changing approaches. Our analyses reveal that this is the case for all of us; both the perspectives were evident in the argumentation of each participant. We move between these partly conflicting perspectives, intrapersonally and interpersonally (Johnsen Høines, 2002, p. 77, 2004, p. 65).

FURTHER QUESTIONS FOR DISCUSSION

When entering into these didactical conversations, it was our intention to initiate a subject-based discussion. Experience had shown that mathematics as a theme was little evident in the post-teaching conversations. We wanted a change and we wanted to identify the preconditions necessary for change. In this respect we see evidence of normative aspects in our conversations: We could be interpreted as saying that we *should not* rely upon an evaluative approach; that we *should* develop alternative and more educative discussions. As we see it, this is true, to some extent. However, it is challenging to find a balance on this issue.

We do not consider the evaluative approach to be irrelevant, nor would we want to eliminate it completely. We wanted to explore the conditions for implying an educative discourse and to develop a common understanding as to what a subject-based discussion could be. In fact we learnt about the tension between the two approaches. We learnt about how an educative approach can be blocked by a pre-existing evaluative approach.

We discussed the preconditions. The participants told us about the restrictions and, about the conflicts they experienced and confirmed that they felt ready for a

[17] and to a lesser degree by the teachers from the university college

change. We discussed the difficulties, the risks and the unpredictable changes this could imply. We discussed the social circumstances, referred to the interpersonal relations in the group and the institutional conditions as well as what they personally interpreted as condition for taking part in a subject-based discussion.

The analysis and the discussions call attention to the participants' readiness as a crucial point. The present analysis provides a foundation upon which to extend the nature of *readiness* as a conceptual tool for further analyses. We discussed readiness in terms of implying, to make room for, to be motivated for, to adopt an expectant attitude, and to be on one's way to realization. This implies to be ready for, to be actively present or actively engaged. This paper does not discuss this issue further, but simply indicates our basis for developing these kinds of discussions. It is intended to provide information about the angle of approach and the early discussions between mathematics educators, students and tutors in the context of practice teaching and the potential for joint investigations.

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STUDENTS' RESPONSE TO ONE-OBJECT STOCHASTIC PHENOMENA

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Four hundred and eighty-five students in lower secondary school (age 13–16) were asked to solve several probabilistic problems in a written test. In this paper I will consider their responses to three of the tasks, which were all related to compound stochastic events. The aim of the investigation has been to consider logical inconsistencies in students' answers. In each task a question was presented to the students in two different ways. A high degree of inconsistency was found in the responses for all age groups. Some results from follow up clinical interview are also reported.

In Norway probability has been part of the national curriculum for many years. The current national curriculum for compulsory primary and secondary education, L97, grades 1 to 10, (KUF, 1996) is structured in such a way as to focus on increasingly complex probabilistic ideas. In the lower grades the focus is on comparing probabilities. It is not until grade 9 that the students work in a systematic way to quantify probabilities and also encounter *compound stochastic phenomena*.

As a part of a larger study focused at students' reasoning about compound stochastic phenomena, 485 students from lower secondary school undertook a written test with problems about fractions and probability. Some of these students participated in clinical interviews where they had to explain and explore further their responses in the written test. In this paper I report on findings from three tasks from the written test and also some preliminary results from the interviews. The first step is to begin by defining in a more precise way the concepts involved.

RANDOMNESS AND INTUITION

The concept of *randomness* is not easily defined. In his well known theory Piaget (1951) proposed that students' probabilistic reasoning is developed through three main stages. He further claimed that their probabilistic understanding is strongly related to their combinatorial competence. Later theories, for example the theory by Jones, Langrall, Thornton, and Mogill (1999), also emphasize formal concepts like

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sample space, independence and assigning numeric values to stochastic events. In Piaget's cognitive theory the important process is that an external action is internalized. But what happens between the external experiences and the internalization? Here Fischbein offers an important contribution by looking at the construction of *intuitions* as an "intermediate stage between outward and internalized action" (Fischbein, 1975, p. 7).

In the study done by Fischbein (1975) where students of different ages participated, it was found that the preschool children showed a better probabilistic intuition than the older students on several of the probabilistic tasks involved. This surprising result was explained as a result of school work – traditional teaching in school does not foster probabilistic reasoning in the way one could hope.

> The teaching process – particularly as it is determined by school – orient the child towards a deterministic interpretation of phenomena, in the sense of looking for and explaining in terms of clear-cut, certain, and univocal relation. The child is taught to seek the causes of phenomena in the form of univocal operating factors. (Fischbein, 1975, p. 169)

Further, Fischbein claims that intuitive knowledge is a "syncretism characteristic" of pre-school children. For this reason the subject probability could be seen as quite suitable in the curriculum for lower grades, with a possible result that many students would be well prepared for working with the subject later in school.

School children's lack of stochastic reasoning is especially evident in connection with *compound stochastic events*. The flipping of a coin several times illustrates such an event.

Research on how well students handle such stochastic events reveals that a large proportion of them have developed what is commonly named the *equiprobabilistic misconception* (EM). These students have a tendency to interpret all outcomes in a stochastic event as being equally likely, not only in a uniform stochastic situation.

In quite an interesting study by Konold, Pollatsek, Well, Lohmeier, and Lipson (1993) the stochastic event "flipping of a coin five times in a row" was used. The following two questions were given to the students:

Q1) Which of the following is the most likely result of five flips of a fair coin?
a) HHHTT b) THHTH c) THTTT d) HTHTH e) All are equally likely.
Q2) Which of the above sequences would be least likely to occur?

Here it would not seem reasonable to expect that the words *most* or *least* would influence the students' responses to the question to any great degree. Especially one would predict that those who opt for alternative e to Q1 would also choose alternative e as an answer to Q2. However, this was not the case. While 72 % of the students correctly chose option e as an answer to Q1, only 38 % chose e when responding to Q2. Konold et al.'s explanation of this surprising finding was that the students' reasoning on the two questions is based on two different perspectives. From the first perspective the task is interpreted as a request to *predict* the outcome, while it is from the second perspective that one makes probabilistic *comparison*. Based on the first perspective the students' reasoning is that all outcomes are possible. To express this they choose the equally likely answer. It is from the second perspective that the students interpret the task as it was initially intended by the researchers.

In describing students' cognitive changes Pratt (2000) introduced the term *internal resources*, defined as incorporating all forms of intuitional and formal thinking. He then summarized the findings about students' inconsistencies in the following way:

> In fact, researchers studying probabilistic thinking often find inconsistencies in what children say about stochastic events. Such inconsistencies are often juxtaposed, and I conclude that different simultaneously existing internal resources led to those contradictory articulations (p. 605).

In the current study I wanted to consider students' inconsistencies in connection with a particular kind of stochastic phenomenon, which I have named *one-object stochastic phenomenon (OOSP).*

ONE-OBJECT STOCHASTIC PHENOMENA

The Robot-task used by Green (1983) is a good illustration of OOSPs.

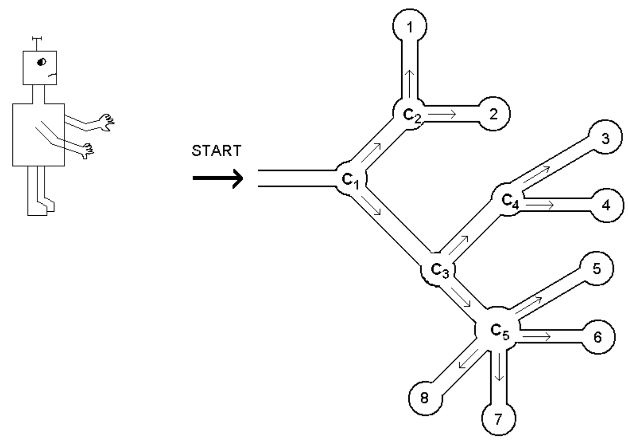

Figure 1. The Robot-task used by Green (1983), page 776. The five crossroads (C_1 to C_5) and the eight rooms (1 to 8), each with a trap, are indicated in the figure.

In this situation (see Figure 1) a robot enters a maze. When the robot reaches a crossroad its further route is decided by a random mechanism. Finally the robot ends up in one of eight traps.

In this study the following question was used:

> At each junction the robot is as likely to go down any one path as any one other (except it does not go back the same way it came). There are eight traps at the ends of eight paths (see picture). In which trap or traps is the robot most likely to finish up, or are all rooms equally alike? (Green, 1983, p. 776)

Only 7 % of the 2930 participants in this study (students aged 11–16) were able to give a correct response to the posed question. It was especially surprising that only 13 % of the oldest students, those of age 15–16, gave a correct response, because these students had already been working with tree diagrams and the product law of probability. Green summarized this in the following way:

> The multiplication principle is poorly understood, and pupils have little in the way of basic intuition on which it can be based. Response to Question 22 [the robot task, author's remark] was very poor. Clearly little conceptual understanding has been achieved, and using a tree diagram was just a mechanical device. (Green, 1983, p. 782)

In an earlier study by Fischbein another type of OOSP was used in an experimental setting. Six different boards were used where a marble could be released from the top (see Figure 2). On its way downward through the device the marble encountered one or more crossroads, and finally ended up in one of several possible end positions. The following question was presented to the students: "If I drop the marble a lot of times, one after the other, will it come out at each place the same number of times, or will it come out some place more often than others?"

The findings in this study were quite contrary to Green's results. A high proportion of the informants (more than 80 %) gave a correct response to the situation presented in boards I, II, and IV, and even in the asymmetric cases (boards III and VI), more than 60 % gave a correct response). Green (1983) comments on this in the following way:

> The abysmal failure of all years on this item is in marked contrast to Fischbein et al.'s results […] in an equivalent but practical setting … (p. 777)

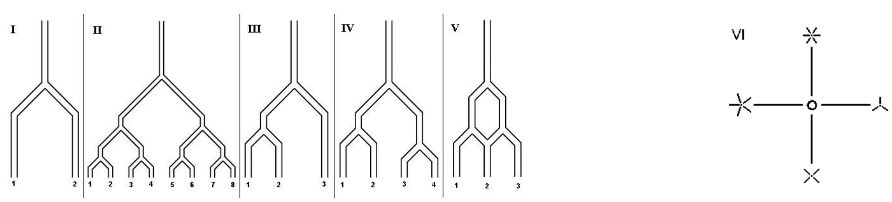

Figure 2. Random devices used by Fischbein a) The five "two-dimensional" devices. When a marble is released from the top it passes through one ore more crossroads on its way down.

b) Schema of a three-dimensional device. The marble has four options in the first; 3, 4, 5 or 6 in the second crossroad.

This difference in results is puzzling and (partly) inspired the current study. It seems reasonable that the difference in result can partly be explained on the basis of differences in research methods, but other factors might also be of importance here.

Both the studies above had as their focus the students' intuitive responses to the tasks. Inspired by the findings in the Konold et al. (1993) study a written test was constructed with two situations from Fischbein's study (III and V in Figure 2) and the Robot-task. In addition to a multiple-choice question another question was added where the students were asked to give a numeric value. In this way it was possible to consider students' possible misconceptions. Some of the students also participated in a clinical interview. Only preliminary results from these are presented in this paper.

AIM OF STUDY

The current study is aiming at further investigation of OOSPs from a quantitative as well as a qualitative perspective. I am particularly interested in the degree of inconsistencies students might have, EM and qualitative explanations for their responses. The following research questions were formulated:

- To what extent do the students give inconsistent answers to problems related to OOSPs?
- What proportion of students shows equiprobabilistic misconception, and is there any difference when it comes to age?
- What qualitative reasons are there for students' inconsistencies?

In this paper my main concern is the first two research questions. Only preliminary and partial answers will be offered when it comes to the last research question.

I now state the questions given to the students in the written test.

THE QUESTIONS USED IN THE WRITTEN TEST

Three different OOSPs were included in the written test, III and V from Fischbein's study and the Robot-task used by Green. In connection with each context a question was presented to the students in two different ways.

Task 1 and 2 (Tasks III and V from Fischbein's study)

a) □ The three options are equally likely.
   □ The marble is most likely to end up in _______ (write down one or more)

b) What is the chance for the marble ending up in box 2? ________

Task 3 (The Robot-task)

a) □ All rooms are equally likely.
□ It is most likely to end up in _______ (write down one or more)
b) What is the chance for ending up in room 6? ________

As in the research by Fischbein and Green the question a) can be answered based on intuition (no explanation is needed). The idea behind question b) was to consider any possible inconsistency in the students' reasoning. It is also reasonable to say that a correct response to b) must involve some reasoning beyond pure intuition.

METHOD

Students from three lower-secondary schools from the same area completed the written test. In comparing the result from different populations the two-sample binomial test was used (Larsen & Marx, 1986, p. 380). The participants were 216 students from grade 8, 97 students from grade 9 and 172 students from grade 10.

RESULT

Task 1

The overall finding is that 39 % of the students gave a correct response to question a), while only 21 % gave a correct response to both a) and b). In the analysis two main types of inconsistent answers were identified – I use the notation I1 and I2 for these.

Inconsistency of type 1 (I1)

These students mark off choice 2 and writing down box 3 (correct response) on question a), but on question b) they write down 1/3 or equal numeric values indicating that the three boxes are equally likely.

Inconsistency of type 2 (I2)

These students mark off choice 1 (equally likely) on question a). But on question b) they write down 1/4 or equal numeric values indicating that they do not see the three boxes as equally likely.

11 % of the students gave a response of the type I1 and 4 % gave an I2 response. That is, a total of 15 % of the responses were inconsistent in either of the two types. (See Table 1)

Grade	8	9	10	All students
Response to question a)				
C (%)	40	40	37	39
EM (%)	36	43	48	42
Response to both question a) and b)				
C-C (%)	18	23	23	21
W-W (%)	56	55	60	58
I1 (%)	11	10	10	11
I2 (%)	4	5	3	4

Table 1. Results in percentage from task 1. C: Correct response in a). EM: Equiprobabilistic misconception. C-C: Correct response to question a) and b). W-W: Wrong response to both questions. I1 and I2: Inconsistent responses.

Equiprobabilistic misconception (EM)

As many as 36 % of the students responded "equally likely" on question a), showing the *equiprobabilistic misconception*. It is an interesting finding that the percentage of students showing this misconception *increases with age*.

Among the youngest students 36 % revealed this misconception, the proportion increasing to 48 % among the oldest students.

That is, there is a significant difference between the two age groups when it comes to EM ($p<0.01$).

Task 2

A much higher percentage of the students were able to solve the second task as 62 % of the students produced a correct response to question a).

Grade	8	9	10	All students
Response to question a)				
C (%)	55	68	67	62
EM (%)	32	25	26	28
Response to both question a) and b)				
C-C (%)	25	46	40	21
W-W (%)	43	30	32	36
I1 (%)	4	4	6	5
I2 (%)	3	3	1	2

Table 2. Results in percentage from task 2. C: Correct response in a). EM: Equiprobabilistic misconception. C-C: Correct response to question a) and b). W-W: Wrong response to both questions. I1 and I2: Inconsistent responses.

Grade	8	9	10	All students
Response to question a)				
C (%)	24	29	25	25
EM (%)	38	38	44	40
Response to both question a) and b)				
C-C (%)	1	5	6	4
W-W (%)	76	70	74	74
I1 (%)	9	8	5	8
I2 (%)	0	2	1	1

Table 3. Results in percentage from task 3. C: Correct response in a). EM: Equiprobabilistic misconception. C-C: Correct response to question a) and b). W-W: Wrong response to both questions. I1 and I2: Inconsistent responses.

Equiprobabilistic misconception (EM)

In this task 28 % of the students responded "equally likely" on question a), and with a decrease in EM with age.

Task 3

In question a) 25 % of the students gave a correct response, while only 4 % were able to give a correct response to both questions. In this task 9 % of the responses were inconsistent. The large drop in correct answers from a) to b) is to be expected since this task is more complex than task 1 and 2.

Also, the increase among the students when it comes to EM in this task is disturbing. The percentage of students showing EM increased from 38 % (8^{th} grade) to 44 % 10^{th} grade). However, this is not a significant difference (p=0.13) between the age groups.

DISCUSSION

Even though the findings in the present study do not correspond directly to the results in Green's and Fischbein's studies (it is "in between"), they support the result in Green's study in that they show students' difficulties in handling OOSPs. Task 1 is related to the simplest possible asymmetric situation. That merely 37 % of the oldest students have a correct intuitive understanding of this situation is a surprisingly poor result. In addition one could have expected that a much higher proportion of the students were able to put a correct numeric value for the event that the marble ends up in box 2. That only 23 % of the students in 10^{th} grade are able to give correct responses to the two questions in task 1 shows that there are great challenges involved in the teaching of this subject.

The results from all tasks might indicate that teaching in fact has a negative effect when it comes to EM for many students. However, it is important to recall that there is not a significant decrease when it comes to the correct response rate. What we see is that in 8^{th} grade there is a variety of different wrong responses, while in 10^{th} grade many of these students show EM.

Inconsistencies are found in all tasks, but the number was especially high in relation to task 1, in which 15 % of the students gave inconsistent responses. In the analysis I have assumed that students that are responding 1/3 to question b) would use the same numeric value about the other boxes (1 and 3). However, it might be the case that these students lack the ability to make such a numeric estimate, in which case the inconsistency rate is too high.

To get some more insight into the students' reasoning I include some preliminary results from several clinical interviews.

PRELIMINARY RESULT BASED ON INTERVIEWS

The analysis of the written test shows that the students give inconsistent responses in the case of OOSPs. It also shows that many students show EM. To get more insight into how the students reason, several different qualitative inquiries were undertaken. Here I report some of the results from two different groups.

Group 1 consisted of 6 students that were mainly asked to explain their answers in the written test. Group 2 consisted of 4 students. These were asked to explain their answers in the test, but they also got access to an ICT environment called Flexitree. Flexitree is basically software that simulates the devices used in Fischbein's study (see Figure 2).

Theoretical framework

As a tool for describing students' different responses, I adopt the theoretical framework offered by Halldén (1999), in which students' different ways of responding are seen as different *contextualizations* of the situation. In this framework three different types of contextualizations are considered: 1) Conceptual, 2) Situational, and 3) Cultural.

I now interpret two different responses in this framework: 1) EM response 2) Correct intuitive response. It is especially hard to understand why students offering a correct response to one of the questions (e.g., question a), later (or earlier) also suggest a response indicating that the student hold EM. One explanation is that words like probability (and "What is the probability for …") are related to and interpreted in relation to the school situation (or school culture) and the mathematical models and formulas emphasized by the teacher. That is, the student's contextualization is framed in the school culture, with a strong link to widely used uniform model. A response indicating EM can therefore be related a cultural contextualization of the situation. Another contextualization appears when the student uses a conceptual contextualization. In this case the students' response is based on their own conceptual understanding – freed from the school culture. In this case their intuition seems to play an important role.

But the qualitative data also offer further insight. Some students seem to lack the ability to make numeric models and others have difficulties in comprising the whole situation, putting predominance on only limited parts of the situation. For example some of the students put very strong weight on "the first crossroads" in task 1, arguing that since the chances for going left or right in this crossroads is equal, the three boxes must be equally likely – thus showing an inability in interpreting any meaning to two consecutive crossroads.

As an example of how these different ideas can be held simultaneously one of the students in group 1 gave the following (first) explanation to task 1:

"I think they all got the same chance, even though when I look at it now it seems to me that box 3 is more likely. But I really believe they have an equal chance since there are equal chance for going there and there [pointing to the two paths from the first crossroads the marble encounters]."

An interesting episode appeared in the interview with Anne (from group 2) where situational contextualization played an important role.

In the first part of the interview Anne had been asked to consider several of the tasks in Fischbein's study (see Figure 2a) and she had been playing with Flexitree for a while to investigate her own suggestions. She had basically responded in accordance with a formal understanding of these situations (offering correct responses to most of the questions). At the end of the session she was asked about the Robot-task with Figure 3 on a piece of paper. To the interviewer's surprise her first answer seemed to indicate that she held EM:

A: I really believe that each of the traps is equally likely, or, that was what I wrote down on the written test. But if one of the marbles comes here [pointing to crossroads to path 5–8], after making two choices. When it comes here it has four options, so then it is suddenly more …

I: Okay. So now you believe that the traps are equally likely, or do you really?
A: They are equally likely. They have to be!
I: They might.
A: Yes. They surely must be!

After some more discussion the student suddenly turns the sheet ninety degrees, and her interpretation of the situation changes.

A: Now I would say that it's more likely to end up in room 1 or 2.

K: Does it matter if the sheet is like this or this [turns the sheet into two different positions]?
A: No, but it was only when I saw it in this position that I understood it.
K: So, do you think the probabilities are the same or not?
A: I think room 1 and 2 are more likely.

After this she was able to offer a sound explanation for her answers and also assign correct probabilities to each of the eight rooms.

SUMMARY

It seems reasonable that a student will make greater efforts during an interview situation than in an anonymous written test. It is therefore not surprising that the results in Fischbein's interviews produced better results than was the case in Green's written test. This view is supported by results in the current study where the result in task 1 (from Fischbein's study) was significantly lower in the written test than in a more practical situation used in Fischbein's study. Even though the students' results in the current study were significantly better when it came to the Robot task, than in Green's study, the result is still quite poor. Together with the surprisingly poor result in the much easier task 1, my study gives support to Green's claim that students have difficulties in handling these kinds of tasks.

By adopting the contextualization perspective I interpret the inconsistencies and the poor results partly as founded in a school culture which seems to foster EM instead of helping students develop sound intuitions in connection to stochastic situations. Another reason for students' difficulties in handling OOSPs seem to be that students' lack of modeling competence. This seems to be especially true when it comes to constructing numeric models.

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LIMITS AND INFINITY – A STUDY OF UNIVERSITY STUDENTS' PERFORMANCE

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Students' work with limits of functions includes the concept of infinity. These concepts are complex but necessary for mathematics studies. A study of student solutions to limit tasks was conducted to reveal how students handled limits of functions and infinity at a Swedish university. As a background to the students' and teachers' situations we have also analysed textbooks and curricula. One result is that many students were unable to handle limits correctly. Some students gave correct answers with incorrect explanations to tasks. Textbooks used at upper secondary schools do not provide much theory or tasks about limits and infinity, so most students new to university do not have a well developed image of these concepts.

BACKGROUND AND QUESTIONS

Students at universities taking courses in analysis are supposed to solve a considerable number of tasks. The students presented in this article were suggested about 30 to 40 tasks per week to solve. Students can check in their textbook if they get correct answers and then continue to work with their problems. An important question is: What if some of the solutions are correct for the wrong reasons? In this article we discuss some of the results from an investigation into students learning limits of functions at university level (Juter, 2003). The main aim of the whole study was to learn more about the development of the students' conceptions of limits of functions. No such study had been done in Sweden before, but there are researchers in other countries (e.g. Cornu, 1991; Davis & Vinner, 1986; Milani & Baldino, 2002; Szydlik, 2000; Tall, 1980; Tall & Vinner, 1981; Williams, 1991) who have investigated students' learning of limits. It appears as if the concept of limits is so complicated for students to handle that some of them develop an incoherent interpretation of it. Infinity is also a concept that causes difficulties for many students when they solve problems about limits of functions (Tall, 2001; Tirosh, 1991).

The research question posed in this article is the following: How do students explain their solutions to limit tasks in relation to their use of infinity? Limits and infinity in curricula and textbooks are discussed in connection with the students' achievements.

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THEORETICAL FRAMEWORK

A concept image is the total cognitive representation of a notion that an individual has in his or her mind (Tall & Vinner, 1981). A concept image is similar to a schema, which is a commonly used word for a mental representation (e.g. Cottrill, Dubinsky, Nichols, Schwingendorf, & Vidakovic, 1996; Juter, 2005a). The concept image might be partially evoked, and different parts can be active in different situations leading to possible inconsistencies. An individual's concept image might differ from the formal concept definition or the concept image in itself can be confusing or incoherent. The students in the study presented here are placed in problem solving situations designed to reveal how their concept images of limits develop during a semester of mathematics studies. The students' solutions and explanations of their solutions are considered to be expressions of their concept images.

The concept of infinity can cause difficulties. It is a notion about which an individual may have one or several intuitive representations (Tall, 1980). If there are multiple different representations evoked simultaneously the individual might be confused. When dealing with limits of functions one has no specific method or algorithm as one has for Diophantine equations for example. The limit process appears potentially infinite, and students can get the impression that there is no end to it (Tall, 2001). It can be difficult for them to work with items that are confusing in identity. Is it an object or a process? Infinity is not dealt with in a systematic manner in school (Skolverket, 2005), so the students have their own constructions of the concept. If defective or limited constructions are not challenged until the students need them, they have a long time to become established in the students' minds.

Tirosh (1991) discussed different types of infinities. Conflicting properties can result in various intuitive meanings, and previous experience can cause conflict with formal theory. Limits are frequently connected to *measuring infinity*, which comes from situations where, for example, a line segment of length L contains half as many infinitesimally small points as a line segment of length $2L$, since limits often are in a context where something becomes larger or smaller (Tall, 1992). This stands in contrast to *cardinal infinity* in Cantor's sense, which, for example, results in the fact that the number of rational numbers and the number of even numbers are the same. Intuitions of cardinal infinity are resistant to development and regular mathematical training, and sensitive to the context in which a problem is posed (Tirosh). These statements point to the importance of an explicit discussion of infinity at universities and schools and careful choice of tasks.

A common obstacle in students' understanding of limits of functions is that they think that functions can not attain their limits (Cornu, 1991; Juter, 2005a; Tall, 1993; Tall, 2001; Szydlik, 2000). There is also a possible mix-up of $f(a)$ and $\lim_{x \to a} f(x)$ (Davis & Vinner, 1986). These two flaws combined can block students in their struggle with tasks that could easily be solved with an equation for instance.

The theoretical framework is presented in greater detail in an exposition of the whole study (Juter, 2003).

EDUCATIONAL CONTEXTS OF STUDENTS AND TEACHERS

Students' and teachers' understanding are products of curricula from earlier years. Teachers are fostered by these documents, and they pass on traditions to textbooks and students. It is therefore important to consider students' achievements in the context of curricula and textbooks. Even if there is a new curriculum, it can take some time to implement it in the classrooms. The intended curriculum, expressed in formal documents, might differ from the implemented curriculum, expressed in, for example, textbooks. There is also a third factor, the achieved curriculum, which is what the students actually manage to accomplish (Valverde, Bianchi, Wolfe, Schmidt, & Houang, 2002). In this article we place our study about students' accomplishments in a broader educational context of textbooks and curricula.

Limits in the curricula

The curriculum of today for upper secondary school mathematics in Sweden states that the concept of limit shall be introduced in an informal way and with the purpose of enabling the pupil to handle derivatives. Limits are not mentioned per se, but required for the concepts that are mentioned, as illustrated by the following passage from the curriculum (Skolverket, 2005, http://www3.skolverket.se/ki03/front.aspx?sprak=SV&ar=0506&infotyp=8&skolform=21&id=MA&extraId=)

> *Matematik C* builds on Matematik B within arithmetic, algebra and functions. It also comprises differential calculus. In the course, problems concerning optimisation, changes and extreme values are treated. (Authors' translation)

The history behind this treatment is interesting. In the curriculum from 1965 (Läroplan för gymnasiet, 1969, p. 266) it is stated:

> Limits: Limits, when x tends to infinity, minus infinity and to a, shall be given a correct definition. As an illustration functions will be drawn in a coordinate system or in set diagrams, whereby the concept neigbourhood can be used. Further, examples shall be given of functions without limit and functions with no finite limit. In some case a calculation of an omega to a given epsilon shall be done such that for $x > \omega$ it follows that $|f(x) - b| < \varepsilon$. Rules for limits shall be presented for the sum, product and quotient of limits…(Authors' translation)

This plan was a consequence of the new math movement and changes followed quickly. The psychological reactions from students and teachers convinced curriculum developers that the plan was not realistic.

A revision came already five years later in Lgy70 (Skolöverstyrelsen, 1971). Then there is a division between central topics and less central topics in the curriculum. Under "Central Topics" it says (Skolöverstyrelsen, p. 105, authors' translation) "Limits and continuity intuitively with support from drawings." As "Less Central Topics" is taken "Formal definitions of limit and continuity." But there was a time when students at the age of 17 were supposed to learn about limits in a formal way based on a definition involving epsilon and omega and formal proofs. Students in the present study show examples of how problematic the learning of the limit concept can be at university level. The study illustrates that there is a need to consider carefully how to treat the concept of limit in the teaching of mathematics at different levels of schooling.

In 1981 a new supplement for the curriculum in mathematics for a natural science programme was agreed on (Skolöverstyrelsen, 1981). In this text the concept of infinity was not mentioned, and the concept of limit was also not mentioned per se. It was implicitly presented through examples both in the definition of the derivative of a function and of convergence of series. "Stringent treatment of convergence is not required" (Skolöverstyrelsen, p. 38, authors' translation). The example given was $\lim_{x\to 0}(1-\cos x)/3x^2$. Thus students are expected to work with the limit concept despite the fact that no information or instruction is given in the curriculum that the concept is supposed to be introduced to them. A consequence of this lack of transparency of the curriculum constructors' intentions could be that the teachers go on teaching as they did before the new curriculum came. Or it could be that the teachers are not enough aware of how to work with limits and do not pay enough attention to the concept.

Limits and infinity in textbooks

Limits of functions are introduced at upper secondary school before the students work with derivatives. The two textbooks (Björup, Körner, Oscarsson, Sandhall, & Selander, 1995; Björk & Brolin, 2000) investigated are commonly used at upper secondary schools in Sweden. They both present the theory of limits of functions very briefly as an introduction to derivatives. One has a definition stated informally, whereas the other does not even have a description of limits. The tasks in the textbooks were designed to develop the students' ability to calculate derivatives, and all examples and tasks with limits are intended to prepare students to meet the definition of the derivative with x or h tending to zero.

Both textbooks present the infinity symbol in examples, but none of the books have a discussion or tasks dealing with different properties of infinity.

In an earlier textbook for upper secondary school (Nyman, Emanuelsson, Bergman, & Bergström, 1980) the concept of limit is introduced as early as on page five in connection with the definition of the derivative of a function. The differential quotient for a tangent is constructed $(f(x) - f(a)) / (x - a)$, and the text says (p. 5):

> We now let B tend to A, that is x tend to a. If the quotient above has a limit, that it tends to a specific value, this is the slope of the tangent. The limit of the quotient is called the derivative of the function $f(x)$ in the point a. (Authors' translation)

The same textbook starts to explain limits of functions about 65 pages later by looking at a sequence of examples that get increasingly complicated. The concept of infinity is introduced without any explanation at all. The same goes for non-finite limits. Thus there is a long tradition in school mathematics of introducing complicated concepts without really discussing them or even mentioning or illustrating their meaning or definition.

Limits are reintroduced at university level. The textbook used in the course which is the basis for the present study (Hellström, Morander, & Tengstrand, 1991), introduces the concept of infinity briefly with some warnings about treating infinity without caution, and one of Zenon's paradoxes at the beginning of the chapter about limits. The texts about limits are rigorously stated with definitions and theorems. The examples show the reader how to use the definition of limits to solve problems and how to carry out various solution methods. There are a few explorative examples. The tasks are of the same character as the examples.

RATIONALE AND DESIGN OF THE STUDY

The empirical study reported here was divided into two parts. The first part was conducted in a spring semester and the second part the following semester, that is, the autumn semester of the following academic year. The students in both parts were reading analysis and algebra at basic university level. Different methods were used to collect data to address the question stated in the introduction: How do students explain their solutions to limit tasks in relation to their use of infinity? Apart from analysing curricula and textbooks, as mentioned before, questionnaires were used to document students' competencies and explanations. The use of questionnaires gives an opportunity to collect more data than, for example, interviews, which were conducted in other parts of the study. One drawback of using questionnaires is that students can not be asked to further elaborate on their answers, but in this part of the study, the students are regarded as a group, and hence there is a value in many student responses to reveal patterns.

In the spring semester 111 students solved five tasks about limits. Each of the first three tasks had four parts for the students to respond to, for example:

Task 1: a) Determine the limit: $\lim_{x \to 3}(2x+3)$ b) Explanation

c) Can the function $f(x)=2x+3$ attain the limit value in 1a? d) Why?

The students had worked with limits of functions in the course, so it was not new to them. Eleven days later they were given two additional tasks. 87 students participated in this second session. The last two tasks provided solutions that could be wrong or incomplete. This was stated on the sheet with the tasks, and the students were asked to give a complete and correct solution for each task. The tasks were selected to show students' responses at different levels of difficulty. The first three tasks were very similar to some tasks in the students' textbook, while the following two tasks were presented in an unfamiliar manner, but with content that was not too dissimilar from some tasks in their textbook. These types of tasks were not common in the textbooks for upper secondary school, which were studied.

A new group of students were given the same tasks the following autumn semester. They were given all the five tasks at the same time, after the notion of limits of functions had been dealt with. The difference in treatment was for practical reasons. 78 students took part in that study.

In both cases the tasks were part of three questionnaires with questions about limits and attitudes towards mathematics. Two interviews with each of the 15 students were conducted in the autumn study (Juter, 2003; Juter, 2005a).

The tasks were constructed to focus on different aspects of the limit concept. The difficulty varied in order to identify the level the students could handle. The students were instructed both orally and in writing at each session when they responded to the questionnaires to make sure that it was clear to them.

The data collected have been rewritten and then categorised with the aid of the computer program, NUD*IST. The categories were decided from the raw interview and questionnaire material. They were not created in detail in advance, except that the right and wrong answers made up different categories. There were sub-categories in each of them based on the students' justifications of their responses. The sub-categories were roughly determined before the data was collected. This process led to a number of categories. Categories with similar types of reasoning were merged together to make the presentation more accessible (Juter, 2003; Juter 2004). This way to work with the data gave different types of category systems for the different tasks. The category systems were flexible so that new categories emerging from the data could be implemented if they were relevant to the study. The work with the categories is similar to the methods of grounded theory (Glaser & Strauss, 1967) where categories evolve from data. It does not stand in conflict with the theories of concept formations presented in the theory section since the aim is to get as much information about how the students learn as possible.

SAMPLE OF DATA AND RESULTS

Examples and student responses are presented in this section to show some results from the study. The selection of tasks was particularly done to bring forward cases were students reasoned about infinity. The results of the first two parts of Tasks 2 and 3 are briefly presented and the last two parts of the tasks are presented in greater detail. The results of Task 5 are also presented in detail.

Task 2: a) Determine the limit: $\lim_{x\to\infty}\frac{x^3-2}{x^3+1}$. b) Explanation

c) Can the function $f(x)=\frac{x^3-2}{x^3+1}$ attain the limit in 2a? d) Why?

Most students were able to solve correctly Task 2a-b. A majority of the students recognised the dominant term and disregarded the constants. Algebra was also used to explicitly calculate the limit. 15 % of the students were unable to handle the algebra required for such calculations, that is divide numerator and denominator with the dominant term. Grevholm (2003) confirmed that university students often have inadequate algebraic knowledge to be able to work with mathematics at this level. The results of Task 2c-d are displayed in Table 1. (R) and (W) denote right and wrong answers respectively.

Category	2c	2d	Spring	Autumn
$x^3-2\neq x^3+1$ (R)	No	$x^3 - 2$ can never be equal to $x^3 + 1$ for the same value of x	7 (6.3)	17 (22)
-2 & 1 (R)	No	terms -2 and 1 will always remain	7 (6.3)	5 (6.4)
No explanation (R)	No	-	18 (16)	8 (10)
Infinity reason (W)	No	Since x never attains the value ∞	16 (14)	21 (27)
Theory (W)	No	The function tends to the limit, it does not attain it	7 (6.3)	6 (7.7)
No reason (W)	Yes	-	15 (14)	7 (9.0)
Empty or misinterpretation		The answer has no connection to the question or is missing	42 (38)	17 (22)

Table 1. Typical student answers to tasks 2c-d. Number of students (% of the group)

There are five categories for the answer "No", which is the correct answer, but the categories labelled "Infinity reason" and "Theory" are marked "W" for wrong. The explanations in these categories do not correctly justify why the function can not attain the limit in the task. Some explanations in the "Infinity reason" category are difficult to label as right or wrong, for instance the example given in the table. The student's explanation is that x never can be equal to infinity, and there is a philosophical dimension to the reasoning. The explanation relies on the fact that infinity is unreachable and a termination of the limit process is therefore impossible, but then no function would be able to attain its limit in similar situations. Table 1 shows that about one quarter of the students gave a correct answer based on incorrect reasoning. The definition of limit and infinity caused many of the errors here. None of the students who answered "Yes" provided an explanation for their answer.

The third task was the following:

Task 3: a) Determine the limit: $\lim_{x \to \infty} \frac{x^5}{2^x}$. b) Explanation.

c) Can the function $f(x) = \frac{x^5}{2^x}$ attain the limit in 3a? d) Why?

Almost all students managed to calculate the limit in Task 3a and to explain their answer with a standard limit in Task 3b. There was a greater variety in the students' answers to Tasks 3c and 3d as shown in Table 2 below. (R) and (W) denote right and wrong answers respectively.

Category	3c	3d	Spring	Autumn
$x = 0$ (R)	Yes	For $x = 0 \rightarrow f(0) = \frac{0}{1} = 0$	15 (14)	16 (21)
No explanation (R)	Yes	-	5 (5)	6 (7.7)
Does not reach limit (W)	No	We can only get infinitely close.	14 (13)	16 (21)
$x^5 \neq 0$ or $\frac{0}{0}$ (W)	No	Then the numerator has to be zero and it never is.	22 (20)	12 (15)
Right for wrong reason (W)	Yes	Because the denominator attains a much larger number for large x.	5 (5)	16 (21)
Empty or misinterpretation		The answer has no connection to the question or is missing.	47 (42)	12 (15)

Table 2. Typical student answers to tasks 3c-d. Number of students (% of the group)

Less than twenty percent of the students delivered a correct answer with a correct explanation even though the task is of standard type and the function is well known to the students.

A large part of the students was unable to correctly decide if the function can attain the limit. The third category includes responses that reveal the confusion of limits of functions and functions as described by Davis and Vinner (1986). The function's ability to attain the value zero is influenced by the context of limits and the properties of limits as the students perceive them. This confusion is apparent in other parts of the results from the study as well (Juter, 2003; Juter, 2005a) where students often think that the limit definition states that limits are unattainable for functions.

The fifth category includes students who demonstrate problems in dealing with infinity. The students in this category had the correct answer but did not offer a correct justification for it. The student who gave the example in Table 2 ("Yes", "Because the denominator attains a much larger number for large x") drew his or her conclusions too far in the reasoning about infinity and stated that the function eventually reaches the value zero as x tends to infinity. It is again a philosophical matter of what happens at infinity. Such an interpretation of infinity is the opposite of the example in the fourth category in Table 1 where the student claimed that x could not reach infinity.

The fifth task was stated as a problem with a given solution:

Task 5: **Problem**: Determine the following limit: $\lim_{x\to\infty} x \sin\left(\frac{2}{x}\right)$.

The students were given the following solution:

Solution:

$$x \sin\left(\frac{2}{x}\right) = \frac{\sin\left(\frac{2}{x}\right)}{\frac{1}{x}}$$

We know that $2/x \to 0$ when $x \to \infty$ and $1/x \to 0$ when $x \to \infty$. The limit $\lim_{x\to\infty} \sin x / x = 1$ implies that

$$\frac{\sin\left(\frac{2}{x}\right)}{\frac{1}{x}} \to 1 \text{ when } x \to \infty.$$

Then the students were asked to make the solution correct and complete.

Table 3 shows the categories for the students' explanations. (R) and (W) denote right and wrong answers respectively.

Category	5	Spring	Autumn
Completely correct (R)	$\frac{\sin\left(\frac{2}{x}\right)}{\frac{1}{x}} = \frac{\sin\left(\frac{2}{x}\right)}{\frac{2}{x}} \cdot 2 \to 2$	10 (11)	17 (22)
Partly correct (R) Incomplete	$2/x$ is not the same thing as $1/x$ therefore you can not use limits on limits	9 (10)	11 (14)
Reasoning (W)	$2/x \to 0$ when $x \to \infty$ so the denominator has to $\to 0$ and the expression $\to \infty$	31 (36)	18 (23)
No change (W)	Nothing	4 (4.6)	1 (1.3)
Empty or unclear	The answer is missing or does not make any sense	33 (38)	32 (41)

Table 3. Typical student answers to Task 5. Number of students (% of the group)

Not many students gave a complete and correct solution to this task. Ten from the spring study and seventeen from the autumn study did. Nine, respectively eleven, students pointed to the error in the given solution but did not give an alternative one. Several students were reasoning about infinity in an incorrect manner as exemplified by the student answer given in the third category in Table 3. Such answers illustrate how some of the students calculated limits locally and did not consider the function as an entity. Student answers in the category labelled "No change" also showed evidence of this local view on limits. The difference is that in the latter case, the solution was given and in the former the students made the solutions themselves.

DISCUSSION

A large number of the students solved the tasks in words of approaching the limit. The idea that limits are not attainable is revealed in quite a few parts of the data. Some students used it as an argument for functions not to attain limits in parts c and d of the tasks. There is well-documented evidence that some students have this misinterpretation of the limit definition (Cornu, 1991; Tall, 1991). Some students did not separate the part with the limit from the part with the function, that is they mixed up $f(a)$ and $\lim_{x \to a} f(x)$, as Davis and Vinner (1986) described. If students believe that limits are unattainable, there might be problems for them to analyse functions.

Infinity, as treated in the fifth category in Table 2, does not seem to be explicitly handled at upper secondary school or at universities, judged from the textbooks studied. Students are influence by their personal conceptions of infinity when they solve problems, which can lead to conflicts (Tirosh, 1991). If the students do not have any prior experience of discussing or even thinking about different features of infinity, they will not have the ability to correctly analyse limits. All tables show problems with infinity. Table 1 (category "Yes") and Table 2 (category "Right for the wrong reason") have categories where the solutions suggest that limits are attainable for very large values of x or if x is equal to infinity. In the first case, this is not explicitly stated and the students could be guessing. When the constants are disregarded in the first part of the task there is a possibility that the students keep the constants away from their reasoning and therefore claim that the function can attain the limit. It can also be that the constants are regarded as embedded into infinity and hence not considered anymore. The absence of explanations makes it impossible to know exactly why the students answered "Yes".

More than one quarter of the solutions were correct with an inaccurate motivation in Task 2, and more than every tenth of the solutions to Task 3 were correct with an inaccurate explanation. Many of the explanations are wrong based on an inadequate conception of infinity, or perhaps an inability to express a correct conception. Experiences of success with incorrect rationale give a false sense of security about the accuracy of the concept image, and our hypothesis is that it can be avoided with an explicit treatment instead of, as suggested in curricula and textbooks, an intuitive approach.

The fact that textbooks do not deal with infinity in a thorough and explicit manner can give students and teachers the impression that it is sufficient for students to work with their possibly vague conceptions from childhood and school. The results presented in this article do not support such impressions. There is a need for students to explicitly learn limits of functions and infinity to prevent erroneous reasoning, as has been exemplified in the present study. This is not the first occurrence of such reports on students' learning difficulties at universities. Högskoleverket (2005) in Sweden recently issued a warning about students' lack of interest in mathematics and about their insufficient pre-knowledge in mathematics. The suggested solution is to have a better cooperation between upper secondary schools and universities. The transitions to the formality of the subject treatment at universities can then be less dramatic. In their conclusion they point to the fact that the problems discussed are not new and that long-term and systematic interventions are missing. It is necessary now to make sustainable initiatives and changes (Högskoleverket, p. 65). We claim that if this is a serious commitment it would be valuable to take results from studies such as this on learning limits.

CONCLUSIONS

Students expose a variety of ways of explaining their solutions to limit problems. Students with adequate reasoning in their explanations are not nearly as many as students who make mistakes, often caused by students' conceptions of infinity, solving these fairly easy tasks. A result such as this can not be accepted as sufficient. One conclusion is that complex concepts such as infinity and limits should be introduced and handled more carefully by teachers and textbook authors at both school and university level. A way to continue the work could be to implement and evaluate some intervention programme in order to construct alternative learning trajectories for students. Teachers at universities must be aware of how students have created their early conceptions of limits and infinity, and try to offer opportunities to develop these, often limited and insufficiently matured, conceptions to more meaningful concepts.

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"MATHEMATICS IS IMPORTANT BUT BORING": STUDENTS' BELIEFS AND ATTITUDES TOWARDS MATHEMATICS

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Students' beliefs and attitudes towards mathematics teaching and learning is the focus of the study described in this paper. Some preliminary results from research carried out in Norway in 2005 are given, which focus on first year students in upper secondary school. The answers from the ninth grade students in 2005 are briefly compared with students' responses from 1995 when corresponding data was collected within the KIM project in Norway. Both of these studies use a questionnaire elaborated in 1995. Some of the aspects related to a similar study amongst Estonian students, that will take place in spring 2006, are also discussed.

INTRODUCTION

Students' beliefs and attitudes towards mathematics teaching and learning play an important role in mathematics education (McLeod, 1989). The learning outcomes of students are strongly related to their beliefs and attitudes towards mathematics (Furinghetti & Pehkonen, 2000; Leder, Pehkonen, & Törner, 2002; Pehkonen, 2003; Schoenfeld, 1992; Thompson, 1992). Like Lester, Garofalo, and Lambdin Kroll (1989) point out:

> Any good mathematics teacher would be quick to point out that students' success or failure in solving a problem often is as much a matter of self-confidence, motivation, perseverance, and many other noncognitive traits, as the mathematical knowledge they possess. (p. 75)

Thus assessing or evaluating students' mathematical knowledge must be made with the awareness of their beliefs. Systematic inquiry into students' affective domain has grown a great deal during the last twenty years and many countries have been included in the research (Lester, 2002). The study described in this paper aims to find out what kind of beliefs and attitudes towards mathematics Norwegian students hold. This study is part of a larger research project that includes two countries – Estonia and Norway. The general idea of the larger study is to expose Estonian and Norwegian students' attitudes and beliefs about mathematics teaching and learning,

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some reasons why these kinds of beliefs are held and any possible relationship between students' beliefs and attitudes and their mathematical performance. This paper presents some preliminary results from the Norwegian study as the study is ongoing. The research question (that is one of the four questions of the larger study) can be formulated: *What kind of beliefs and attitudes towards mathematics teaching and learning do students from one urban area in Norway hold?*

SOME THEORETICAL CONSIDERATIONS

To find a well-developed, well-defined theoretical framework in the study of beliefs and attitudes is a challenge and the endeavour to develop one coherent framework for this area has been the aim of several workshops in different conferences, for example, the 28th Conference of the International Group for the *Psychology of Mathematics Education* (PME) in 2004. According to Hannula (2004) 'there is a considerable diversity in the theoretical frameworks used in the conceptualisation of affect in mathematics education' (p. 107). Goldin (2004) supports that point of view by acknowledging that 'we do not have a precise, shared language for describing the affective domain, within a theoretical framework that permits its systematic study' (p. 109). Therefore we do not present one final coherent theoretical framework for our study in this section but rather describe different notions and discuss the relationship between their conceptions.

As *belief* and *attitude* 'are not directly observable and have to be inferred, and because of their overlapping nature' (Leder & Forgasz, 2002, p. 96) it is problematic to have a common definition of these notions. Several researchers do not see the possibility of isolating these concepts and define an attitude as a collection of beliefs (e. g. Rokeach, 1972; Sloman, 1987) or classify belief as one component of attitude (e. g. Aiken, 1980; Statt, 1990). Cooper and McGaugh (1970) make a useful clarification when they note that

> … one has an attitude *toward* and a belief *in* or *about* a stimulus object… Belief connotes an attitude which involves or identifies the subject deeply with the object. (p. 12, emphasis in original)

Different researchers relate the notion belief to different aspects. For example, Kloosterman (2002) sees a direct connection between belief and effort. Some researchers classify beliefs as a subclass of conceptions (Hart, 1989; Thompson, 1992), others explain conceptions as a subset of beliefs (Pehkonen, 1994). Schoenfeld (1992) talks about beliefs as an individual's feelings and understandings. A considerably wide definition is given by Rokeach (1972) who claims:

> A belief is any simple proposition, conscious or unconscious, inferred from what a person says or does, capable of being preceded by the phrase "I believe that…". (p. 113)

The definitions of belief and attitude formulated in this paper are adopted from Pehkonen (2003) and Triandis (1971) as we found them most suitable for our study. Triandis (1971) explains the concept of an **attitude** by the following words:

> Attitudes involve what people *think* about, *feel* about, and how they would like to *behave* toward an attitude object. Behaviour is not determined by what people *would like* to do but also what they think they should do, that is, social *norms*, by what they have usually done, that is *habits*, and the *expected consequences of behaviour.* (p. 14, emphasis in original)

This definition takes into account two aspects that are relevant for us. Firstly, it points out two basic verbs. *To think* relates to the person's cognitive domain. *To feel* can be considered to be relevant when the affective domain is under discussion. The study described here uses a questionnaire for illuminating mostly students' *thinking* about the mathematics, however the larger study includes lesson observations and interviews to expose some *feelings* that will be interpreted based on the behaviour. Therefore, both conceptions are relevant. Secondly, the definition includes the social perspective that points out several factors that influence students' beliefs and hence their behaviour (see Pehkonen (1995) for more details) and how cautious one must be in drawing conclusions based on the sources (questionnaire for example) as one can never claim that the environment around the informant does not have any influence of her/his thinking and behaviour (answering to the questionnaire for example).

Pehkonen (2003) follows the similar idea and does not situate **beliefs** in the human affective domain but somewhere between the cognitive and affective domains, in what he calls the "twilight zone", as he argues that beliefs have 'a component in both domains' (p. 1). He understands

> beliefs as an individual rather stable subjective knowledge, which also includes his/her feelings, of a certain object or concern to which tenable grounds may not always be found in objective consideration. (Pehkonen, p. 2)

We agree as the interplay between *the thinking* and *the feeling* is unavoidable because on the one hand beliefs are a part of persons' knowledge that is highly subjective and on the other hand the conceptions *feelings* and *beliefs* are often overlapping and cannot be distinguished.

THE KIM PROJECT

One of the studies of which the results are discussed briefly in this paper was carried out in Norway in 1995 amongst students from the grades 6 and 9. It was called the KIM[1]-project and it collected data on students' understandings of key concepts in the national mathematics curriculum (Streitlien, Wiik, & Brekke, 2001). This project elaborated a questionnaire (later called the KIM questionnaire) that contained 125 items designed to expose students' beliefs about mathematics, mathematics teaching and learning. The KIM questionnaire used Likert-scale type responses. Leder and Forgasz (2002) describe the Likert-scale in the following terms:

> A series of statements about the attitude object comprise a Likert-scale. Items are typically rated from "strongly agree" to "strongly disagree" and five divisions are very common. (Leder & Forgasz, 2002, p. 98)

The questionnaire was administered to 1482 students from 6th grade and 1183 students from 9th grade. Some of the results are compared with the results from year 2005 and will be reviewed later in this paper.

Within the same project students' mathematical performances and their attitudes were investigated. Students' (273 from grade 6 and 234 from grade 9) responses to mathematics tests were compared to the questions about their professed beliefs and attitudes towards mathematics and its teaching and learning. The study concluded: Those pupils who state a positive interest towards mathematics, on average, performed better on the mathematics tests than their fellow students (Streitlien, Wiik, & Brekke, 2001).

METHOD

Research participants

Our study, carried out in spring 2005 in Norway, was one part of the LCM-project within the Norwegian Research Council's KUL[2] programme. Our interest embraced students' beliefs and attitudes towards mathematics teaching and learning and our informants were students from grades 7, 9, and first year in upper secondary school. Six schools from one urban area in Norway that are partners in the KUL-LCM project took part, and about 370 students were involved in our study.

[1] KIM – Kvalitet i Matematikkundervisningen, translated as 'Quality in Mathematics Teaching'.
[2] KUL – Kunnskap, utdanning og læring, translated as 'Knowledge, education and learning'.

Data sources

Based on the literature, the questionnaire can be considered as a common instrument to study beliefs and attitudes (Leder & Forgasz, 2002) and several researchers use it as a main tool (e.g., Graumann, 1996; Pehkonen, 1994; Pehkonen, 1996; Pehkonen & Lepmann, 1994; Perry, Howard, & Tracey, 1999; Tinklin, 2003; Tsamir & Tirosh, 2002; Vacc & Bright, 1999; Williams, Burden, & Lanvers, 2002). In our study we used virtually the same questionnaire as the KIM study. Some changes were carried out in response to our research questions and the data from the pilot study (see Kislenko, Breiteig, & Grevholm, 2005). Hence, our questionnaire contained 126 statements divided into 13 groups which were beliefs about: mathematics as a subject (16 statements); learning mathematics (14); own mathematical abilities (11); own experiences (security) during mathematics lesson (4); teaching of mathematics (17); learning a new topic in mathematics (8); environment in class (10); teaching tools in mathematics lessons (6); using computer in spare time (2); using ICT in mathematics (computer) (18); own evaluation of importance of mathematics (8); evaluation of the teacher (10); mathematics and the future (2).

For example, statements in the class "new topic in mathematics" were: "the teacher starts by giving us rules"; "we start with a practical problem from a daily life"; "the teacher asks us what we know about the new topic"; "the teacher leaves the textbook to decide what to do", etc.

Following Gorard's (2001) advice 'since there may be so little similarity between responses to forced-choice and open-ended questions it is probably advisable to mix the types of questions in any instrument' (p. 93), some open ended questions were included too in the questionnaire. One other dissimilarity with the KIM questionnaire was that our questionnaire was web based, that is students filled up the questionnaire which was made available on an Internet webpage. Each student was issued with a unique code which she/he used to log into the questionnaire page.

RESULTS

As mentioned above we collected our data in spring 2005 and closed the questionnaire on 15th June. Therefore, the analysis presented below is preliminary. Deep and structured data analysis is due to start during the autumn 2005. But some initial and tentative results are given in the following subsections. First, the data from the KIM project is compared with the data from our project with focus mainly on grade 9. Then, there are some preliminary results from our research with focus on students in upper secondary school. The answers from grade 9 students and first year students in upper secondary school are compared. And finally, some considerations concerning a relationship between mathematical performance and beliefs are discussed.

Comparison between the studies

Out of 90 grade 9 students included in the study, 85 completed the questionnaire. Thus, the response rate (c 94.4 %) was very high in these classes. Based on the approach by Streitlien, Wiik, and Brekke (2001) we formed 5 groups of statements about mathematics: interest, usefulness, self-confidence, diligence, and security. In the following table (Table 1) some results from our study are compared with the results from the KIM study. The statements are taken from the groups *interest* and *usefulness*. The columns in the table refer respectively: totally agree (Ta), partially agree (Pa), uncertain (U), partially disagree (Pd), and totally disagree (Td). The following discussion is not only based on Table 1 but on all the data from the questionnaires. Only a small part of data is presented due to limited space.

Interest and usefulness	Ta	Pa	U	Pd	Td
1b Mathematics is exciting and interesting.	13	50	18	12	7
1b Mathematics is exciting and interesting KIM.	8	36	18	22	15
1i I never get tired of doing mathematics.	8	9	15	38	29
1i I never get tired of doing mathematics KIM.	3	11	12	36	38
1j I like to do and think about mathematics also out of school.	2	19	13	32	33
1j I like to do and think about mathematics also out of school KIM.	3	9	13	31	44
1k Mathematics helps me to understand life in general.	18	31	26	15	10
1k Mathematics helps me to understand life in general KIM.	6	20	31	23	19
1l Mathematics helps those who make important decisions.	15	32	29	18	6
1l Mathematics helps those who make important decisions KIM.	13	28	38	12	8
1m Mathematics is boring.	15	37	15	18	14
1m Mathematics is boring KIM.	25	29	13	22	10
1p Good mathematical knowledge makes it easier to learn other subjects.	25	32	21	14	7
1p Good mathematical knowledge makes it easier to learn other subjects KIM.	14	29	36	14	7

Table 1: Interest and usefulness 1995 and 2005. Frequencies in percentages.

As can be seen from Table 1 and our data generally there seems to be a small tendency towards more positive beliefs about mathematics amongst students from the 2005 survey. Students' views tend to be more radical and they are more certain in their statements, especially in relation to the usefulness of the mathematics. Students agree more that mathematics is useful in different situations in life and acknowledge that being good in mathematics helps to learn other subjects. The comparison in the groups of *self-confidence* and *diligence* did not give any considerable dissimilarity. Most of the students find that mathematics is a difficult subject and they have to work hard and solve many exercises to be good at mathematics. Students from both studies (around 85 %) understand that it is their responsibility to learn mathematics and acknowledge mathematics to be a subject which increases in difficulty as they progress through the grades. Nevertheless, there is still a close match of agreement that mathematics is boring. This conclusion is rather striking in the situation where 97 % of students say that mathematics is important. It means that students have a high motivation to learn but for some reasons they are bored in the mathematics lessons. Indeed, the technology that can help teachers to make mathematics lessons more challenging and fascinating has developed enormously during these 10 years (1995–2005) but the phenomenon of "being bored" in mathematics lessons is still quite common amongst the students in grade 9. What can be the reasons for this phenomenon? Is it only the matter of mathematics as a school subject or is it something else that is hidden behind the scenes?

Upper secondary school compared to 9th grade – comparison within the study

There were 160 first year students from upper secondary school who completed the questionnaire. The following table (Table 2) has a similar construction to Table 1 but now the students from 9th grade are compared with students from first year of upper secondary school (later UppSec) inside of our study to see if there are any tendencies towards students' progress through the grades. The group of *usefulness* is chosen since the differences within this group are most noticeable.

Usefulness	Ta	La	U	Ld	Td
1a Mathematics is important G. 9.	75	22	0	0	2
1a Mathematics is important UppSec.	54	32	9	5	1
1e Mathematics is useful for me in my life G. 9.	66	20	7	5	2
1e Mathematics is useful for me in my life UppSec.	37	40	8	10	5
1f It is important to be good at mathematics in school G. 9.	51	36	8	2	2
1f It is important to be good at mathematics in school UppSec.	26	53	10	9	2
1g I need mathematics in order to study what I would like after I finish school G. 9.	41	27	18	5	9
1g I need mathematics in order to study what I would like after I finish school UppSec.	31	21	28	12	9

Usefulness	Ta	La	U	Ld	Td
1k Mathematics helps me to understand life in general G. 9.	18	31	26	15	10
1k Mathematics helps me to understand life in general UppSec.	6	27	22	24	21
1l Mathematics helps those who make important decisions G. 9.	15	32	29	18	6
1l Mathematics helps those who make important decisions UppSec.	9	28	35	14	13
1o I do not need to know mathematics G. 9.	4	6	11	26	54
1o I do not need to know mathematics UppSec.	8	10	17	30	35
1p Good mathematical knowledge makes it easier to learn other subjects G. 9.	25	32	21	14	7
1p Good mathematical knowledge makes it easier to learn other subjects UppSec.	19	48	21	8	5

Table 2: Usefulness. Grade 9 and first year in upper secondary. Frequencies in percentages.

An aggregate value was formed for this group of items to get a better insight into the differences. This technique follows Streitlien, Wiik, and Brekke (2001). The items were coded so that positive interest always gave high values, for example: *Mathematics is important* (from 5 = "totally agree" to 1 = "totally disagree") and *I do not need to know mathematics* (from 1 = "totally agree" to 5 = "totally disagree").

A scale was made from the 8 items of Table 2. A new variable is an opinion that mathematics is useful. Table 3 shows the average of the responses to the eight items distributed by grade.

Usefulness of mathematics	
Grade 9	3.96
First year in upper secondary school	3.62

Table 3: Usefulness of mathematics. Average of items in Table 2. Neutral value 3.

Two issues in Table 3 drew our intention. Firstly, both values are higher than the average value (which is 3) which indicates that students at both levels consider mathematics to be highly useful and this corresponds to the conclusions made earlier in this paper. Secondly, there is a considerable decrease (0.34) in perception of usefulness between the students from grade 9 and first year in upper secondary school. These results are similar to the results from the KIM study. Streitlien, Wiik, and Brekke (2001) point out that one of the possible reasons could be an effect of a decreasing general motivation for schooling. It also might be related to the content of the subject as mathematics becomes more difficult and abstract as students progress through the grades.

To illustrate other aspects we made the same kind of comparison within the other four groups and made a similar synthesis (Table 4).

Groups of items	Grade 9	First year in upper second.
Interest	2.75	2.79
Self-confidence	3.09	3.09
Diligence	4.02	4.00
Security	3.47	3.40

Table 4: An average of the items in different groups. Neutral value 3.

Table 4 reveals that there appears to be little difference between the students from grade 9 and first year in upper secondary school.

Some interesting results from our study include:

- 86 % of students agree that mathematics is important and 77 % acknowledge the usefulness of mathematics in their lives.
- 48 % claim that mathematics is boring, while 65 % are sure they need to know mathematics.
- Most students understand that they have to work hard even if they do not enjoy working with mathematics in lessons (76 %) and it is their responsibility to learn mathematics (89 %).
- All the students, except one, find it important to know something about numbers and calculations and only two students think it is unimportant to know how to solve practical problems.
- There is still a huge emphasis on "mental calculations" amongst the first year students in upper secondary school as 96 % acknowledge that it is important to become good at this.

These are only some general results from our study in one urban area in the southern part of Norway. The deeper data analysis will enlighten more the issues of our interest, and some reasons behind the kinds of answers that we got from our respondents will be discussed in the future.

The analysis of the relationship between students' performances and their beliefs

Andreassen explored the mathematical performance of the same students who participated in our attitude study using a written mathematics test and presented the results in her master's thesis (Andreassen, 2005). We plan to complement her study by investigating the relationship between students' performance and beliefs as the study progresses. The coding process of the students in both studies makes that analysis possible. Streitlien, Wiik, and Brekke (2001) note that there occurred a significant connection between the performance of the mathematical test and the self-confidence in mathematics in the KIM study. We have an idea that a positive attitude towards mathematics and the teaching of the subject leads in general to the motivation of students to learn more, and conversely, high performance in mathematics, combined with the experience that one achieved well in the subject, leads to positive attitudes towards mathematics. Presenting our results according to a possible relation between attitude and achievement we take seriously into consideration a caution by Cockcroft (1982) who warns that despite the "teacher's perception that more interesting and enjoyable work will lead to greater attainment … research certainly suggest caution against overoptimism in assuming a very direct relation between attitude and achievement" (p. 61).

BRIEF INTRODUCTION TO OUR FUTURE RESEARCH IN ESTONIA

The research in Estonia is planned to take place in the spring 2006. About 10 schools from one urban area in Estonia will participate. These schools will be selected from schools which collaborate and take part in a school-practice program with Tallinn University. Part of the data will be gathered through the same questionnaire (translated into Estonian) which we used in our Norwegian study. To get insights into the reasons behind the attitudes, qualitative methods will be included. Therefore, lesson observations and interviews are planned (in grade 7, grade 9, and first year in upper secondary school) during a two months period. The supposed number of lessons observed in a week is about 10. Individual interviews with the teachers will start before the lesson observations and will be carried out frequently during the observation process. Students will be grouped according to their answers in the attitude questionnaire and these groups will comprise students who hold more or less similar attitudes towards mathematics (for example, "positive", "rather neutral", and "negative"). This in order to get a common characteristic of groups for a better understanding of the reasons behind students' attitudes. Personal interviews with the students are not planned yet, but if there turns out to be a need for having them, it is not excluded.

According to the TIMSS study in 2003, Norwegian students tend to have a significantly higher level of self-confidence than students from Estonia (Mullis, Martin, Gonzalez, & Chrostowski, 2004). We hope that our Norwegian and Estonian studies will help to illuminate the reasons for this phenomenon.

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A STUDY OF TEACHERS' VIEWS ON THE TEACHING AND LEARNING OF MATHEMATICS, THEIR INTENTIONS AND THEIR INSTRUCTIONAL PRACTICE

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This is an ethnographic study of how teachers in lower secondary school are implementing the current mathematics curriculum, L97, in Norway. The methodology in the study includes focus groups, individual conversations, self estimation and classroom observations. I found different degrees of coherence between what teachers say they do and what they actually do in the classroom. In this paper I focus on the methodology and the analysis of data from one teacher who wants to focus on students' conceptual understanding in mathematics as L97 does, rather than exercising drill and procedures. Also possible constraints prevented him from carrying out "ideal" teaching and some of L97's recommendations are considered.

OBJECTIVES AND SIGNIFICANCE OF THE STUDY

Reform 97 (R97) is the educational reform in Norway that took place in 1997. As part of the more wide-ranging Reform 97, which affected the whole of the compulsory education system, a new curriculum was implemented in August 1997. This curriculum or syllabus for grades 1–10 (age 6 to age 15) is referred to as L97 (Hagness, Veiteberg, Nasjonalt læremiddelsenter, & Kirke-, utdannings- og forskningsdepartementet, 1999).The curriculum describes different working methods in all subjects in general and in mathematics in particular. According to my interpretation of the curriculum, it encourages an investigative approach to teaching. It stresses that pupils shall be active in the learning process. They shall be experimenting and exploring and through collaboration with each other acquire new knowledge and understanding.

My research questions are:

How are teachers in their mathematics teaching practice responding to the L97's recommendations?

What kinds of interactions between the teacher and the student are observable in the mathematics classroom?

How are teachers' practices in the classroom related to their beliefs about teaching and learning mathematics and to their goals for students in the subject?

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UNDERLYING THEORETICAL FRAMEWORK

The mathematical part of the curriculum, L97, can be seen to reflect a constructivist view on learning and teaching mathematics. I see several principles which I interpret as reflecting a constructivist view: Pupils shall be encouraged to build up knowledge largely by themselves, they shall be active, enterprising and independent, they shall acquire new knowledge by exploring and experimenting and pupils' previous knowledge is regarded important in the learning process. L97 encourages discussion and reflection and emphasises how students' misconceptions can be ground for further learning. This seems to indicate an approach to learning reflecting Piaget's notions about assimilation and accommodation, which are two central concepts in Piaget's constructivist theory. Development of cognitive constructions implies both an assimilation process and a process of accommodation. Assimilation means integrating new elements into existing structures. In terms of learning mathematics the process of assimilation occurs when a mathematical challenge can be solved by already existing cognitive structures. The challenge is to fit new conceptual structures into already existing structures.

> The general concept of assimilation also applies to behaviour and not only to organic life. Indeed, no behaviour, even if it is new to the individual constitutes an absolute beginning. It is always grafted into previous schemes and therefore amounts to assimilating new elements into already constructed structures (innate as reflexes are, or previously acquired). (Piaget, 1970, p.707)

Through this quote Piaget indicates that a person's conceptual structures don't have a fixed starting point in its developments and that some kinds of cognitive structures are always in a person's mind. This implies that students always have some already developed conceptual structures in mathematics, and it is this previous knowledge L97 puts weight on. In my study I found that the teachers often referred to students' prior knowledge. "Refprioknow" (referring to prior knowledge) is one of the codes I have used in the analysis process. A prerequisite for learning is that not only a process of assimilation but also a process of accommodation must take place. Referring to the biological assimilation and accommodation Piaget says: "Similarly in the field of behaviour we shall call accommodation any modification of an assimilatory scheme or structure by the elements it assimilates" (Piaget, 1970, p. 708). Thus a process of accommodation is a "process of continual revision of structure" (Noddings, 1990, p. 9). Referring to Piaget, von Glasersfeld (1995) says that "the most important occasions for accommodation arise in social interaction" (p. 11). This underpins the importance of interactions that occur both between students and between teacher and student(s). This I address through my second research question (what kinds of interactions between teacher and students are observable in the mathematics classroom) and the analysis of these interactions are outlined later in this paper.

My interpretation of L97 in constructivist terms is underpinned by Confrey (1990) who emphasises that according to a constructivist epistemology pupils themselves can be seen to construct their understanding through a process of reflection, by von Glasersfeld (1985) who explains how teachers should be more interested in children's misconceptions, and by Davis, Maher, and Noddings (1990) who say that learning mathematics requires constructions and that mathematical activity in a mathematical community is a common thread in what a constructivist view implies. From a constructivist view on teaching and learning mathematics, an important consideration within a teaching situation is that there is a closeness in perspectives between the participants in a discussion (Jaworski, 1994). This is important both when the teacher is the speaker and the student the listener and vice versa. Von Glasersfeld (1995) puts weight on the teacher being concerned with what goes on in the student's head and not only focusing on the trainee's performance as the trainer does. "The teacher must listen to the student, interpret what the student does and says, and try to build up a "model" of the student's conceptual structure" (von Glasersfeld, p.14). Cobb (1988) introduces the term miscommunication and addresses the issue of how miscommunication occurs in the classroom. During the classroom observations I experienced lack of closeness in perspectives and miscommunication between the teacher and the students. One of these occurrences is presented in the Findings section of this paper.

Teachers' own beliefs about teaching and learning of mathematics and about the nature of mathematics determine whether opportunities for learning mathematics take place (Davis, Maher, & Noddings, 1990). Teachers' beliefs influence the features of the classroom they create (Goos, Galbraith, & Renshaw, 1999). However, Goos et al. acknowledge that there exist constraints and pressure that may prevent teachers from acting according to their beliefs. This is in accordance with my findings that are outlined later in this paper.

METHODOLOGY

In the study, I use research methods fitting largely into an ethnographic approach (Bryman, 2001). I have worked with four selected mathematics teachers in lower secondary school. I arranged three focus groups with a total of fourteen teachers from both primary, intermediate and lower secondary school from which I selected the four teachers fitting some criteria. After the first focus group I decided to narrow my study to lower secondary school, I wanted both male and female teachers, and the teachers had to give their consent to participate in my study. Based on focus-groups, individual conversations with teachers, teachers' self estimation and classroom observations I investigate how teachers' practices are related to their beliefs about teaching and learning mathematics and to their goals for their students. A simultaneous use of several data-gathering methods gives me the opportunity to grasp a complex reality (Pehkonen & Törner, 2004).

I interviewed the teachers before and/or after the lessons I observed. The interviews were unstructured and conversational in style and I therefore refer to them as conversations. The conversations were audio taped and the transcriptions of the tapes are subject to analysis. Through these conversations I got information about what the teacher intended to do in the lesson I was going to observe and also the teacher's reflections on how a lesson had been. We also talked about L97, and how they responded to its recommendations.

I also have information from the teachers obtained through self estimation. I have drawn on Pehkonen and Törner (2004). They used Dionne's (1984) perspectives of mathematics (traditional, formalist and constructivist perspective) in terms of toolbox aspect T (mathematics is a toolbox, doing mathematics means working with figures, applying rules, procedures and using formulas), System aspect S (mathematics is a formal, rigorous system, doing mathematics means providing evidence, arguing with clear and concise language and working to reach universal concepts) and process aspect P (mathematics is a constructive process, doing mathematics means learning to think, deriving formulas, applying reality to mathematics and working with concrete problems). My four teachers were asked to distribute 30 points corresponding to their estimation of the factors T, S and P in which they should value their actual teaching, what they mean is ideal teaching and what they mean L97 reflects with regard to these aspects. To obtain more detailed information about their views on mathematics teaching, I also asked them to write down, about one page long, their personal opinion of what good or ideal mathematics teaching is.

The main sources of data in my study are classroom observations. The interactions between the teacher and the learner I am referring to in my second research question take place in the classroom. I observed each teacher one lesson a week for 3 months. The lessons were audio taped. During the lesson I wrote field notes; both from students' activities during the teacher's presentation and teacher-student actions. I also copied what the teacher was writing on the board.

Analytical Process

To outline a portrait or a characterisation of one teacher I have to take into account the different kinds of data I have collected. The analysis of the data from focus groups and individual conversations are compared with the analysis of the data from classroom observation and I relate these to the estimation form. The outcome of this is mapped with L97 and its recommendations.

Teachers' own utterances

Focus group conversations, teachers' self estimations and writings about good mathematics teaching, conversations with the teachers before and after lessons give the information about their response to the curriculum which is addressing my first research question (how are teachers in the mathematics teaching responding to L97's recommendations). Focus groups and individual conversations with the teachers are transcribed. Extracts from the conversations and his own writings are presented to underpin the portrait I am presenting of Bent in the Findings section. How Bent estimated himself with regard to aspects of L97 and how he looked upon L97 in the estimation form are also used to underpin the characterisation of him. When I selected codes to do the coding of the transcripts of the conversations I found that similar codes occurred both when analysing the transcripts of the conversations and of the classroom observations. Some of these codes are presented in the next section.

Classroom observation

Five of the eight lessons I observed with Bent, are fully transcribed. The field notes I wrote during the lessons together with what I copied from the board, have given me useful information during the analysis process. The analysis process started already while I was sitting in the classroom and continued while transcribing audiotapes from the lessons. I wrote narrative summaries and made an overview of each lesson. The narrative summaries are divided into sections and subsections where I am saying briefly and as factual as possible without interpretation what is going on.

Coding

Prior to the coding process a general issue is if categories or codes are derived from the data or if they are brought to the data a priori. Wellington (2000) outlines three possibilities: the categories can be pre-established, they can be derived from the data themselves, or they can be both, that is some codes are pre-established and some are derived from the data.

During the coding process I had to go back to and reflect upon my research questions to decide what codes to use (Wellington, 2000). The second research question (what kinds of interactions between the teacher and the student are observable in the mathematics classroom) is mainly answered through classroom observation. Questions like if I found any patterns of interactions and how was the communicative approach, needed to be answered. The first research question has been addressed through conversations and also through classroom observations. Are the students encouraged to work in an investigative way (as L97 does) or is the teacher directing them what to do? Is the teacher focusing on conceptual understanding (as L97 does) or exercising skills and procedures? Is the teacher encouraging the students to be active in the learning process (as L97 does)?

The codes I have been using have mainly emerged from my data. Thus the theory derived from the data is what Strauss and Corbin (1998) label as “Grounded Theory”. In such studies the researcher begins with a study and allows the theory to emerge from the data. However, I didn’t read through the transcripts with a blank mind. Before I started the coding process I had had conversations with the teachers, I had studied L97, and I had been sitting in the classroom. All this is part of my data. In addition to that my own experience as a teacher and a teacher educator has an influence on what codes I have been using. Some of the codes are derived from past research done by other researchers that I have studied. Thus the codes I have used are in the third category described above – a mixture of pre-established codes and codes derived from the data. When I started the coding process, I took one lesson at a time and made a list of suggested codes to use. I suggested codes both for conversations and for lessons.

Based on conversations with Bent, I have the impression that he wants to focus on students' conceptual understanding in mathematics and that he encourages students to be active in the learning process. I found that several codes that emerged from my data underpinned the focus on conceptual understanding:

- Wants the students to understand the answer (Reason-answer)
- Use of concretes / illustrating (Illustrating-Concretes)
- Refers to prior knowledge (RefPrioKnow)
- Structures students' thinking (StructStudThink)
- Questions students' thinking (QuestStudThink)
- Positive Confusion

Bent wants the students to be active in the learning process. Both through our conversations, through what he wrote about ideal teaching and through what he did in the classroom he gives that impression:

- He invites students to participate (InvStudPart)
- He tells a student to repeat something aloud or he repeats a contribution so everybody can participate (Sharing)
- Students are encouraged to work with concretes
- Students' contributions are taken into account also during whole class teaching

FINDINGS – A CHARACTERIZATION OF BENT

Bent's utterances

Based on focus groups, conversations with Bent before and after lessons, what Bent wrote about ideal teaching and the estimation form I found that Bent thinks it is important to focus on students' conceptual understanding. During pre conversation 19/2 we talked about conceptual understanding of dividing fractions as opposed to method mastering of doing it. Bent claims that the brightest ones automatically will gain conceptual understanding from mastering the method but more students will understand why you have to turn the second fraction upside down if you explain why. However, he is afraid that some students then will be confused when focusing on the why and that it will take too much time. He says:

> If I had an infinite amount of time, I could have spent lots of time on it and may be got everyone to understand it, but I haven't. The time I have got to spend on it may rather lead to a certain extent of confusion among some who are happily living with just using the rule and are managing well and not wanting to be engineer or mathe-

> matics teacher. They know the rule and are happy with that. Maybe some more will understand it, but some will be frustrated. [ ] I haven't reached any conclusion what is the smartest to do. (pre-conversation19/2).

In this lesson Bent is planning to use area to illustrate multiplication and division with fractions. He says that he believes that that will increase students' conceptual understanding. He says that the challenge is how to find good ways to combine using the rule and explaining the background for the rule without making the students to be confused. He expresses uncertainty about how wise it is making connections to other parts of the mathematics after they have learned the rule. He is afraid that some will be confused, for example starting talking about area when you are working with fractions. He is worried that some students will think: "Area I don't know, but fractions I know because there I know the rules**"** (pre-conversations 19/2). He says that in such cases he'd like to be able to differentiate more between students who are having different abilities because the clever ones understand while it might lead to confusion among others.

Also when working with the volume of solid blocks Bent says that he wants to focus on conceptual understanding by using concretes. During pre conversation 8/1 he says that he wants them to use the concretes, measure the sides and then calculate surface area and volume.

> The transition of knowledge from seeing drawings in the textbook, knowing the formula, to sitting with it in their hands, doing the right measurements and calculate the surface area and volume, I'll spend some time on. They know how to do it with drawings in the textbook, but from that to have it in their own hands. (Pre-conversation 8/1)

Bent thinks that students might become confused through this activity. However he says he wants the students to master a *positive confusion* and that they also have to reflect upon how big a square centimeter is and if the answer they get is reasonable. During this pre-conversation he also says that he wants to make a link between formulas and equations, that they can use a formula as an equation to find an unknown quantity.

Bent also wants the students to be active in the learning process. The following extract from what he writes about ideal teaching underpins this.

> I think teaching ought to be experimental. If the student can find out the knowledge / the rules / the formulas through own activities I believe that it will last longer and that they get a greater ownership to the knowledge.
>
> To learn mathematics the student has to work as much as possible on his own. Teaching from the board has to be followed by students' own activity either at school or at home. The student him/herself must work hard. The student often wants the teacher to show as many solutions as possible. It is a challenge to motivate students to work hard to overcome difficulties. Parents also want the teacher to teach from the board.

> However, that is doing them disfavour. They must experience the difficulties on their own. (From Bent's writing about ideal mathematics teaching)

During conversations Bent said he wanted the students to be active in the learning process. During pre conversation 8/1 he says that he wants the students to invent the formulas for volume and surface of solid blocks before they shall use it and that they shall not only find it in the textbook. Also when he is in charge of the lesson, teaching from the board he will invite the students to participate: "It will be much activities steered by oneself and I will do something on the board in conversation with the students". (Pre-conversation 8/1)

In the classroom

During plenary parts of the lessons Bent invites the students to participate and I can identify a sequential pattern of discourse when analysing the lessons. Bent's invitation to the students to participate (1) is followed by some kinds of students' response (2). Then Bent either structures students' thinking (3) or a consolidation takes place (4). When B has "structured students' thinking" a new contribution from the students (2) takes place followed by a new "structuring of students' thinking". The shift or alternation between (2) and (3) may occur several times before a consolidation (4) takes place. The 1-2-4 sequence can be related to what Mehan (1979) describes as an IRE (Initiation – Reply – Evaluation) pattern of discourse. The 1-2-3-2-3-2-4 sequence corresponds to Mehan's IRFRFRE pattern of discourse where F is feedback.

In my project, the probing or feedback phase (3) is the core of the teaching – learning process. What the teacher does or says here are traits that tell or illuminate what type of teacher he is. In Bent's lessons I can identify traits that for me seem to illuminate a teacher who is focusing on students' conceptual understanding. By illustrating, referring to prior knowledge, questioning students' contributions and thinking, sharing, and highlighting key information Bent is structuring students' thinking so they can build up their conceptual knowledge.

An overview of this pattern of discourse can be presented as follows:

1. Bent invites students to participate
   a) Asking a closed or open question or a question involving conceptual understanding
   b) Demanding a participation
2. Students contribute
   a) With a comment
   b) With a statement
   c) With a question
   d) With an answer
3. Bent structures students' thinking
   a) By referring to prior knowledge

   b) By illustrating
   c) By questioning students' thinking
   d) By highlighting key info
   e) By sharing
4. Consolidation
   a) Consent. Bent is praising a student for a right answer or they agree on a solution
   b) Convention. For example that in a triangle we call it base line and height not length and breadth
   c) Authorship of knowing. Bent claims something that is either right or wrong based on his knowledge as a mathematics teacher.
   d) Method of mastering. A method of solving a mathematical challenge is either agreed upon or Bent is doing it.

The following episode from the plenary part of the lesson 16/1 illustrates this pattern of discourse. It is an interaction sequence between the teacher and 4–5 students which starts with the teacher addresses the question to one of the students (invstudpart). Other students pose questions and contribute and teacher structures students' thinking and probes conceptual understanding by illustrating and by highlighting key aspect –the difference between the height in the triangle-shaped ground and height in the prism – and by referring to what students are supposed to know (refprioknow).

1	B:	Then volume is length multiplied with breadth multiplied with height. This is old news, isn't it? This you know quite well. Can I implement this on the other ones I've got here, the one with triangle shaped base and on the cylinder and on this funny one (showing a prism with a trapezoid ground base)? Is it possible to carry out the same principle for all of them? What do you mean, Andy? ***(Refprioknow) 1 – Asks question*** (Then there are some questions about something that is written on the board earlier) Let us carry on with the triangle Andy!
2	Andy:	Has to multiply the height with the base line. ***2 – Answer***
3	B:	Say it ones more, I don't think everybody could hear it ***3 – Sharing***
4	Andy:	Has to multiply the height with the baseline ***2 – Answer***
5	B:	Yes, because that is the ground base, isn't it? We just have to include that it is the ground base multiplied with the height. It is the triangle there multiplied with the height that still is the volume. Then you can carry on. ***4 – Consent, authorship, 3 – Highlightskeyaspect, 1 – Invstudpart***
6	Andy:	To find the ground base, you have to multiply the height with the base line and divide by two. ***2 – Studstatement***
7	B:	Okay, base line multiplied with the height in the triangle. (Writes on the board) ***3 – Sharing*** (Can hear a student asks if baseline and ground base is the same, however, the teacher either doesn't hear it or

he ignores the question) How can you separate between the height in the triangle and the height in the whole solid block, A? ***3 – Highlights key aspect***

8 Andy: Hmm? ***2***

9 B: Can you separate between the height in the triangle and the height in the whole triangle shaped prism? Can you separate between that? ***3 – Highlights key aspect (rephrased)***

10 Andy: Because the height that is the height in the triangle-shape, that is that on the side there, because that is the ground base. Then there is the height of, from the one side of the base to the other side of the base, it becomes like breadth. ***2 – Statement, a mathematical argumentation***

11 B: Can label it h1 and h2 ***3 – Highlightskeyaspect*** (*the teacher wrote* h^1 *and* h^2 *on the board, not* h_1 *and* h_2)

12 Stud: Can call it breadth? ***2 – Stud-quest***

13 B: Breadth, how? ***3 – Queststudthink***

14 Stud: The breadth of the triangle ***2 – Stud-statement***

15: B: Okay, the breadth there. However, I want to stick to what you have learned about ground base and height in a triangle. ***3 – Refprioknowl authorship***

16 Stud: It will only be height if you lay it down, won't it? Because the height,… it might be discussed. ***2 – Studquest***

17 B: It can be discussed, but when does it become a height? Isn't it a height if it lies down in a special way? ***3 – Queststudthink***

18 Stud: It is okay saying that if you lay it down then you say that it is the height. Then the height is upwards and h2 is then the breadth ***2 – Stud-statement***

19 B: Could have called it h1 and h2, the height in the triangle and the height in the whole thing, do you agree? ***3 – Structstudthink***

20 Stud: Can say breadth in the triangle ***2 – Studcomment***

21 B: I don't want to introduce breadth in the triangle, because then you'll become confused. Shall see that h1, the first height in the triangle and multiply that with the base line. ***4 – Authorship, convention***

22 Stud: H1, you mean the height within the triangle? ***2 – Studquest***

23 B: The height within the triangle, yes ***4 – Consent***

24 Stud: And h2, that is the height …***2 – Studquestn***

25 B: That is the height in the whole figure. And then you have to divide by two, don't you? ***4 – consent***

26 Stud: Is it h1 down there now? ***2 – Studquest***

27 B: It says h1. Then you multiply with h2. It becomes easier when we shall put numbers on it. ***4 – Consent, motivates for furtherwork***

During this episode, the teacher has drawn a prism with a triangle shaped base on the board and written:

V= ground base • h V=baseline • h^1 / 2 • h^2
(The teacher wrote h^1 and h^2, not h_1 and h_2)

The interactions between the teacher and the students in this episode can be described by the pattern of discourse presented above. I have put the analytical codes in bold italics after each statement. Students' contributions are taken into account. They are commented or questioned by the teacher and we can perceive the circular shifts between the response phase – students' contributions (2), and the probing phase – teacher structures students' thinking (3). It is possible to notice that the teacher wants to put weight on students' conceptual understanding of the similar principles in the formulas for volume of different solid blocks and also of the differences between the two heights in the formula they are working with. Finally he motivates for further work by initiating how this can become clearer for the students.

The communicative approach in this episode is dialogic, meaning that there is a dialogue between teacher and students going on. However, it is clear what the teacher's goal of this dialogue is: to generalise the formula for the volume of all prisms and to develop the formula for the volume of a prism with a triangle shaped ground base. I therefore claim that the communicative approach is dialogic but closed. The teacher comments the students' contributions but his goal of the lesson is the focus of the sequence. In statements 12–21 it is possible to perceive what Jaworski (1994) calls lack of closeness in perspectives, a miscommunication (Cobb, 1988). My field notes tell me that some students had discovered that the triangle the teacher was presenting as the ground base in the block they were working with was right angled. And if you chose one of the other sides of the block as the ground base, then one of the smaller sides in the right angled triangle became the breadth of a rectangular ground base, the height h_2 of the block with the triangle as the ground base became the length of the same rectangle and the other smaller side of the triangle became the height of the block. The volume of the block then became length • breadth • height/2. Possibly because Bent was focusing so much on generalising the formula for volume of different prisms the desired closeness in perspectives between the students and the teacher didn't take place. The teacher didn't succeed in building the desired model of the student's conceptual structure (von Glasersfeld, 1995).

CONCLUSION

From the estimation form below we see that Bent estimates his own teaching closer to the toolbox aspect than what he thinks is ideal teaching and what the recommendations of L97 are. His distribution of points to ideal teaching and recommendations of L97 tell us that Bent is aware that he is not teaching so much according to ideal teaching and L97 as he ought to. From what he says in conversations and his writings there are several constraints preventing him from doing that: students want him to teach from the board and also parents have said that they prefer him not to reduce the time he spends teaching from the board presenting and explaining mathematics. Time pressure is another constraint preventing him from doing more process work in the teaching of mathematics and finally he says that lack of knowing methods teaching more according to the process aspect L97 recommends prevents him from doing it.

Bent	Mathematics as a toolbox	Mathematics as a system	Mathematics as a process
My real teaching	18	5	7
Ideal Teaching	10	7	13
L97's view on teaching mathematics	13	5	12

Table 1: Bent's estimation form

This teacher is "a teacher with potential", meaning: with potential to implement L97 to a greater extent than he currently does. He consciously knows that his teaching is more traditional than what L97 recommends and his view on ideal teaching is very close to L97. However, he is trying and he wants to switch towards more process oriented teaching.

Our challenge as being mathematics teacher educators is how to work with such teachers so their teaching can be further developed according to their ideals and according to the current curriculum.

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TEACHER RESEARCH IN MATHEMATICS TEACHING

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The article accounts for an elementary school teacher's experience from working at developing her way of teaching mathematics. Sketches from her teaching are given to highlight the processes she went trough. They are a part of a three year long study of her mathematics teaching where she worked on improving her way of communicating in the classroom and using her experience with the children to make decisions about her teaching.

INTRODUCTION

In recent years there has been a growing interest in teachers as learners; how teachers learn to think and act in particular ways and what contributes to their learning. There has been a shift away from the dominant positivistic view of the learning and teaching process to a constructivist view of the learning that takes place in the classroom. "Where once the process-product research paradigm was paramount, qualitative studies are now central to most research in mathematics education" (Dawson, 1999, p. 7).

In this article I will give examples of my attempt to improve my mathematics teaching in elementary school. The examples are from a three year long study on my mathematics teaching. In the study I focused on my development as an elementary teacher during a three-year period of teaching mathematics. I accounted for the interaction between me and the students and the effect it had upon my development as a teacher. The purpose of the research was to study how enhanced meta-cognitive thinking can result in development that is valuable for the teacher's work. I described my attitudes to the profession and teaching mathematics in particular, and analyzed the effect of my own teaching and learning experience on my attitude development.

Other aspects that had an effect upon my professional development were described. The observation of colleagues' teaching and joint reflection on children's learning, learner/teacher interaction, and personal development, were important factors of the study. Furthermore I described how reading about research on learning and teaching mathematics helped me to analyze the children's learning and my interaction with them. Finally, I described the significance of my experience as a teacher trainer for interpreting and analyzing my teaching.

Action research and self-study models were used to analyze the research process. At the beginning of the work my focus was on the children's learning. Later the

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focus shifted to my own development as a teacher and finally to the research process. The main results were that I learned to reflect in action, use my understanding of the children's way of constructing knowledge and analyze the impact of teacher/learner interaction on classroom learning.

At the time the research formally started I was actively reviewing my practice and my mathematics teaching in particular. I wanted to be able to capture what influenced the development of the teaching process. To help me focus more clearly I asked the following questions:

– In what ways do my beliefs on mathematics education affect my teaching?
– How can I use my knowledge on mathematics teaching and children's thinking and understanding of mathematics to improve my teaching?
– In what ways can I use my learning from experiences with my students in class to reflect on my teaching?

For three years time I collected data from my mathematics teaching in primary school by writing notes, video recording in class, gathering written materials from the children and writing reports. Parents were informed about the work through written information, visits to the classroom and information evenings. They were also encouraged to take active part in their children's homework in mathematics. Two parents were interviewed on their work with their children. Notes were kept from meetings with the parents as well as from meetings with colleagues and discussions with teacher students.

In this article I will focus on the third research question, my learning in the classroom and the interaction with the children in their mathematics learning. What is accounted for here is only a part of the whole study and I therefore find it necessary to start with a description of the three year process. Then I give examples from my teaching in two classrooms and analyze the learning that took place.

UNDERSTANDING CHILDREN'S WAY OF CONSTRUCTING KNOWLEDGE

In order to better understand my students' thinking I looked for research on children's mathematical thinking. The research findings of the Purdue Problem-Centred Mathematics Project (Cobb, Wood, & Yackel, 1992; Yackel, Cobb, & Wood, 1999) had a great influence on my way of planning my teaching and listening to the students. The description of the communication of children dealing with mathematical problems helped me focus on my students' communication in the classroom and find ways to enhance discussions and reflection.

I also studied Inger Wistedt's research on the quality of children's thinking in problem solving activities (Wistedt & Martinson, 1994) and Ann Ahlberg's studies on children in different problem solving situations (Ahlberg, 1995) in order to broaden my knowledge in the field.

At the beginning of my research I got the chance to participate in a course on Cognitively Guided Instruction (CGI). The course was based on representing to teachers the findings of research on young children's thinking about whole numbers (Carpenter & Moser, 1984; Carpenter, Fennema, & Franke, 1995, 1996; Carpenter, Fennema, Franke, Levi, & Empson, 1999).

> Within this domain researchers have provided a highly structured analysis of the development of addition and subtraction concepts and skills as reflected in children's solutions of different types of word problems. In spite of differences in details and emphasis, researchers in this area have reported remarkably consistent findings across a number of studies and have drawn similar conclusions about how children solve different problems (Carpenter, Fennema, Peterson, Chiang, & Franke, 1989, p. 500).

I had a long experience of working with children and trying to understand their mathematical thinking. Learning about the research on children's thinking and construction of mathematical concepts helped me to better understand my own students' thinking, and analyze their development. Working with the findings introduced at the course also helped me realize that I was able to learn from my experience in class and to use my analysis of the children's development to make instructional decisions.

Barbara Jaworski's research and her description of an investigative approach to mathematics teaching was also inspiring (Jaworski, 1994). She worked with her colleagues on understanding what investigating teaching is about. They came to the conclusion that it means 'opening up' mathematics, asking open ended questions and encouraging students' inquiry rather than straightforward 'learning' of facts and procedures. Reading about their work helped me in staying loyal to my own views of mathematics and teaching, and my role both as a teacher and a learner in the classroom.

UNDERSTANDING THE LEARNING IN THE CLASSROOM

Beginning the research my aim as a teacher was to try to understand the thought processes of my students as they constructed their knowledge of mathematics. I wanted to be able to learn from my experience in the classroom, interpret what I observed and use my analyses to make decisions about my teaching. Later I realized that what I was aiming at was to try to understand how my interaction with the students proceeded and interpret what I learned. To be able to understand the process I went through I realized that I had to learn more about teacher development.

Reading about the Purdue Problem-Centred Mathematics Project I found articles on the researchers' study on the teacher cooperating with them. They learned that not only were the children learning by engaging in the activities but so were their teacher (Wood, Cobb, & Yackel, 1991). The teacher's main concern was to change her way of teaching in accordance with her change of view. To shift the focus from telling the children how to calculate to respecting the children's way of explaining their procedures and accepting them even if the children got wrong answers to the problems.

I also read about the research on teachers in connection with the CGI-project. The research results from Wisconsin on the children's understanding of whole numbers were introduced to teachers at in-service courses. Several studies were made on how the teachers used this knowledge in their teaching and what influence it had on their way of teaching and their attitudes to mathematics teaching (Carpenter et al., 1989; Fennema Carpenter, Franke, & Carey, 1993; Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996; Knapp & Peterson, 1995). "When children begin to show increased learning, teachers continue to implement new methodologies that result in improved learning, and so the circle continues." (Fennema et al., 1993, p. 580) Their results were in coherence with my experience and helped me interpret my learning in the classroom.

John Mason (2002) discusses the learning of teachers that research their own practice.

> The more you listen to students working together in groups, the more you realize the complexity of being 'taught'. The more you probe children's thinking, the more you realise how sophisticated and powerful children's thinking can be. (p. 27)

He stresses the risk of making habits of how you respond to your students, instead of responding sensitively to situations, we frequently react according to established patterns without realizing it. We continue to believe that we act freshly all the time, when in fact much of the time we react rather than respond. But when we have acted freshly and appropriately there is a sense of freedom and these moments keep us going.

FINDING A RESEARCH METHOD

Interpreting my findings in a way that explained the process I went through I went back to Jaworski's writing. In her book: "Investigating Mathematics teaching", she discusses her way of finding a research method for her thesis when she gained her PhD from the Open University (Jaworski, 1994).

> In it I tried to set out in a style acceptable to the examiners the theory, research and conclusions of five year's work and thinking. It proved impossible to do this from any position other than that of the centrality of the researcher to the research. I saw the research then as being ethnographic, or interpretivist, but realize now that an alternative term is *constructivist*. I took constructivism as the central tenet of my theoretical position on teaching and learning, and tentatively recognized its internal consistency with the way my research evolved. (p. xiii)

Her explanation helped me realize that finding a research method is a complex process and that it's important to be loyal to your philosophy of understanding. The dilemma Jaworski describes, led me to look further for the roots of my beliefs and way of understanding and interpreting what I experienced.

At the beginning of the research I considered my work as action research. I gathered information, reflected on what I experienced in the classroom and tried to use what I had learned to make modifications in my work with the students. At the same time I was reading about children's learning, teacher development and discussing my thoughts with others. According to Kemmis (1999) my research at that time could be labeled as action research.

However I felt that I needed further help with interpreting the data. I found that Jaworski's description of evolutionary action research was in coherence with my experience. She describes teachers that are developing knowledge and awareness through enhanced meta-cognitive activity as reflective practitioners (Jaworski, 1998). I was constantly reflecting on my work, both in the classroom and when reviewing my work afterwards. She discerns evolutionary research where the teachers' development is at the core and guides the way of how the research progress develops from a more formal action research where the structure of the research is clearly defined in the beginning. She refers to Schön's (1983, 1987) interpretation of knowing, thinking and reflecting and their relation to action and practice. A teacher who moves from being able to use her knowledge in the classroom (knowing-in-action) to reflecting afterwards on her actions in the classroom (reflecting-on-action) and also being able to reflect in the classroom and making decisions based on her analysis of what she notices in the classroom (reflecting-in-action) is developing in action.

Looking further for tools to help me with the interpretation I found writings about self-study useful. According to Loughran and Norfield (1998) self-study defines the focus of the study (i.e. context and nature of a person's activity), not the way the study is carried out. They also discuss the difficulties individuals face in trying to change their interpretations (frames of reference) when their own experience is being examined. In my struggle to understand what was happening in the classroom and how my interpretation of my experience changed as I reflected more on it I found their writings helpful. They stress that self-study outcomes demand immediate action and thus the focus of study is constantly changing.

Bullough and Pinnegar (2001) stress that biography and history is a necessary part of self-study. When the issue confronted by the self is shown to have relationship to and bearing on the context and ethos of time, then self-study moves to research. What is presented in this work is the story of my struggle with my teaching.

Analyzing my results was an ongoing process through my research. I examined the notes I kept from the classrooms, parents meetings, discussions with colleagues and student teachers, as well as the children's work and the video-tapes that I studied, alone and together with parents, colleagues and student teachers.

SKETCHES FROM THE CLASSROOM

The first year of my research I was a class teacher in fourth grade (nine year old children). I had been their teacher since they started school and knew them well. This year I experienced how cooperating with them gave me confidence in choosing subjects and deciding the way of working in class based on our joint experience. I had been experimenting with giving them homework assignments once every second week where their task was to gather information at home to work on. Sometimes I gave them problems to solve and discuss with their parents. We discussed their findings at school and they reported on their solutions. At the beginning our discussions were rather short, but gradually the time we spent on discussing their findings increased and the focus of discussions changed from first discussing mainly how they gathered information to later focus on their mathematical findings. I realized how important these discussions were both to me and them. While listening to them explain their way of working I learned much about their thinking. This experience made it easier for me to make decisions about our further work.

Geometry was a popular subject in class this year. Our discussions about their work gave us new ideas to work on and we found more interesting tasks to deal with. The more open tasks I gave them and the more investigations they made, the more new ideas were born.

In the textbook we were using that year there were some tasks about studies on two-dimensional shapes. The studies of the shapes and their properties aroused many questions. The triangles interested many of the children and they found it challenging to find out how other polygons can be made by laying two or more triangles side by side. The regular hexagon made of six congruent regular triangles was a popular shape. These explorations led us to study tessellations. The children made explorations with different shapes and were happy to find out that they could tessellate with many types of polygons.

One day we were discussing the children's solutions to a problem they had solved at home.

> A family of five likes to change seats at the dinner table. Each night one member of the family changes seat with another member. How many days will it take until they have all changed seat with each other?

Peter told us how he and his father had worked on the problem. He asked if he could show us his solution on the blackboard. He started by drawing a circle and then placed five dots around the circle representing the family around the dinner table. He then drew a line from one of the dots to the other four. Then he drew lines from another dot to the three dots that were not connected to it yet. He went on until he had connected each dot to all the other dots around the circle. While doing this he told us how he and his father had worked on the problem and found this solution. From his story it was obvious that they worked on the problem together and both of them had enjoyed the experience.

When he had finished his drawing there was a pentagon on the blackboard with a five armed star inscribed. The other children found his solution very smart and asked how he got the idea to solve the problem this way. Peter answered that he liked to draw pictures while solving problems. It helped him to see how things worked out. The pentagon and the five armed star aroused a lot of discussions and some of the children experienced with drawing a five arm star and the pentagon circumscribed by connecting the arms of the star.

I decided to use this opportunity to study the pentagon further. I asked the children if it is possible to tessellate with a (regular) pentagon. By experiencing with some cardboard pentagons they soon found out that it's not. After a break the same day Bjorn said: "The soccer ball is made out of pentagons" and showed us the soccer ball he had been playing with. By studying the ball we soon found out that the ball was made of both pentagons and hexagons. I asked the children why they thought that the soccer ball was made of both pentagons and hexagons and not only of either pentagons or hexagons. Using the cardboard polygons they soon found out that it would be possible to make a ball out of pentagons. They realized quickly that it was impossible with only hexagons because they tessellate. "There is no space between them and therefore it will newer get curved", Dora explained.

Later I brought a dodecahedron into the classroom and the children studied how many pentagons were needed to make the solid. The children saw that the surface of the dodecahedron wasn't as smooth as the surface of the soccer ball and realized how smart it was to use both hexagons and pentagons to make the ball. After studying the dodecahedron we decided to make Easter baskets this year that were half of a dodecahedron (made of six pentagons).

The children's interest in the studies with the polygons and tessellation resulted in two tessellation projects on mathematics and arts. One was a tessellation with either one or two types of polygons and the other was made using the technique the Dutch artist M. C. Escher is known for (Schattschneider, 2004). They made their tile from a cardboard square, cut one piece from each of two adjacent sides and glued them to opposite sides. These two projects took a long time and many of the children spent hours on finishing their pictures. When the children introduced their projects to the class they used mathematical concepts to describe their work. Following is Anna's and Helga's description of their tessellation with two types of polygons:

> We decided to use triangles and squares and placed them like this", they said while showing us how they placed cardboard polygons on a paper and used a pencil to trace them. "We went on until we finished the circle. That's how we got the dodecagon. And then there was this hexagon inside. We decided to use the triangles to fill the hexagon. We started by coloring the squares red and the triangles blue. Then we found out that it was smarter to use different shades of the colors and also used yellow and green for the triangles.

By the end of the year I gave them a homework assignment where I asked them to look for tiles in their surroundings, on walls, pavements etc. I asked them to draw the pattern made with the tiles and draw two or three different patterns. Then I asked them to imagine that they were making their own tiling, choose a tile and draw a sketch of the pattern they would like to make.

While discussing their findings and the pattern they made I saw how deep understanding they had gained on the attributes of different types of polygons and the tiling process. They were capable of using geometric concepts to describe their work and the way they discussed the findings of their classmates showed me that their knowledge had broadened. Peter was not eager to show his tiling pattern. He thought it was ugly. His pattern was reflected through a diagonal and therefore gave an image of a three-dimensional pattern. His picture aroused interesting discussions about three-dimensional patterns and how they are made. I would have liked to go further into research on three dimensional patterns but since the school year was almost over it had to wait.

The geometry projects I have accounted for were not in the curriculum for 4th grade. All of them resulted from discussions in the classroom and the children's ideas and interest in the subject. The discussions were a natural component of the children's explorations and research into the world of geometry. The more open the assignments given to the children were, the more explorations they made and more interesting new projects aroused. I soon found that the more I was willing to rely on the children's explorations and listening to their ideas the more learning took place in the classroom. The children were all engaged in the projects because they were about their own ideas. From their work and discussions I learned about their understanding of the concepts they were working on and was able to analyze their learning.

Peter learned much from participating in this work. He was very shy and unconfident about himself. He found himself rather clumsy and was afraid of showing his classmates his work. But his solutions were often those which the other children admired most. The two episodes I described where he introduced his solutions to the homework assignments are examples of events I learned from how important it is to be accepted by the classmates. Praise from me would not have raised his self-confidence as much as did his classmates admiration. He also showed us a different view of the problems we were working on and brought new ideas into the classroom.

The second year of my research I got a first grade class again. It took some time to learn to know the children. Soon I realized that being open to their ideas, listening to them and observing their way of working helped me to decide what kind of mathematical tasks to give to them. The CGI findings were helpful in analyzing the children's solution strategies and I tried to make decisions about my further work with the children based on my interpretation. I made problems about our joint experience in class or about something that I knew was a part of their daily lives. I felt confident with my experience and was satisfied with how much I was learning from my work with the children. But there were exceptions and

I will give an example of my effort to understand one of my student's ways of thinking.

When Ole started school I soon realized that his number sense was rather limited. To me it seemed confusing because he had good skills in verbal communication and he was also very creative in his drawings. In September I posed the following problem to the class.

> Einar had five toy cars. On his birthday he got two more toy cars. How many cars does he have now?

The problem was about one of the boys in class and most of the children were quick to solve it and got another problem to work on. Ole started drawing. First he drew a jeep with many small details. Then he started drawing another car that was totally different from the jeep and with even more details. While working he discussed his drawing with the children in his group and explained to them the function of the different details. It didn't seem to interest him that they were solving the problem and discussing their way of solving it. I felt impatient and asked him if he could find out how many cars Einar had now. He told me that he first had to draw them all. I then asked if he could first draw all the cars without the details and add them afterwards. He seemed offended and answered that it was important to draw all the details first. By the time the class was over he hadn't finished his drawing. I asked him if he could use counters to find out how many cars Einar has but he was unwilling to do that and kept working on his drawing. Finally he accepted to use the counters. First he counted five and the he counted two, joined them all together and counted them all again and got the answer seven.

Learning from this experience I decided to make problems with things that were easier to draw. But Ole always found some details to add to his drawings and never had time to finish them all. He never tried to count on his fingers or use anything else to count. In December we rode a bus together and passed his home. He was eager to tell me about a family that lived next door to him. They had just had a new baby. "First they had that many children" he said and showed me three fingers, "and now they have so many children" he added and shoved me four fingers. I was surprised that he didn't use numbers to tell how many the children were so I asked him how many they were. "So many" he said and showed me the four fingers again and seemed surprised to get this silly question. I asked if he could count them for me and he quickly said 1, 2, 3, 4, 5 while touching the fingers, but didn't keep track of the counting.

I realized that even if I had succeeded to have him count for me he was still not aware of that counting is not just a sequence of numbers but is used to tell how many there are in a set. I realized that I had to give him time to develop his number sense and not press him too hard to solve the problems the same way as the other children were doing.

In March I posed the class the following problem.

> Five children baked buns at school. Each of them baked three buns. How many buns did they bake in total?

Ole counted cubes. First he counted five cubes and connected them to a bar. Then he counted three cubes and connected them. He laid the two bars together and counted the difference. He was proud to announce that the total was two. When I asked him if he could tell me how he knew how many the buns were he showed me how he found his solution. I asked if he could show me the buns and he showed me the bar with the three cubes and then the bar with the five cubes and said that they were the children. I pointed out to him that each of the children baked three buns and he showed me the bar with the three cubes and said that they were the buns. When I asked if all the children baked only these three buns he added that the total was two. He was consistent with his answer and it didn't seem to bother him to hear the other children count fifteen buns. I asked him if he could draw a picture of the children and the buns but he said he was too tired. I decided not to pose him any further questions on this problem.

This experience showed me that even though I had tried to respect his way of working I had been too eager to encourage him to solve problems the same way as the other children instead of letting him work at his own pace. He was obviously trying to use knowledge from earlier experience with comparing problems to solve this one. He knew that I expected him to try to solve the problem. Instead of drawing a picture of the situation which was his way of understanding a problem; he counted cubes because he thought that I expected him to do so. I had not relied on that he could construct his own knowledge about numbers and operations even though I had tried hard to respect his thinking.

RESULTS

The examples I chose to describe here are representative for the processes I have analyzed that I went through. They are a part of a long story of my confrontation researching my own teaching (Bullough & Pinnegar, 2001). The episodes from the fourth grade classroom working on geometry show how my reflection on what was happening in the classroom resulted in changing in the focus of my mathematics teaching. The focus of my study was changing in coherence with what I learned from my reflection on my teaching (Loughran & Norfield, 1998). When the children brought new ideas into the classroom or had different aspects to the problems, than I had myself, I used it as an opportunity for me and the children to learn something new. When I decided what kind of projects I wanted the children to work on, I used their ideas to guide me both in what kind of mathematics to focus on and how to formulate the tasks. I often responded right away to the children's ideas and used them as an opportunity to focus on important mathematical aspects. I was not only using my knowledge of children's learning and mathematics education in my teaching but also making decisions about my teaching based on my reflection on action and in action. The examples I gave from the changing of the focus of the discussions on the children's homework assignments is an example of my ongoing reflection on the focus of discussions and how I used what the children brought into the classroom to help me make further decisions about my teaching. This is in coherence with Jaworski's description of a reflective practitioner (Jaworski, 1998).

My own interest in mathematics helped me respond to the new ideas the children came up with. I was able to see the richness of their contributions and use them as a new learning opportunity. An example of that is Peter's solution to the dinner table problem. The problem was not about geometry but his solution resulted in a drawing of a pentagon and a five armed star. I decided instantly to use his solution to connect to the children's earlier studies on tessellation. This is one example of my reflection in action. In my analysis of the process I went through I saw that the more I reflected on my action afterwards the more I was able to reflect in action (Jaworski, 1998). The more I dared to rely on the children's way of constructing their own knowledge by engaging in meaningful activities, the more I learned about their thinking and understanding of mathematical concepts (Mason, 2002). It helped me to make decisions on further work with the children. This example also highlights that not only were the children learning by engaging in the activities in the classroom, but so was I (Wood et. al., 1991). When Bjorn brought the soccer ball into the classroom I realized that I had succeeded in waking the children's interest in polygons. I was happy to see how quickly they connected their tessellation explorations to the studies of the ball. Dora's utterance about the hexagons with no space between them and her realization that the surface therefore would never get curved showed a deep understanding of tessellations and two- and three-dimensional properties. This experience is in coherence with the findings of the CGI project that when children begin to show increased learning the teachers continue to implement new methodologies that result in improved learning (Fennema et al., 1993).

I found it important to help the children experience that geometry is a part of our life and we can make use of our knowledge in geometry in many ways. It was therefore natural to give them the opportunity to experience with tessellation and make their own piece of art. By asking them to study tiling in their surroundings and then make a sketch of their own tiling I was encouraging them to use their knowledge to construct new ideas. The more knowledge and deeper understanding the children were showing the more I felt that I could challenge them (Fennema et al., 1993).

The case I described on my experience from the first grade classroom is different from the episodes from the fourth grade classroom. The fourth graders I knew very well. I had been experiencing together with them for more than three years and felt free to explore new ways of working with them. When I got a first grade class again I faced the challenge to get to know the children and understand their way of learning. My knowledge about the CGI studies was a great support to me and I felt that it helped me in my attempt to analyze their understanding of numbers and the mathematical operations.

Ole's way of thinking about numbers was very different from the other children's and from what I had learned from the CGI studies and. I chose to describe my experience from working with him to give a different example to the previous one in order to highlight different phases of my progress. I was often frustrated during the lessons and felt that I was not capable of dealing with him. It irritated me that he didn't make any effort to count in order to solve the problems I posed to the class. He was good at telling stories and had a far more advanced vocabulary than most of his classmates. I was quick to decide that he was just stubborn and wanted to decide himself what to do at school. My interpretation of the episode I gave from my interaction with him while he was drawing the cars is that I was reacting rather than responding as Mason described (Mason, 2002). When he told me the story about the family who had just got a new child I realized how limited his number sense was. The story was advanced and he used a lot of nuances in his telling, but had no word for the numbers he needed to express. Our discussion on the bus was one of the highlights of my experiences this winter. I had never listened carefully enough to the boy to realize why he didn't use counting to solve problems. I had never met a child at the same cognitive level as he was that had so limited understanding of numbers and the way we use them to describe the world around us. This experience forced me to review my way of working with him and my expectations to his mathematical explorations. Although I tried hard to give him the space he needed to develop his understanding I realized when he described to me his solution to the buns' problem that I had been expecting him to develop his mathematical thinking at the same pace as the other children. I expected him to go through the stages of development described by Carpenter et al. (1999) as did his classmates and he was trying to please me by giving me an explanation that he thought I was asking for. I was not satisfied with my way of handling this dilemma and it forced me to revise my understanding of his way of learning.

Facing the challenge of changing my teaching in coherence with what I felt was important to do; I often came across obstacles that were difficult to overcome. I believe that by constant reflection on her experiences in class the teacher is capable of improving her teaching. John Mason has given me a good explanation by writing "Change is not something you do to other people, but something you do to yourself. I cannot change others but I can work at changing myself" (Mason, 2002, p. xii). By being sensitive to the children's responses I may be able to help them on their way to improve their mathematical understanding but they must work on it themselves in a way that has a meaning to them.

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TEACHING "MATHEMATICS IN EVERYDAY LIFE"

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This paper presents a case study of one teacher, the aim being to distinguish strategies of connecting mathematics with everyday life and the underlying beliefs. A case study provides us with the opportunity to portray, analyse, and interpret real individuals and situations in their uniqueness. The article presents practical examples of how a teacher connects mathematics with everyday life by arranging small-projects and using other sources than the textbook.

INTRODUCTION

In the Norwegian national curriculum of 1997 (called L97 for short), "Mathematics in everyday life" was presented as a new main area. As opposed to the other areas, like numbers and geometry, mathematics in everyday life was supposed to be more of a meta-topic, establishing the subject in a social and cultural context (cf. Hagness & Veiteberg, 1999, p. 168). The idea was that mathematics in everyday life should be incorporated in the overall teaching of mathematics, rather than being a distinct topic on its own.

An evaluation study of the mathematics plan in L97 (Alseth, Breiteig, & Brekke, 2003) concluded that teachers, although having more knowledge about the curriculum than before, still teach in a traditional way. Only occasionally, when organising projects or larger activities in their classes, do they teach according to the suggestions and aims of the curriculum. The connection between mathematics and everyday life, as described in the curriculum, is thereby not implemented by most Norwegian teachers.

This article refers to the case study of one experienced teacher who appears to follow the ideas of the curriculum. The main aim is to distinguish and analyse the concrete strategies of this teacher when it comes to connecting mathematics with everyday life. The study had a split focus on the beliefs and practices of the teacher, and the connection between the two. In this article the main focus will be on the practices, and the results concerning the beliefs will only be briefly referred to.

There are many variations of the concepts "belief" and "belief systems" in the research literature (cf. Furinghetti & Pehkonen, 2002; McLeod & McLeod, 2002). A distinction between beliefs and knowledge is often made, and beliefs are said to be the filters through which experiences are interpreted (Pajares, 1992), or dispositions

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to act in certain ways (Scheffler, 1965). Research has shown that there appears to be a link between teachers' beliefs about mathematics and their teaching practices (Wilson & Cooney, 2002), and Thompson (1992) suggests that teachers' beliefs influence their future teaching practices. It therefore appears to be sensible to study teachers' beliefs and teaching practices together.

The issue discussed in this article is strongly related to the intentions of the Norwegian curriculum. The curriculum intends to, *"create close links between school mathematics and mathematics in the outside world"* (Hagness & Veiteberg, 1999, p. 165), and learners should construct their own mathematical concepts. *"The starting point should be a meaningful situation, and tasks and problems should be realistic in order to motivate pupils"* (Hagness & Veiteberg, p. 167). The curriculum implies a distinction between the physical world (outside the classroom) and the world of school mathematics (cf. Penrose, 1994), and the relationship between these two worlds is often recognised as problematic (cf. Smith, 2000). The connection between mathematics and everyday life, which is so strongly emphasised in L97, appears to be far from trivial, and the transfer of knowledge from one context to another is often considered to be problematic (cf. Boaler, 1993; Ernest, 1998; Evans, 1999).

When the phrase, "mathematics in everyday life" is used here, it is the connection between mathematics and everyday life which is in focus. The seemingly similar term "everyday mathematics" is used by many researchers (cf. Arcavi, 2002, Brenner & Moschkovich, 2002; Wistedt, 1992), and this term has been defined as: (a) mathematics that we attain in our daily lives, or (b) mathematics that we need in our daily lives (Wistedt, 1992). The definition of mathematics in everyday life used in this article comes from L97, and it has a focus on how teachers might organise their teaching in order to bridge the gap between the everyday practices and school mathematics (cf. Arcavi, 2002; Civil, 2002).

"Everyday life", as the term is used in L97, refers to the physical world outside the classroom, and it is therefore not limited to what the pupils encounter every day or what is known and commonplace to them (like their classroom experiences). There is therefore a rather wide and inclusive definition that is implied in the curriculum.

The connection between mathematics and everyday life has received much attention in international curriculum reforms in the last decades. There does not, however, appear to be a consensus about how this connection is supposed to be carried out. Some believe that pupils should learn to understand mathematics by applying it to realistic word problems, while others believe they should apply mathematics to problems from everyday life (i.e. outside the classroom) (Cooper & Harries, 2002). Brenner and Moschkovich (2002) make a distinction between academic mathematics, school mathematics, everyday mathematics, and workplace mathematics.

It is possible to claim that the connection between mathematics and everyday life has been integrated in classroom practice through word problems, although these word problems have often been determined by the mathematical theories that the pupils were supposed to be exercised in (Brenner & Moschkovich, 2002). Mathematization and contextualisation are also proposed as important practices to create a bridge between everyday and academic practices in mathematics, but familiar contexts do not always make life easier for the pupils (Arcavi, 2002). Civil (2002) claims that the concern should not be to bring everyday tasks to the classroom, but rather to create a learning environment where learning is viewed as: (a) occuring mainly by apprenticeship, (b) involving work on contextualised problems, (c) giving control to the person working on the task, and (d) often involving mathematics that is hidden. With this model as a reference, burning questions in this article will be: What kind of learning environment is the teacher creating in order to bridge the gap between the everyday practices and school mathematics? How can a teacher organise the learning activities in order to create a learning environment as described above?

METHOD

Some researchers consider case study to be a methodology, while others have a focus on "the case" being an object of study (cf. Creswell, 1998). More generally, case studies are used when the "how" and "why" questions are being posed, and in situations where the researcher has little control over events (Yin, 1994). A more technical definition states that:

> 1) A case study is an empirical inquiry that
>   - investigates a contemporary phenomenon within its real-life context, especially when
>   - the boundaries between phenomenon and context are not clearly evident
>
> (...)
>
> 2) The case study inquiry
>   - copes with the technically distinctive situation in which there will be many more variables of interest than data points, and as one result
>   - relies on multiple sources of evidence, with data needing to converge in a triangulation fashion, and as another result
>   - benefits from the prior development of theoretical propositions to guide data collection and analysis. (Yin, 1994, p. 13)

This study investigated a contemporary phenomenon in its real-life settings. It involved in-depth data collection involving multiple sources of information (cf. Creswell, 1998), namely classroom observations (with extensive field notes and audio recordings), interviews, and a questionnaire survey. The teacher presented as the case in this study was selected because he was an experienced teacher who was known to focus on connecting mathematics with everyday life.

In the starting phase, a planning meeting was arranged, where the teacher (called Harry) and the principal at his school were informed about the research project, the ideas and aims. At this meeting, the particularities of time and place were discussed.

The classroom observations took place in a period of four weeks, during which I followed all mathematics lessons that Harry taught. In the second week of this period, Harry and his colleagues (the mathematics teachers at his school) were given a questionnaire, which they individually answered and returned. This questionnaire was not analysed until just before the interviews at the end of the observation period, to prevent the classroom observations from being biased. In the last week of the observation period, all lessons were recorded with a mini-disk recorder. At the end of the observation period Harry was interviewed twice. These were semi-structured interviews, and a list of questions had been prepared and discussed with a group of fellow doctoral students. These questions were used as a starting point for further discussions.

The field notes (which were taken throughout the entire observation period) and all the audio recordings were transcribed, and the results of the questionnaire were written down in electronic format. All the data material was then analysed. From this analysis, a list of themes and categories appeared. These themes and categories were used in a second phase of analysis and discussions of the findings. This article has a focus on three categories, one from each main theme (the main themes were: "activities and organisation", "content and sources", and "practice theories"):

- Projects
- Other sources
- Connections with everyday life

When I refer to the results in the following, I try to tell a story of what Harry's teaching was like. Here I have used the field notes as well as the transcripts from the lessons as a source.

RESULTS

Harry mainly taught mathematics and natural sciences, and he was regarded a good teacher by his colleagues as well as by his pupils. He had a strong interest in technology, and he often made links between mathematics and technology. The school in which Harry was teaching is located in a small Norwegian town. It is the main lower secondary school in the town.

Harry's beliefs were investigated through interviews and a questionnaire, and the practices through classroom observations. During the interviews he explained that he was concerned with the connection between the mathematics taught in school and everyday life, and in the questionnaire he also claimed to focus a lot on this. When planning lessons, he often had this connection in mind. He focused on activating the pupils because he believed that pupils in activity and interaction would learn more than if they were passively observing and listening to the teacher's explanations. He believed that blackboard teaching represents an active teacher and passive pupils. His main purpose was to activate the pupils.

During the interviews Harry explained that he did not follow the more formal project ideology a lot. He rather had a focus on small-projects and activities. He gave some examples of projects that his pupils had been involved in, and they included finding geometric shapes in industry sites, drawing shapes and patterns on a quadratic piece of paper with only pencil, ruler, and compass. When the teacher training college (in their neighbourhood) was last rebuilt, he took a group of pupils to the construction site and asked them to make all kinds of measures and estimations as to size, weight, and volume. Another example of how he used projects was in the bicycle assignment.

The bicycle assignment was the last activity I observed in Harry's classes. In this small-project, Harry asked some of the pupils to bring along their bicycles to class.

> When you are now going to work with the bicycle assignment, you are going to draw the bicycle on the scale of 1:5. You are going to collect as accurate measurements and angles as possible! And I just said: draw a sketch today, a rough draft! So that you can sit down later and make an accurate drawing. And then you will pick out as many geometrical shapes as possible. (Transcriptions 13.5.03)

Harry spent some time introducing this assignment before he let the pupils start measuring the bikes. In the introduction he talked about several issues related to everyday life, such as how the digital speedometer works, the brakes, and so on. The pupils decided for themselves whether they wanted to cooperate or not, and some pupils worked outside in the schoolyard.

One of the first practical issues that came up was how large they should make the sketch for the drawing to fit into one page.

Teacher:	Yes. Perhaps we should say something about how big this bicycle will be. Is there room for you to draw it on a page?
Pupil:	No!
Teacher:	If it is 1 metre high, how big will it be in your book then?
Pupil:	5, 50, no 5 centimetres.
Teacher:	Remember to draw on a scale of 1:5.
Pupil:	I don't have a clue. 5, 15, something else, 5 then! I haven't got a clue! Asking me about these things…
Teacher:	No. No. Do you follow, Sandra? If that one is 1 metre high, if we say that the bicycle is 1 metre high now. How high will it be in the book then?
Pupil:	Yes of course. You just divide by 5!
Teacher:	Yes, of course.

Pupil: But I don't know what that is. Fourteen.
Teacher: 20 centimetres. (Transcriptions 13.5.03)

Several other issues also came up, and the pupils got several opportunities to discuss different mathematical concepts in natural and realistic contexts. They also got the opportunity to measure different geometrical objects in practice, using measuring instruments like the slide calliper. Since Harry was also a teacher of natural science, he often got into discussions about technological and physical issues concerning the bicycle. There were several discussions that involved references to everyday life. An example of this is the following:

Teacher: For instance – you are going to find as many geometrical figures as possible – and for instance, I would have included the length of that [the pedal]. Yes, so you must write the length of it. But why is it necessary to include the length of it, why is that a point?
Pupil: Yes, because you must see how much force … no, I don't know.
Teacher: Yes, you are on the right track. If it was shorter, what would it be like to cycle then?
Pupil: Hard. (Transcriptions 13.5.03)

On a couple of occasions one of the pupils came up to Harry and asked about some technical issues regarding the bicycle, like: "Why do the pedals have exactly that length?" Then they got into an interesting discussion on this. These were not purely mathematical questions, but they were related to technology. There were also questions on measuring, sizes, lengths, other geometrical phenomena, and so on.

The textbook has been, and still seems to be, the most important tool or source for the mathematics teacher. The Norwegian national curriculum, L97, gives a description of mathematics and mathematical activities that strongly implies a different way of working with mathematics. The curriculum also presents several concrete suggestions of sources to use, such as situations from the media, bringing in the pupils' own experiences, small-projects, and computer tools.

In the questionnaire Harry claimed to use other sources than the textbook very often. In 21 of the 22 lessons that we observed in Harry's classes, the main activity involved work with other sources than the textbook. Harry explained during the interviews that his pupils solved many problems from the textbooks, but they mainly did this for homework. Other activities and small-projects were emphasised in the lessons. On some other questions in the questionnaire he replied that he often used open tasks in his classes, situations from the media were often used, and the pupils sometimes got the opportunity to formulate problems from their everyday life. During the interviews, Harry explained that he used old exams and tests as a source for problems and tasks. He also let his pupils solve problems from the KappAbel contest, which provides a database of problems (http://www.kappabel.com/).

According to Harry, there are not many good computer games for use in school. Two games with which he had good experiences are "The incredible machine" (http://www.abandonia.com/games/en/25/TheIncredibleMachine.htm) and "Crocodile Clips" (http://www.crocodile-clips.com/index.htm). The teacher manual was also a source that he frequently used, and he had gained many ideas from a book called "Den matematiske krydderhylle" (Rossing, 2000). Sometimes he used the Internet in his teaching, but he explained that it is hard for the pupils to find appropriate information there without help from the teacher. Much of the information that can be found on the Internet is simply too difficult to understand for the pupils, and it can be hard to decide what information that is trustworthy. When Harry used the Internet in his teaching, it was mainly to wake the pupils' interest.

DISCUSSION

A main challenge for teachers is to bridge the gap between school mathematics and everyday practices. More generally, one might describe this process as the transfer of learning between a practical and a theoretical context. This transfer process is often regarded as troublesome (cf. Evans, 1999).

A main concern for teachers should be to create a learning environment that involves contextualised problems; the control of the learning process should be with the pupils working on the problems rather than with the teacher, it would often include hidden mathematics, and learning would often be organised as apprenticeship (Civil, 2002).

Harry often organised activities or small projects for the pupils. These activities mostly involved contextualised problems. In the bicycle assignment referred to above, the context was the starting point for discussions rather than the application of some mathematical theories, as suggested by the curriculum (cf. Hagness & Veiteberg, 1999, p. 167). These projects also included elements of hidden mathematics.

Although one might argue that the activities which Harry organised involved elements of apprenticeship learning, this appeared to be an area that could have been improved when compared with the model presented by Civil (2002). Apprenticeship learning has traditionally been described as a master-disciple relationship where a master is tutoring his small group of disciples. It often involves a workplace context. Although the teacher might be described as a master and the pupils as his disciples, the traditional classroom context does not fit the apprenticeship model very well. A possible way of implementing this could be to organise the class in small groups with a focus on cooperative learning. In this way the pupils could cooperatively move towards mastery within the groups. Although Harry replied in the questionnaire that he often let the pupils work in groups, this appeared to be something he did not focus on. He explained in the interviews that the pupils worked in pairs or in groups if they wanted to, but this was not something he organised, and this was also consistent with our observations in the classroom.

Projects are important according to the Norwegian curriculum and the use of projects and small-projects are explicitly mentioned in the area of mathematics in everyday life. The curriculum states that the pupils in grade 9 should have the opportunity to *"use mathematics to describe and process some more complex situations and small projects"* (Hagness & Veiteberg, 1999, p. 180). In grade 10 they should have the opportunity to *"work on complex problems and assignments in realistic contexts, for instance in projects"* (Hagness & Veiteberg, p. 182). The last quote indicates that projects can actually be a way of implementing problems with realistic contexts, or more generally to make connections with everyday life. Using projects, like we have seen in Harry's class, could also be a way of creating a learning environment that would meet some of the demands made by Civil (2002).

There are many ways for a teacher to connect mathematics with everyday life. Some of these possibilities are presented in the national curriculum (L97). The syllabus aims at creating close links between school mathematics and mathematics in the outside world. In order to build up the concepts and terminology of mathematics, day-to-day experiences, play, and experiments are proposed (cf. Hagness & Veiteberg, 1999, p. 165). L97 seems to distinguish between school, leisure time, working life, and social life, and the so called "outside world" should thereby include the latter three, since these are the parts of (everyday) life that take place outside of school. (It is because of this distinction in the curriculum that we have defined "everyday life" to imply everyday life outside of school in this thesis, although school life is certainly part of the pupils' everyday life more generally.) Mathematics is supposed to be useful in all these areas.

The connection between school mathematics and mathematics in the outside world, which is visible in the area called "mathematics in everyday life", is strongly connected with a more general aim of the curriculum. The pupils should not only develop skills in the subject, but also understanding and insight, so that they can use mathematics in different contexts. Pupils should be given the chance to experience and become familiar with the use of mathematics at home, at school and in the local community. This should be done in a process of reinvention where the pupils create their own concepts, and the starting point for such a process should be a meaningful situation or problem.

In the questionnaire, Harry replied that his pupils were very often involved in a process of reinventing the mathematical theories. One might argue about how reinvention is supposed to be understood in mathematics education, but this was still an area that we did not see implemented in Harry's teaching a lot. Only one example (that we have not described here) could be interpreted as reinvention, but Harry gave the pupils quite clear hints as to where the activity was supposed to bring them, so this example would probably not fit the definition of reinvention shared by many people.

The pupils are intended to construct their own concepts in a process of reinvention, with meaningful and realistic contexts as a starting point, and the pupils are also going to learn mathematics that they can use in "the outside world". These issues are complex, and they imply ways of teaching and learning that are quite different from what is normal in Norwegian classrooms (cf. Alseth et al., 2003). The curriculum suggests that teachers use projects and small-projects, but it remains for the teachers to figure out how these projects can be organised and applied in order to reach the curriculum intentions.

When organising activities as a process of reinvention, the control of the learning process is gradually moved from the teacher to the pupils. It appears that this is one point where Harry did not reach the ideals proposed by Civil (2002). The way he arranged his lessons with activities and small-projects fits the model quite well, but the control of the learning situation would often still be with him rather than with the pupils. Civil's model describes some important ideas to focus on in order to create a good learning environment, and several of her points were visible in Harry's teaching. The aim of this article was not to present the perfect teacher. Starting with a model, we have analysed the findings from a case study of an experienced teacher in order to find some elements or practices that can contribute to creating a good learning environment. The focus has been on the connection of mathematics with everyday life.

CONCLUSIONS

For the findings of this study to be really useful, they should be generalised into theory that teachers can apply in their teaching. The initial aim presented in the beginning of this article was to distinguish strategies and ideas for teaching, in order to implement the ideas of the curriculum as far as mathematics in everyday life is concerned, and to create a good learning environment (as presented by Civil, 2002). The examples from Harry's teaching have illustrated two strategically related issues; use of projects and other sources.

It seems evident that using the textbook as the only source, following a teaching strategy that involves lectures from the blackboard and letting the pupils solve exercises from their textbooks, does not fulfil all the intentions of the curriculum. Harry claimed that such a way of teaching results in active teachers and passive pupils. This is supported by a strong constructivist tradition which describes learning as a process of active construction.

Harry responded to this by arranging small-projects that involved the pupils being active, and this would sometimes imply taking the class out of the classroom. Instead of just explaining how carpenters use their knowledge about Pythagoras' theorem to construct right angles, he let the pupils try this out for themselves and made them construct right angles of wooden sticks in the woodwork room. Instead of just telling the pupils about the geometrical shapes that were hidden in a bicycle, he let them measure and draw some of their own bicycles. In all these small-projects, the pupils were given the opportunity to use their mathematical knowledge in practical activities, and there were several examples of pupils who made personal discoveries of mathematical nature through these projects. After the pupils had been engaged with these projects and practical activities, and their interest had been raised, Harry lets them practice solving textbook tasks (mainly for homework) related to the same issues. This approach can thereby be described as:

– start with practical activities and small-projects,
– let the pupils discover mathematical concepts and theories for themselves (guided by the teacher),
– connect the practical activities with mathematical content knowledge, and then
– let the pupils practise solving more traditional textbook tasks.

By using such an approach, the teacher will have to use other sources than the textbook. The main challenge is therefore to come up with (or have access to a source of) ideas for such activities and small-projects.

When compared with the model presented by Civil (2002), there were some elements that Harry's teaching lacked. A process of reinvention, which could be an important way of giving learners more control of the learning process, and apprenticeship learning, which could have been organised in cooperative groups, were not so visible. There are probably other approaches that might work as well, but Harry's teaching has given us some interesting examples that can contribute to a more practical model, describing how teachers can organise their teaching in order to create a good learning environment.

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EFFICIENCY OR RIGOR? WHEN STUDENTS SEE THE TARGET AS 'DOING' MORE THAN 'KNOWING' MATHEMATICS

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The object of this paper is to present the analysis of an observation of three student teachers' collaborative work in a problem-solving context. Two episodes are presented which reveal incidents of specializing, generalizing, conjecturing, and convincing. I begin my analysis by looking at the student teachers' difficulties in changing representational system from natural language to formal language (using a symbolic notation including fractions and letters). Further analysis describes how reasoning processes involved are constrained by the didactical contract.

I believe that it is useful for mathematics learners to engage in problem solving processes in order to obtain a deep understanding of the whole process of doing mathematics. However, processes of generalizing and justifying in mathematics are often perceived as problematic to students (e.g., Chazan, 1993; Almeida, 2001). Joe Garofalo and Frank K. Lester (1985) have suggested that students are largely unaware of the processes involved in problem solving and that addressing this issue within problem solving instruction may be important. In this paper I will present two episodes from a video recorded small-group work session at a university college, in which three student teachers are supposed to collaborate on a generalizing problem in algebra. The research reported is part of my ongoing PhD project. The analysis of the episodes shows how the students[1] are unsuccessful in shifting from arithmetic to algebraic thinking, and how this prevents them from constructing the intended (by the teacher) mathematical knowledge. In this situation the didactical contract is threatened, and the analysis indicates how the reasoning of one of the students becomes directed by striving for fulfilment of the didactical contract (through a solution of the given problem), at the cost of mathematical rationality.

[1] I will refer to the student teachers as 'students', and the teacher educator as the 'teacher'.

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THEORETICAL BACKGROUND

When writing about mathematical problem solving, it is relevant to clarify what is meant by a mathematical problem. I agree with Alan H. Schoenfeld's definition of what a mathematical problem is:

> For any student, a mathematical *problem* is a task (a) in which the student is interested and engaged and for which he wishes to obtain the resolution, and (b) for which the student does not have a readily accessible mathematical means by which to achieve that resolution. (Schoenfeld, 1993, p. 71)

Many writers have attempted to clarify what is meant by a problem-solving approach to teaching mathematics. Schoenfeld (1985) emphasizes that defining what a problem is, is always relative to the individual. In a problem-solving approach the focus is on teaching mathematical topics through problem-solving contexts and inquiry-oriented environments which are characterized by the teacher "helping students construct a deep understanding of mathematical ideas and processes by engaging them in doing mathematics: creating, conjecturing, exploring, testing, and verifying" (Lester, Masingila, Mau, Lambdin, dos Santos, & Raymond, 1994, p. 154). Specific characteristics of a problem-solving approach include:

- Interaction and negotiation between students and between teacher and students (Van Zoest, Jones, & Thornton, 1994)
- Teachers providing just enough information to establish background of the problem, and students clarifying, interpreting, and attempting to construct one or more solution processes (Cobb, Wood, & Yackel, 1991)
- Establishing a conjecturing atmosphere (Mason & Davis, 1991)
- Teachers knowing when it is appropriate to intervene, and when to step back and let the pupils make their own way (Lester et al., 1994)
- The approach can be used to encourage students to make generalizations about relations and concepts, a process which is central in mathematics (Even & Lappan, 1994; Schoenfeld, 1994).

In one of the episodes to be described, the conjecturing of one of the students is interpreted in terms of fulfilling the didactical contract (Brousseau, 1997).

> In a teaching situation, prepared and delivered by a teacher, the student generally has the task of solving the (mathematical) problem she is given, but access to this task is made through interpretation of the questions asked, the information provided and the constraints that have been imposed, which are all constants in the teacher's method of instruction. These (specific) habits of the teacher are expected by the students and the behaviour of the student is expected by the teacher; this is the didactical contract. (Brousseau 1980, as cited in Brousseau 1997, p. 225)

Guy Brousseau (1997, p. 30) defines an *adidactical situation* to be a situation in which the student is enabled to use some knowledge to solve a problem "without appealing to didactical reasoning [and] in the absence of any intentional direction [from the teacher]". The *devolution* of an adidactical learning situation is the act by which the teacher makes the student accept the responsibility for an adidactical learning situation or for a problem, and (the teacher) accepts the consequences of the transfer of this responsibility (Brousseau, p. 230). Negotiation of a didactical contract is a tool for the devolution of an adidactical learning situation to the learner. When the devolution is such that the learners no longer take into account any feature related to the didactical contract but just act with reference to the characteristics of the adidactical situation, an ideal state is accomplished. Nicholas Balacheff (1991) argues that beyond the social characteristics of the teaching situation, we must analyze the nature of the target it aims at: "If students see the target as 'doing', more then 'knowing', then their debate will focus more on efficiency and reliability, than on rigor and certainty" (p. 188).

The students in the reported research do not succeed in representing by formal language an enlargement of a quantity in terms of an arbitrary percentage. According to Raymond Duval (2002), transformations between different representational systems can, from a cognitive point of view, be considered of vital importance with respect to "the mechanisms underlying understanding" (p. 4).

Drawing on Mary F. Belenky, Blythe M. Clinchy, Nancy R. Goldberger, and Jill M. Tarule (1986), Hilary Povey (1997), and Hilary Povey and Leone Burton (1999), I work in my PhD study with the concept 'student authority' as an integration of the concepts 'authority' and 'agency'. Student authority is about the learner's authoring of mathematical knowledge, where agency is seen as a link between the learner's understanding of mathematics and his/her role in that construction. This paper aims to give insights into how mathematical meaning is negotiated through collaboration.

METHODS

My PhD project is an *educational case study* (Stenhouse, 1988, as cited in Bassey, 1999) with two cases, each constituted of a group of students' learning of mathematics in a teacher education programme. Lawrence Stenhouse's characteristic of an educational case study corresponds well with the intention of my project:

> Educational case study [is where] many researchers using case study methods are concerned neither with social theory nor with evaluative judgement, but rather with the understanding of educational action…They are concerned to enrich the thinking and discourse of educators either by the development of educational theory or by refinement of prudence through the systematic and reflective documentation of evidence. (Stenhouse 1988, as cited in Bassey 1999, p. 28)

The cases in my study are instrumental to the general understanding of authoring processes in mathematics. Participants in the research reported in this paper are three female students in their first year of a programme of teacher education for primary and lower secondary school. At the time the data was collected, they had been collaborating on several tasks in different topics during the seven months they had been on the programme. The data is collected in order to answer the research questions of my PhD project, which is about how mathematical knowledge is authored by the learner, and how mathematical meaning is negotiated through collaboration.

The episodes which will be described are taken from a video recorded small-group work session at a university college, in which the students are supposed to collaborate on a generalizing problem in algebra. There is no teacher intervention during the part of the lesson described in this paper. One of the teachers has designed the task aiming at the students' recognition of the connection between natural language and formal language – formula and context, in addition to strengthening their general understanding of how percentage enlargement of lengths of a two-dimensional figure influences the area of the figure. Also, he wanted them to work on problem solving processes like going from the particular to the general, and testing hypotheses. I have analyzed the episodes from the perspectives of reasoning processes and the impact of the didactical contract.

DESCRIPTION AND ANALYSIS OF TWO EPISODES

Three students, Alise, Ida, and Sofie (pseudonyms), are sitting in a group room adjacent to a big classroom, in which the rest of the class (about 40 students) are working in groups on the same collection of tasks, which has been handed out. There are two teachers present in the big room, observing the work of the students, helping them, and participating in dialogues with them, so also with Alise, Ida, and Sofie. The teachers are my colleagues, and I have also been involved in mathematics teaching in the class. I observe and video record the work of the students. My role is to be an observer and neither to interfere with their work nor to help them. This role is justified and explained to the students as necessary because the data collection should be in as naturalistic a setting as possible. In this way I try to minimize the disturbance of my presence in the room.

The students have been working for 14 minutes on the problem set out in Figure 1.

Imagine you have a square. Make a new square of which the side length has increased by 50 %. How many percent has then the area increased? How many percent has the perimeter increased?[2]

Imagine now that the side length increases by p %. How many percent will the area and perimeter consequently increase?[3]

Figure 1

They have chosen the problem from a collection of tasks handed out, and are supposed to present their work for another group of students who work on a different task from the same collection. They have drawn a 2 x 2 cm^2 square and have increased the lengths by 50 percent and drawn the result, a 3 x 3 cm^2 square as illustrated below.

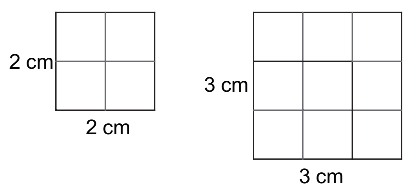

Figure 2

The students have found out that when the side of a 2 x 2 cm^2 square is increased by 50 percent, the area will increase by 125 percent. Now they focus on the next part of the task, which involves an enlargement of the side length by *p* percent.

[2] During the lesson described the students work only on the concepts length – area, not perimeter, though perimeter is part of the given task.

[3] When the side of a square increases by *p* percent, the scale factor of the enlargement of the area is given by:

$$\left(1+\frac{p}{100}\right)^2 = 1+\frac{2p}{100}+\frac{p^2}{10000} = 1+\frac{2p+\frac{p^2}{100}}{100}.$$

In general, when lengths increase by *p* %, the area increases by $(2p+\frac{p^2}{100})$ %.

Episode 1. Trying to represent the result of an enlargement by p %

Excerpt I from transcript[4]

154 Ida: You know, that was what I was thinking about, that it was one plus or minus *p* hundredths.
155. Alise: Yes, yes.
156. Ida: I don't know, that is what I first thought about, but I don't quite understand…Imagine that the side length increases by… (Repeats the task handed out) we might as well call it *x*, too.
157. Alise: Yes (Sofie nods).
158. Ida: Isn't it just tackling it the same way, except that we use *p* instead of a number? (Draws a new square, which is not visible from the video tape) (13 seconds later)
159. Ida: I have drawn it twice as large [] (smiles)
160. Sofie: The side length increases by *p*. But we have to take as a starting point… shall we start with the one which is 2 centimetres then? It increases by *p*. How many percent will the area and the perimeter increase? (Repeats task) Then we must take four, and then we must take…then we must take…This will be four, and then we must take *p* minus four over… four.
161. Ida: Why do you do this?
162. Sofie: Because [nine minus four
163. Ida: [It's supposed to increase by…
164. Sofie: nine minus four equals five.
165. Alise: []
166. Sofie: What?
167. Ida: No, because *p*…
168. Alise: We can't use *p* here. No, because this is not *p*, because…
169. Sofie: No, because *p* is the *percentage*.
170. Alise: Yes, by which it increases.
171. Sofie: yes, yes, yes
172. Alise: So, actually, you must have…have two multiplied by… or two plus *p*… and minus four (Sofie writes down what Alise expresses)
173. Sofie: Two plus *p* minus four?
174. Alise: No, two multiplied by *p*…(looking out into the air) because what we now did was, fifty percent of this (side of length 2 cm) was one centimetre (Sofie: mm), then we added one centimetre (Ida: yes). So this will be two plus one (Ida: yes)… in order to get that (the square with side length 3 cm) (Ida: yes). So therefore you must have two plus *p* … percent… (Looking at Ida and Sofie)

[4] The transcript is translated from Norwegian by the author. For transcription conventions, see appendix.

From Ida's utterance in turn 156, I interpret the pronoun 'it' in turn 154 to refer to the side length. Ida is in turn 154 seen to bring in a formulation of a half remembered concept, 'one plus or minus *p* hundredths'. The fact that she is not able to decide whether it should be plus or minus, indicates that Ida does not have a very deep understanding of the (use of the) formulation. This interpretation is strengthened by the fact that she suggests the expression 'one plus (or minus) *p* hundredths' to represent the enlarged side, not the scale factor of the enlargement of the side. Nevertheless, 'one plus (or minus) *p* hundredths' is Ida's conjecture about the representation of the enlarged side. Her question in turn 158: "Isn't it just tackling it in the same way, except that we use *p* instead of a number?" I interpret as a call for looking back, in order to try to generalize by using *p*, on the basis of what they have done in the concrete example with a certain percentage. In turn 160 Sofie is seen to follow up Ida's call when she suggests that they specialize to the case with an original square with side 2 cm. Sofie tries to represent the percentage of the enlargement of the area when the lengths of the original square with side 2 cm are increased by *p* %. In the first part of the task they have calculated the percentage of the enlargement of the area when the lengths of a square with side 2 cm are increased by 50 % (Figure 2). The enlargement of the area in terms of percentage was found to be $\frac{9-4}{4}\cdot 100 = 125$, which explains Sofie's attention to the numbers nine and five in turns 162 and 164. She represents the enlarged area by *p*, and thus conjectures that "[] *p* minus four over… four" expresses the percentage of the enlargement of the area. In the subsequent turns Sofie tries to convince Alise and Ida that this is correct, but in turns 167 and 168 Ida and Alise convince Sofie that her conjecture is not valid. Sofie realizes that it is wrong to let *p* represent the enlarged area, "[] because *p* is the *percentage*" (emphasis in original).

Sofie's dealing with the mentioned algorithm, despite the lack of success, I interpret as important for Alise's further reasoning. In turn 174 Alise uses the same example (from Figure 2), and recollects how they calculated the enlargement of the side when it was increased by 50 %. In spite of the fact that she starts turn 174 by saying "two multiplied by *p*…", she conjectures that a side of length 2, when increased by *p* %, is represented by $2 + p\ \%$. Thus, Alise does not succeed in following up her own reasoning about the enlargement as the product of the original length and the factor *p* percent. The transformation from natural language to formal language (using letters) has not worked well for Alise. When we in natural language talk about an enlargement of a quantity, here two, by *p* percent, we are likely to say "two plus *p* percent", meaning "two plus *p* percent of two". When we use formal notation, we have to write $2 + 2 \cdot p\ \%$.

What makes an enlargement by an arbitrary percentage so complicated for these students in Episode 1? There is a lack of transparency concerning what concept they are talking about, lengths or area, which is partly because of the use of the indefinite pronoun 'it'. In turns 154–158 the attention is on lengths and enlargement of lengths. In turn 160 the attention starts on length and continues on area and enlargement of area. Then there is a shift back to a focus on lengths in turns

167–174, just interrupted by a hint at enlargement of area by Alise in turn 172, " … or two plus *p*… and minus *four*" (my emphasis). These students lack a technique of representing in terms of mathematical symbols a quantity (here length) which is increased by *p* percent, and consequently, they do not succeed in representing the corresponding enlarged area. In the first part of the task they used mental calculations to find that fifty percent of two was one, a calculation they did not write down. Therefore they did not have anything in front of them which could help them from the particular to the general, with an arbitrary enlargement in terms of *p* percent. Another question is why Alise and Sofie do not respond to Ida's conjecture when she suggests that "[] it was one plus or minus *p* hundredths"? Drawing on several observations in the past of Ida's participation during small-group work on mathematics, I had got the impression that Ida is a student who is likely to let the other ones have the initiative. I had perceived her to be less confident in mathematics, but responding to the others' initiatives. In the past I had observed her computing relevant calculations on her calculator, and trying to sum up results from the collaboration. I assume that Ida's peers might have got a similar perception of her, and that this might have influenced the ignorance of her initiative, a conjecture which directs attention to the scale factor of the enlargement, and thus could have been decisive for their work.

Episode 2. Ida challenging Alise's rationale

In Episode 2 the students still work on the same task; to find a formula for the enlargement of the area of a square in terms of percent, when the lengths increase by *p* %. I will, through my analysis of this episode, focus on Alise's rationale for a conjecture and the way that Ida challenges this conjecture.

Excerpt II from transcript

190. Alise: But what we know about the area (Ida: yes) is that it increases by hundred and twenty five percent when you double it, in a way.
191. Ida: But we don't [know…
192. Alise: [What will the length be?
193. Ida: Well… but the only thing we know here is that the side length increases by *p*.
194. Alise: Yes, but then we can say that it…that it increases by hundred and twentyfive over fifty or the other way around, fifty over hundred and twenty five. (Alise and Ida use their calculators to compute the quotients $\frac{125}{50}$ and $\frac{50}{125}$)
195. Alise: [Because then we know that it increases by…
196. Ida: [Two point five or zero point four (result from calculation)
197. Alise: Yes, if we say that it resulted in two point five, which is the most probable one…(Ida: yes). Then the area increases, if you

increase, increase … the side length by p (Ida: yes?) then the area increases by two point five p.

198. Ida: But then … is it because you assume that it is fifty?
199. Alise: What?
200. Ida: How…
202. Alise: The reason why, becau…the way we found that it was two and a half, because I think that (quotient) is most likely compared with zero point, was it zero point four?
202. Ida: Yes
203. Alise: I assume that two and a half is most… (Sofie smiles)
204. Ida: Yes, but I was just wondering why you took hundred and twenty five over fifty?
205. Alise: Because it was these (quantities) which were present in that example. Because it increased [by…the area was hundred and twenty five
206. Ida: [but are you sure that we should continue with that (quantity) in the rest of the task?
207. Alise: But a little while ago, when I computed this you know (Ida: yes?) then I nevertheless got fifty six percent. Even if I had computed it wrong, and I got the same [] (Ida: yes, Sofie nods). So then we can say… we assume that it is a rule … or?
208. Ida: Well, do you think we can do that…or?
209. Alise: I don't know.
210. Sofie: Well, let's do that now. Then we will see how it turns out, and then we do it differently afterwards if it turns out to be wrong.
211. Alise: No, because when you take the enlargement of the area, which is hundred and twenty five (Ida: mmm) over the enlargement of the side length (Ida: mmm) then you get two and a half (Ida: yes), then you get the ratio of these two.
212. Ida: And then you have found out…it is two and a half percent, the area has increased by two and a half percent?
213. Alise: No, two and a half *times* (Ida: yes), not the percentage.
214. Ida: It is written here (looks at the task): How many percent will the area and the perimeter in this case increase?
215. Alise: Well, by two and a half then? (Lifts her left hand, shrugs her shoulders a bit, shakes her head slightly, smiles and blows out air through her nose). This I don't know. Two and a half times. Works well for me.

(Alise laughs, Ida smiles and shakes her head several times)

Then five turns between the students follow, in which Alise argues that when they take two and a half and multiply by p percent, they will get an answer in terms of percentage.

221. Alise: …(Writes $p\,\% \cdot 2.5 = 2.5p\,\%$ in her notepad)

> They then decide to test the formula for $p = 25$. When they compute the enlargement of the area, starting with a 2 x 2 cm^2 square, they get 56.25 %. However, when they use the conjectured formula, they get 62.5 %, and hence the conjecture is proved to fail. Accordingly they agree on asking a teacher to help them.

Ida is in turn 193 emphasizing that the only thing they know is that the lengths increase by *p*. Alise suggests in turn 194 that the area increases by $\frac{125}{50}$ or $\frac{50}{125}$. The quantities 50 and 125 originate from the first part of the task in which the students have found out that a 50 percent enlargement of the lengths of a square will result in a 125 percent enlargement of the area. When Ida tells what the quotients are in terms of decimals, Alise responds in turn 197 that 2.5 is the most probable one, and that this means that if the side length of a square increases by *p* percent, the area increases by 2.5 *p* (which is not correct). Later (turn 221) Alise writes the expression $p\,\% \cdot 2{,}5 = 2{,}5p\,\%$ in her notepad, conjecturing that it represents the percentage of the enlargement of the area of an arbitrary square, when the side length is increased by *p* percent. Alise's proposed formula can be seen as a result of doing mathematics in the classroom, where the rationale for the student in investigating a quantity or relation is not within mathematical rationality. The ratio of the enlargement of the area to the enlargement of the side length does not make mathematical sense. A trustworthy explanation for Alise's choice of 2.5 in preference to the other, may be that the quantity 2.5 is close to the quantity 2.25, which denotes how many times the area has increased when $p = 50$. This is the only value of *p* for which the proposed formula gives the right answer, a fact Ida hints at in turns 198 and 206.

It is relevant to search for Alise's driving force and rationale for the apparently irrational conjecture with a wider focus of the lens (Lerman, 2001), to focus on the social characteristics of the teaching/learning situation, and not just focus on Alise's individual learning trajectory. Considering Alise's enterprise from the perspective of the didactical contract (Brousseau, 1997), her task is to give a solution to the problem given to them by the teacher, a solution which is acceptable in the classroom context. In this situation she acts as a practical person, for whom the priority is to be efficient, not to be rigorous. The aim is possibly to produce a solution, not to produce knowledge, a situation described by Balacheff (1991).

"It works well for me", Alise says in turn 215. This utterance together with the subsequent gestures and laughter I interpret as evidence of Alise being aware of the weakness of the conjecture and the detrimental effect of her attitude. When Alise chooses one of two quotients with the reason that it "is the most *probable* one" (turn 197, my emphasis), she is concerned about reliability, not rigor and certainty. Ida wants to go beyond probability when she asks Alise the reason *why* she "took hundred and twenty five over fifty?" (turn 204). This question I interpret to be crucial, because it reveals that Ida is not satisfied with Alise's arguments on probability and thus faces Alise with her lack of mathematical rationality. Alise's answer I consider as evidence of Alise not being concerned about rigor and certainty: "Because it was these (quantities) which were present in that example…" (turn 205). This utterance, together with Alise's conjecture that 2.5 is as many *times* the area has increased in the case where the side length has been increased by 50 percent (turn 197), exemplify a possible consequence of the didactical contract: The students are supposed to do calculations with the numbers present in (or calculated through) the given task. Having a mathematical rationale for using them is likely to be secondary (as here for Alise). According to Brousseau (1997, p. 228) a *learning situation* is one "in which what one does appears to be necessary with respect to obligations which are neither arbitrary nor didactical". Alise's obligations are here seen to be both arbitrary and didactical – didactical in the sense that she sees the situation as being only justified by the teacher's expectations, and hence it is not a learning situation for Alise.

While Alise's goal is to be efficient and *do* the task, Ida is seen to be more focused on *knowing* and certainty, and is therefore seen to challenge Alise's rationale and reasoning, as for instance when she asks Alise *why* she "took hundred and twenty five over fifty". I consider, further, as evidence of Ida's concern about knowing turns 198, 206, and 208, which indicate Ida's insights into the limitations of Alise's formula (that it is only valid when $p = 50$). Turns 212 and 214 indicate Ida's call for knowing manifested as a wish to understand the proposed formula and how it relates to what the task asks for; the enlargement of the area of a square when lengths are increased by an arbitrary percentage.

DISCUSSION AND CONCLUSION

The students in the episodes described participate in the practice of collaborating on a mathematical task set up by one of their teachers, and are all seen to contribute to the joint activity of solving the problem. The problem of expressing what effect an arbitrary percentage enlargement of the side length of a square has on the area of the square, I interpret to be a mathematical *problem* for the students according to Schoenfeld's (1993) definition. Even if I cannot know whether they are genuinely interested in the task or not, they are engaged, and wish to obtain a resolution. The students do not have a readily accessible means by which to achieve a resolution of the problem, therefore the second part of Schoenfeld's definition is satisfied.

There seems to be evidence that the students' collaborative work has contributed to mathematical progress in the discussion, which ends with the rejection of Alise's conjectured formula. They have used processes of specializing, generalizing and conjecturing, and tried to convince themselves and each other, and they have got stuck (Mason et al., 1982; Mason & Davis, 1991). Although the picture is not sharply outlined, the students might be seen to play different roles during the collaboration in both episodes presented: Alise is likely to make conjectures (turn 174 in Episode 1, turn 197 in Episode 2), Sofie to specialize (turn 160 in Episode 1, turn

210 in Episode 2[5]), and Ida to ask for justification, and thus she can be viewed as a warrant of rationality by the way she challenges her peers.

In what follows I want to focus on two points. Firstly, the analysis reveals serious difficulties which these students experience with the meaning and representation of an enlargement by *p* percent. The students lack a technique of representing an enlargement of a quantity in terms of an arbitrary percentage. They can handle the special case when lengths increase by a particular percentage, but when lengths increase by an arbitrary percentage, they do not succeed in the transformation of the representational system from natural language to formal language (using a symbolic notation including fractions and letters). Hence the analysis of the data presented points at the importance for learners to engage in the activity of transformations between different representational systems, in order to develop mathematical fluency. Relevant transformations in this case would be changes between representations in the form of natural language, notation systems using fractions, decimals, and symbols, in addition to illustrations and empirical situations. This would be a way of engaging students in activities which promote a shift from arithmetical to algebraic thinking.

Secondly, the analysis points at the strong impact of the didactical contract, which is traced back to Alise's apparently irrational proposal for a formula. What at first seemed to be meaningless manipulation of symbols, turned after a closer examination out to make sense when taking the student's point of view. The teacher has designed an adidactical situation which presupposes the students' mastering of a technique of representing by formal language an enlargement of a quantity in terms of an arbitrary percentage. Because the students do not master this technique, there is a didactical problem. The teacher has not "contrived one [adidactical situation] which the students can handle" (Brousseau, 1997, p. 30); the devolution of the problem specific to the construction of the target knowledge has not worked well.

Negotiation of a didactical contract takes place in a metadidactical situation (see Brousseau, 1997, p. 248), outside the didactical situation, in which the teacher reflects on and prepares the lesson he must construct, and the student looks at the teaching situation from the outside. In teacher education the analysis of these episodes could be used at a metadidactical level to reflect on the didactical situation; the devolution of the learning responsibility to the students, and the validation and institutionalization of knowing and meaning. Reflection on the didactical situation would contribute to a better understanding of the didactical phenomena (Brousseau, 1997, p. 247) related to the didactical contract, and reflection at the metadidactical level could predict expected outcomes of the didactical contract – which is specific for the target knowledge in play.

[5] Sofie follows up this suggestion in the continuation of the dialogue (turn 227, complete transcript) by initiating that they test with a value of *p* different from 50.

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APPENDIX

(Transcription conventions)

[	B interrupts A or speaks at the same time as A:
	A: Because [nine minus four
	B: [It's supposed to increase by
[]	inarticulate or inaudible utterance
…	pause (up to 3 seconds)
italics	emphasis
(text in parenthesis)	representation of non-verbal action or comment on action

GIRLS' BELIEFS ABOUT THE LEARNING OF MATHEMATICS

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There has been an increased attention to research on beliefs about mathematics and mathematics education and it has become one of the central elements of study in mathematics education. This paper reports from a qualitative research study on the beliefs of four Icelandic teenage girls about mathematics, the study of mathematics, and themselves as learners of mathematics. Their descriptions and thoughts are viewed in the light of theories and newfound results of overseas quantitative research on girls' beliefs about mathematics and the study of mathematics. The main conclusions of this research are that these girls:

- *view mathematics as a process*
- *place emphasis on understanding and solving the problems at hand*
- *are self-confident, well organized and study hard*
- *do not often use elaboration strategies.*

INTRODUCTION

Mathematics is one of the main subjects in primary and lower secondary school. Extensive research has been made in the field of mathematics education in the last thirty years. The research has contributed to the understanding of how people learn and to finding new ways of organising teaching. Research in this area has drawn people's attention towards the influence of beliefs on how people learn and the outcome of learning. I have been a lower secondary school teacher for several years and became interested in learning about beliefs and their importance. I wondered how much I knew about the beliefs of my pupils over the years and discovered that my knowledge is limited. My main effort had been to get to know my pupils as individuals and analyse their mathematical knowledge. Therefore, I was interested in deepening my knowledge on some of the already existing research on beliefs and to conduct my own research.

My main research question was: *How do pupils in lower secondary schools think about mathematics and their mathematical learning?*

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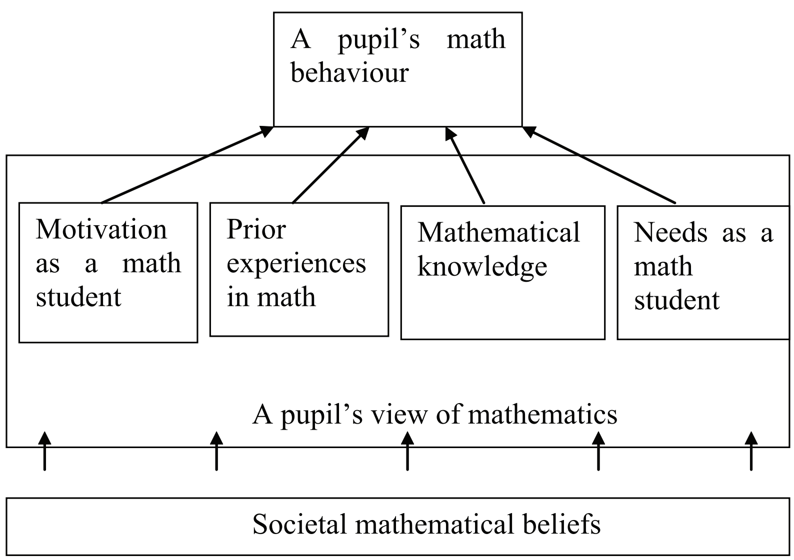
This research question was divided into three areas, the beliefs about mathematics, the study of mathematics, and the pupils themselves as learners of mathematics. My experience had taught me that teenage girls often ask many questions about learning and their interest in mathematics is decreasing. Consequently, I decided to focus on the beliefs of only a few girls. A qualitative research method was used and the study was based on individual interviews with four Icelandic girls. They were all in their final year of lower secondary school.

The following definition of the concept beliefs is used in the study:

> Students' mathematics-related beliefs are the implicitly or explicitly held subjective conceptions students hold to be true about mathematics education, about themselves as mathematicians, and about the mathematics class context. These beliefs determine in close interaction with each other and with students' prior knowledge their mathematical learning and problem solving in class.
>
> (Op 't Eynde, De Corte, & Verschaffel, 2002, p. 27)

THEORETICAL BACKGROUND

Since the seventies, great interest has been shown in the study of students' beliefs and ideas. It can be expected that this research-interest is based on the attitude that beliefs influence how people understand themselves and their surroundings, and how they deal with their lives.



(Pehkonen & Safuanov, 1996, p. 34)

The main view is that on the basis of ideas on a specific matter every individual develops beliefs about it. His beliefs evolve from simple perceptual beliefs, experience, ideas, and expectations. Beliefs are built up from many factors and their interactions are complicated. Pupils develop their beliefs in interaction with their surroundings and they also influence their surroundings. Mathematical ideas and beliefs act as a filter that influences all their thoughts and actions concerning mathematics. Prior experience of mathematics and the learning of mathematics influence both beliefs towards learning and the use of mathematics. Societal mathematical beliefs also influence pupils' beliefs.

More factors could be mentioned that influence the mathematical behaviour of students. A network of influences from the people in one's surroundings influences the individuals' beliefs and how or if they try to learn mathematics. Beliefs towards subjects and learning are, in addition to cognitive factors, the basis of learning. Beliefs have to do with factors such as motivation, self-confidence, and how positive students are. These factors do not only support the learning, they are a part of it (McLeod & McLeod, 2002; Pehkonen & Safuanov, 1996).

Gender is one of the factors that has been found to be of great influence, but not to the same extent on the performance as on the beliefs and thereby on the motivation and the purpose of learning mathematics. Around the turn of the century some research (Brandell, Nyström, & Staberg, 2002; Pehkonen, 1994) showed that beliefs towards mathematics, the study of mathematics, and the experience of being a learner of mathematics, which were held by pupils in lower secondary schools, were changing. The pupils expressed beliefs indicating that mathematics was more for girls than for boys, and the research showed that girls worked better in maths-class and were more successful. More awareness had risen that some social factors, inside and outside the classroom, had some influence. Some research had shown that the majority of pupils no longer saw mathematics as a male-dominated subject. There was, however, a clear difference in what the sexes considered important in the learning of the subject. In the reports from PISA 2000, similar conclusions were drawn (Centre for Educational Research and Innovation, 2003, pp. 7–25, 82–90, 127–142). In PISA 2000, the boys scored higher, but the difference was not significant. The ways the sexes studied were different. The boys were more confident and showed more interest in mathematics. They believed they could cope with learning difficulties, used elaboration strategies and enjoyed competition. The girls were more concerned with what to learn and used more energy. They paid more attention to organising their study, were able to concentrate more easily and used more control strategies. The gender difference was there, but it was in beliefs and ways of learning. The conclusion was that in order to work for equality it was necessary to work with beliefs and learning methods.

A big research project on teenagers' beliefs in mathematics, the GeMa-project, has been conducted in Sweden, from which Brandell and her colleagues (Brandell et al., 2002) have reported. The focus is on gender and comparison of the beliefs of girls and boys. From this, many things of interest appeared, underlining that the beliefs of teenagers are very diverse. More than half of them thought that mathematics is neither a male nor female domain. To give some ideas of how Swedish teenagers thought, I have chosen three examples:

- The girls are considered to work hard in lessons, get encouragement from the teachers, and the expectation is that they will do well.
- The boys are supposed to be disturbing in class, assumed to like using computers, like challenging problems, expect mathematics to be easy and that they will need mathematics in their future jobs.
- Girls think that it is important to understand mathematics and get worried if they are not succeeding.

These studies use quantitative methods and they gave me a good overview over this research field. Pekhonen´s theories also gave me some inspiration as to what questions to ask.

METHODS USED

Very little research has been done on the mathematical beliefs of Icelandic teenagers. Iceland has participated in some multinational research, recently PISA 2000 (Centre for Educational Research and Innovation, 2003) and PISA 2003 (Björnsson, Halldorsson, & Olafsson, 2004). This research is entirely quantitative. I thought it would be interesting to conduct a qualitative research where I could study thoroughly the beliefs of a few individuals and give some ideas of how they express their beliefs. Many studies look at gender as a factor of great influence (Brandell et al., 2002; Gothlin, 1999). So I decided to narrow my research and study only girls and relate my findings to the big multinational studies and some studies of girls' beliefs about mathematics. The four girls I interviewed were all 15 years old attending a lower secondary school in the capital city of Iceland, Reykjavik. They were volunteers from a class of 12 girls (and 10 boys).

There are many different research approaches in the field of qualitative research. In interviews, participants have good possibilities to use their own words and the interviewer gets real examples of how the participants express their experience and what concepts they use. In an interview, the participant also gets a chance to add new elements and the interviewer can ask new questions to get a clearer picture of the participant's ideas. I found this an interesting approach for me as a researcher entering a new field. I wanted to find out what ideas the girls had, interpret them, react and ask further. I prepared some open questions and divided the subject into three main areas, the subject, the study, and being a learner. The research question was as well divided in three main questions: What is mathematics? What is important in the study of mathematics? How does it feel to be a mathematical learner?

DESCRIPTION AND ANALYSIS

The first interview was about the girls' beliefs about mathematics. I asked how they would describe mathematics and mathematical knowledge and how they felt about it. I used in my analysis three main perspectives of the nature of mathematics, traditional, formalist, and constructivist perspective (Pehkonen & Törner, 2004). In the traditional perspective mathematics is seen as a set of skills or a toolbox. In the formalist perspective mathematics is logic and rigour and the focus is on the system. In the constructivist perspective the process in building up understanding is most important.

Helga, one of the girls, was asked the question: *What is mathematics?* She answered:

> Mathematics makes it possible for you to calculate sizes and helps you in your daily life to know what you need to know and it also tells you why things are the way they are.

In the interview Helga mainly showed the traditional perspective. She

– talked about the importance of arithmetic
– said that geometry was about using the right formulas
– thought everybody used mathematics
– saw mathematics as a big subject and thought she had still much to learn
– said that mathematics was about explaining relations

The other girls expressed similar beliefs. However, they had different views as to how they valued the usefulness of mathematics and influence in the society.

Mathematical knowledge must be the aim of the study of mathematics. Some mathematics educators have found it useful to distinguish between two types of mathematical knowledge, conceptual knowledge and procedural knowledge. In his book *Elementary and Middle School Mathematics,* Van de Walle (2004, p. 27) writes about these two main perspectives about the nature of mathematical knowledge, Conceptual Knowledge and Procedural Knowledge. Conceptual knowledge consists of logical relationships constructed internally and existing in the mind as a part of a network of ideas. Learning is based on the individual as he is building up his own knowledge. He needs to be reflective and active. He must ask his own questions and think about how he can use his knowledge to understand new things. The learning of mathematics is about making sense of how the mathematical ideas connect. The understanding will be better as the individual makes more connections. The mathematics teacher gives out problems that help the student to construct knowledge on a specific area. Discussions are important both between students and also between the student and the teacher. Elaboration strategies are in focus and students have to draw conclusions from their work.

Procedural knowledge is the knowledge of the rules and the procedures that one uses in carrying out routine mathematical tasks and also the symbolism that is used to represent mathematics. Students have to build up their knowledge step by step. Students have to be active in memorising the procedures. They use various ways to improve their memory without focusing on their mathematical understanding, as when they use rhymes to remember how to calculate fractions. The aim of procedural knowledge is to build up instrumental understanding that students can use quickly when solving routine problems. The teacher's role is to build up a logistic sequence of small pieces of mathematical information. Most important is that the teacher is able to explain and make the learning easier for the students.

In the interviews on mathematical learning the girls found it hard to talk in general terms about mathematical learning. They could nevertheless describe clearly their ideas of a good mathematics teacher and how he should play his part in the learning process.

Lara, one of the girls, said that in teaching the teacher should go from the simple to the more complicated. The teacher's role is to organise and build up a learning process. The learner's role is to follow the process and practice on many problems. She considered the role of the teacher to be very important. She thought the relationship between the learner and the teacher was meaningful. She found that the learning is most successful if the learner studies under the guidance of the teacher. Lara described a good teacher of mathematics like this:

> Mathematics teachers should take one step at a time when explaining. He should be patient and have various ways to explain the same concept. It is an advantage if the learner knows the teacher and the teacher knows the learner on personal terms.

The girls all showed a strong tendency to look at mathematical knowledge as procedural knowledge. Their view was that the learning of mathematics was important for everyone, and in order to succeed in the learning the student should work well every day and organise his work precisely. They thought that good performance involves the mastering of arithmetic skills and strategies. However, they did express, especially Soffía, that it is important that the learning advances the thinking skills and logical thinking. They also proclaimed the importance of independent students' work when building up an understanding of the procedures.

When discussing with the girls how it felt to be a mathematical learner, I used as a background some research on how people sense their own ability (Fennema, 2000; Guðbjörnsdóttir, 1994; Magnúsdóttir, 2003). This research shows that a person's estimation of one's own abilities influences how one organises and performs an act. It also has an influence on what kind of problems one deals with, how much effort one puts into it, and for how long time one tries. Further examination of the research results shows that gender influences the students' choice of challenging mathematics courses. These ideas are in coherence with the results from the PISA 2000 study (Centre for Educational Research and Innovation, 2003).

Meece discusses the studies by Fennema and Peterson (1985, as cited in Meece, 1996) in the nineties. They developed a model for learning behaviour to try to explain the gender difference in how the sexes succeeded in solving mathematical problems. They proclaim that in order to be able to solve complicated problems you need to act in a specific way. The students have to be able to get deeply involved in the problem, be able to work alone and show persistence, and concentrate. The model consists of four elements:

– Evaluation of the student's own mathematical abilities
– Sense of how useful mathematics is
– Learning abilities
– Sense that mathematics fits your gender-role

(Meece, p. 117)

These are the main components of the Fennema and Peterson model. It has been used in many research studies. The fourth element in the model seems to have little influence, though the results from the other parts of the model seem to be gender biased. Fennema and Peterson claim that the difference in learning style is due to the socialisation in the classroom. Reports show that it is very common that teachers support boys in problem-solving strategies. It is also found that girls rather than boys avoid risks and problem-solving and seek guidance. This has influence on how teachers encourage their students to take part in mathematical discussions and thinking. According to Fennema and Peterson, boys are more enterprising, more active in their study, and more likely to start discussions with their teacher about what they are studying, than girls.

The interviews on how it felt to be a mathematical learner were lively. The girls' expressions were strong and personal. Their vocabulary was greater and more varied when discussing this question. They became eager in the conversation and their thoughts went deeper. Helga said:

> I find it is rather easy to understand mathematics though I sometimes need some explanations. I think it is very important that the teacher is able to explain the same thing different ways. I feel irritated if I don't understand.

Helga felt that mathematics can give both positive and negative feelings. She felt happy when she succeeded in solving difficult problems but unhappy when she failed. From her point of view mathematical learning is about solving the problems that the teacher gives.

Lara finds it important to have peace, quiet and space when she is studying because then she finds it easier to think. She claimed she was good at understanding mathematics.

> Usually I find it rather easy to understand mathematics. It is very important to understand because otherwise you will get problems later. It is also a fact that if you understand it, there is no problem to remember the strategy.

Lara emphasized the importance of understanding. She said that understanding was a precondition for her to find it interesting and challenging to do mathematics. When she was asked about the text in the mathematics books she said:

> I often read the explanations in the mathematics books but I find the strategies and methods they describe complicated. I like it much better when people show me how I am supposed to solve the problems.

Sigrun said that she found it very important to understand the things with which she was working in her mathematics study. She did not feel comfortable if she finished a problem without understanding what she was doing. The mathematics was sometimes troubling her, not least the algebra. When the learning was going badly, she felt she was “a loser” but when she understood she felt like “a genius”. She connected these emotions mostly to dealing with new material. She said:

> ...always when you understand you get a good feeling, you become proud of yourself and you are confident that the rest will be easy.

Sigrun emphasized the importance of organising her study and found it easier to concentrate in peace and quiet.

> I find it most rewarding to work on projects that are challenging but not too difficult so that you can solve them after trying for a while. Then I have to use my brain and I try hard to solve problems like that.

Soffía did not find it hard to learn mathematics.

> I think I have mathematics in my genes. I understand everything, at least when I have given it a thought. It doesn’t take a long time for me to learn something.

Soffía thought that in the learning of mathematics analysing what you are looking for is most important. She did not find it useful to read the text in the mathematics books or listen to the explanations from the teacher. She believed in dealing with the problems. Soffía thus described successful learning:

> I find it most rewarding to work on problems where I really have to think. I prefer problems that take me up to 30–60 minutes to solve ... and I gain most if I find the solution without any help.

The main conclusions from the girls’ beliefs about themselves as mathematical learners were that these four girls felt very confident about their abilities to learn mathematics and to use them. Sigrun and Soffía expressed that it was most rewarding to struggle with the problem solving them by themselves, while Helga and Lara thought it was best to get at once an explanation if they found it difficult to understand. All the girls found it easy to organise their study and felt good when they were working on mathematical problems. All of them experienced joy when they understood things they had been working on. They all thought they were able to work independently in their study within the teacher's framework.

My main findings are that these girls express similar beliefs as girls in the overseas research-studies I used as a reference. Their beliefs about mathematics, the study of mathematics and themselves as learners of mathematics are in tandem with the conclusions of the PISA-study from 2000 (Centre for Educational Research and Innovation, 2003) and the Swedish GeMa-project (Brandell, et al., 2002).

DISCUSSION AND CONCLUSIONS

Qualitative research study is a way to give expressions of some individuals a space and a depth so that their views can be analysed from many angles. A qualitative research study based on interviews also gives the researcher opportunity to ask for the meaning of the words the individual expresses. In my research, I focused on getting a description of the girls' beliefs and not so much on how they had developed those beliefs or why. There is always a dilemma how to interpret research results and how you can use them to understand the rest of the world. In qualitative research, the main conclusion is saying something about the individuals involved, but, at the same time, being careful about using the results to draw conclusions about other people. The results can, however, often be used to understand how people think because a good insight and understanding of one individual can make it easier to understand how others think. I feel at least that my experience has made it easier for me to discuss beliefs on mathematics and mathematical learning with other teenagers and also other mathematical students. I also feel that my research has validated the idea that a research study in the Scandinavian countries and some other European countries can give some ideas of the situation in Iceland. I know that it is not possible to compare results from a qualitative research to results from quantitative research. Nevertheless, I have found it helpful to use results from a quantitative research in my analysis and to gain some knowledge in that field.

Conducting a research study is very rewarding for the researcher. I learned a lot while I was looking into these girls' minds and their ways of thinking. It would be interesting to make a follow-up study discussing those girls' beliefs at a later stage in their lives and also to use the experience from this study to interview several Icelandic teenagers.

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THE TEACHER AS RESEARCHER: TEACHING AND LEARNING ALGEBRA

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Together with a colleague I performed a study of which factors that facilitate or obstruct students' learning of algebra. This research is the context for my reflections in the paper. Thus I will briefly present the aims of the study. Starting with three examples from my research I describe and discuss how these, together with the theories connected with them, helped me to interpret and understand better what happened in the classroom and how theories and research could be used to improve my own teaching practice. I also try to show the relationship between the challenges of having the double role as a teacher and also an observer and the benefits of doing action research, both for the students and for the teacher/researcher. Finally, I discuss how the process of going through a research education has affected me as a person and a newcomer in the didactical field.

INTRODUCTION

Is an upper secondary school teacher who has been working as such for more than 20 years really in need of professional development such as in-service or supplementary training? His/her teaching qualifications include solid knowledge of mathematics as well as pre-service training in pedagogy and teaching methods. No major problems with teaching have occurred during the years, and the results in forms of students' performance in tests and their marks have been rather satisfactory. He/she has a genuine interest in working with younger people and really enjoys teaching. Would it not be sufficient with some occasional 'brush-up'-course or some single discussion days for example about changes in the syllabus for a course? Then the teacher's long experience will guarantee the rest.

Society has gone through many changes during the teacher's 20 years of service, and so has the school environment. New curricula have been introduced with a new system of criterion-referenced grading. The public view of teaching and of knowledge in general has changed, and it is a real challenge for an ambitious teacher to renew himself in order to cope with the ever-changing conditions for his work with the students.

The teacher's central role is discussed in a report from the Swedish National Board of Education (Skolverket, 2003, p. 34): 'The teacher is unanimously mentioned by

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Relating Practice and Research in Mathematics Education. Proceedings of Norma 05, Fourth Nordic Conference on Mathematics Education (pp. 427–439). Trondheim: Tapir Academic Press

the students as the most important factor for their desire to learn'(author's translation). But a solid pre-service training and long experience as a teacher does not suffice. Pettersson, Kjellström, and Björklund (2001) describe in another report the aim of a broader competence development. Their implications are that teachers should, from recent research and theories of knowledge, teaching and assessment in mathematics, both deepen and broaden their skills and knowledge in relation to the type of school within which they are working.

Seven years ago I was, to a great extent, in the above mentioned situation. I was steadily working as a teacher of mathematics and physics and was mainly satisfied with my job. Then something happened that changed my life. I was involved in a didactical project, led by professor Barbro Grevholm from Kristianstad University. The aim was to increase the participants' knowledge of the didactics of mathematics and to inform of research in the subject. It was also meant to stimulate contacts between different categories of teachers as well with the university and to make it possible for teachers to work with development projects at their own schools.

At the time, there had been some public debate in the media about the decreasing skills in algebra of the beginner students at the Swedish technological universities. I had discussed this with a colleague of mine, Tomas Wennström, who was also involved in the didactics project, and we both thought that it would be of great interest to investigate our students' algebraic knowledge.

The overall intention with this paper is to describe and discuss my own action research and my transition from being a teacher to also being a researcher. Firstly, the algebra project with its aims is very briefly introduced. Secondly, three examples from the study form the background for how theories and research results could be used to improve my own teaching practice. Thirdly, action research with its challenges and benefits will be discussed, especially with the teacher as researcher. Finally, my professional development and its consequences for me as a person conclude the paper.

The paper does not aim to give a research report of the algebra project itself, with a deeper description of theories, research questions, methods or results. It will, however, in the conclusion present some research based recommendations for teaching.

THE ALGEBRA PROJECT

The teaching and learning of algebra is a most difficult matter in school mathematics. Teachers struggle with a great variety of problems with students' understanding of algebraic expressions, equations and functions. These difficulties are often accompanied by problems of legitimacy. To motivate students for learning algebra, to promote interest and curiosity for it, and to create learning situations that enable students to both develop and succeed, are crucial characteristics of 'good' teaching. These goals are most complicated and hard to achieve and therefore necessary for every mathematics teacher to work with in their classroom practice.

With the aim of making a survey in which factors that facilitate or obstruct students' learning of algebra were investigated, Tomas and I initiated a longitudinal study of a group of students from the Natural Science Programme at an upper secondary school in 1998. In 2000 a supplementing group was included in the study for the possibility of making comparisons and also for an extension of the research questions. The investigations concluded in 2003, when the last group left school. Results of the study were presented in seven reports ('Algebraic knowledge in upper secondary school, I–VII', Persson & Wennström, 1999–2001; Persson, 2003).

Data was collected during a period of five years, using a variety of methods: tests, inquiries, interviews, essays, observations and results like marks for the different mathematics courses. It was analysed and combined in different ways in our reports, and some of the results will be presented below in the examples.

I completed the later part of the study alone, and in March 2005 it resulted in a licentiate thesis (Persson, 2005). In this, all seven reports are put into a more general frame of mathematics education research. In particular, a synthesis is made of the relevant research questions, of results and findings and of general conclusions of the study. Moreover, a discussion of teachers as researchers and suggestions for further research in the field are included.

The purpose was to show a comprehensive view of students' learning, but in order to accomplish this, the research questions had to be structured into smaller parts. A number of main factors that influence students' algebraic learning were identified:

- What *pre-knowledge* the students had, entering upper secondary school, especially in algebra.
- Which *concept* development they went through during the three years, both individually and as a group.
- The forms of *instruction* the students actually met.
- The *time for learning* mathematics in general and particularly algebra.
- What importance *affective factors* such as interest, attitudes and feelings, have in the learning process.

For each of these factors, our own teaching experience and the findings of the investigations were compared with existing theories and results in the didactical field. As examples can be mentioned Sfard and Linchevski's (1994) discussion of the conceptual difficulties with the reification of algebraic objects and Walberg's (2003) educational productivity factors, amongst which time plays an important role. Some of the methods used in the study were directly set up against the background of relevant theories and earlier research. Important questions here were: What was the effect of our findings on our own teaching? Did it indeed help us to better understand what happened and to really improve our work?

THREE EXAMPLES

It is not possible to present all the findings and conclusions of the study in detail in this paper. Nor could all important aspects of the changing process from teacher to researcher easily be described. Instead, I have chosen some exemplary episodes and single results from the study to illustrate what actually happened, how this affected me as a teacher and what consequences it had for my teaching practice.

Misconceptions of a sign rule

One part of the study was organized as a case study of ten students. Their background, pre-knowledge, attitudes, performance in tests and how they managed in the different mathematics courses was noted and discussed. One of the students, 'Student B', showed relatively good skills at the pre-tests and had a positive attitude towards algebra and mathematics in general. She managed most of the algebraic rules well, yet she got wrong answers to many of her tasks because of the same mistake over and over again. For example:

$$2a(5a-3b)-3a(3a+b)=10a^2-6ab-9a^2-3ab=a^2+3ab$$

$$4x^2-9+16x^2-1=20x^2+8$$

Apparently the mistakes depend on a misconception of a sign rule. There is some confusion between $-9-1$ and $(-9)\cdot(-1)$, which causes one term to be positive.

Student B, whose results in mathematics normally were good, was very frustrated and it took time before we together could come to the bottom of the problem. She was even irritated when we tried to pinpoint her misconception. In such a situation it is vital that the teacher do not leave the student until both understand the error. After a thorough discussion about negative numbers, student B had overcome her block, and then showed a rather rapid development of her algebraic knowledge and skills.

Episodes like this are well known by teachers, and are almost part of their everyday practice. I myself had encountered it on numerous occasions, but this time I also could relate it to relevant theories of learning. What could be observed here could be interpreted as a small leap in the learning process, which for example Vygotsky (1986) has described. Freudenthal (1978, p. 78) wrote: 'If learning process is to be observed, the moments that count are its discontinuities, the jumps in the learning process.'

What I learned from theory is that these thresholds in the development of concepts are normal and that they are individual. They often occur in a social interplay, and in what Vygotsky (1978) would call 'the proximal zone of development'. So here the theory of social constructivism got real, and I could consider it when planning my teaching. It is advantageous to organize the work in the classroom in such a way that it promotes discussion in this zone.

The misconception could be identified as an example of 'rote learning' (Novak, 1998) of the sign rule for multiplication of negative numbers, which made it possible for the student to use it in an inappropriate way. Rote learning occurs when the learner memorizes new information without relating to prior knowledge. It was also a basic error in understanding the different roles of the negative sign (Gallardo, 2001), which leads to the semiotic problems with the algebraic language (Drouhard & Teppo, 2004). These various ways of looking at the problem made me aware of important aspects of my instruction and of how I should and, indeed, how I should not organize my teaching.

Support time

It is not uncommon at upper secondary schools that some sort of differentiation, such as ability grouping (streaming), is used for certain subjects like mathematics. Pre-tests are used to place the students in the 'right' groups. There are advantages with such an organization, but there are also serious disadvantages (see for example Wallby, Carlsson, & Nyström, 2001). One is the 'locking-in'-effect, which happens when a student is placed in a 'lower-achieving' group. In this, mathematics teaching is adapted to a 'low-ability level' and at a slower pace. It will then be almost impossible for him/her to make considerable progress and rise to a 'higher level'. Based on our own professional experience and beliefs, we thought ability grouping was a totally wrong way to organize mathematics classes.

On the other hand, without any organized differentiation it used to be quite common at the Natural Science Programme that beginner students 'fell out' after rather a short time, because of problems with the mathematics courses. This was almost seen as normal, but for Tomas and me it was highly unsatisfactory. We discussed another construction, which we called the 'support time'. Those students who had problems with arithmetic, such as fractions or powers, or with basic algebra, got an extra hour of mathematical instruction each week during the first year. This time was compulsory for the students and it was organized in small groups (< 10) where they could get individual instruction from their usual teacher. Our intention was that the students would have sufficient support in their mathematics learning to 'catch up' with the regular course. They also had the opportunity to discuss mathematics in a less stressful situation.

Time is one of the most important factors for learning. This has been described in the report from the Swedish National Board of Education (Skolverket, 2003). Walberg (1988, 2003) has identified nine psychological factors for learning, amongst which time takes an important place.

> The positive effect of time is perhaps most consistent of all causes of learning. (2003, p. 7)
> …then it can be seen that time is a central and irreducible ingredient among the alterable factors in learning. (1988, p. 78)

But he also emphasizes that the quality of the time is essential:

> ”Productive time” is the time spent on suitable lessons adapted to the learner – in contrast to “engaged” or “allocated” time, which may be futile in the content or method of instruction, is inappropriate for individual students. (1988, p. 80)

For the individual student it is important that enough time is available for meaningful learning and then, of course, also of algebra. Some students need more time then others, and theory clearly supports our construction. It also showed Tomas and me the importance of this aspect in all our teaching. The ‘support time’ became a success at our school. During the first year, we actually had no single ‘drop-out’ because of problems with mathematics at the Natural Science Programme, and ‘support time’ is now a regular part of the organization of mathematics education at the school.

Affective factors

One of the other cases was ‘Student F’. His initial position in mathematics was extremely difficult. The results of the pre-tests were very poor, with great problems in arithmetic and number sense, which also affected his algebra tasks. He also had problems with variables and simplifications. He answered for example:

$$4x - x = 4 - x$$
$$a - 3a + 2a = a - 5a$$
$$10x + 3(4 - 3x) + 8 = 11x + 11$$

Equations were almost impossible for him, and functions totally unknown. In our study we compared our students’ conceptions of variables with the hierarchic levels, suggested by Quinlan (1992), and with Küchemanns’s (1981) categories. Student F placed himself in a very limited way into these. Questions in the pre-test, aimed to test deeper conception of algebra, were mainly left unanswered by him. His attitudes towards mathematics were also not favourable. In an inquiry he wrote about mathematics:

> Rather difficult and not especially fun. (All citing of the student is author’s translations)

He said about algebra:

> I really see no usefulness of it whatsoever. I have never encountered it in normal life and I will probably never do it either.

Maybe student F was a 'hopeless case' and should have been advised to rethink his choice of a programme with so much mathematics in it? In our first aims for the study, Tomas and I wanted to find a lowest possible level of pre-knowledge for succeeding in mathematics ('success' was defined as at least passing the four compulsory courses Mathematics A-D of the programme). Surely, this student must come below such a level?

Of course, student F had to attend the 'support time', where he could discuss basic mathematics and algebra with the teacher and other students. Crucial was that he had a genuine will to pass the mathematics courses, so he really struggled hard and put much time and effort in doing exercises. An additional important factor is that besides my efforts as a teacher, he had also had a good peer support. As a consequence of my understanding of social constructivism, I had more and more tried to organize the classroom work so that it would be possible (and often compulsory) to cooperate in different constellations, two-and-two or group wise. For student F this was indeed positive. He started to succeed with mathematics in various ways, and this changed his attitudes considerably. After a while he wrote the following in an essay about algebra:

> I think I have learned rather much since we wrote the last time. Before it seemed almost impossible with some tasks that now only are the beginning of more difficult ones.

After passing the three first courses he wrote about algebra:

> Useful, I can manage it better now, but it is still difficult. Algebra is getting more and more clear to me. It is even beginning to be fun to work with.

He finally passed the fourth course, which was our definition of success. With hard work and a strong belief from both himself and me as the teacher that he would succeed, together with a good social climate in the classroom and much cooperation, it was after all possible. This showed that our aim to establish a lowest level of pre-knowledge was basically wrong, and we had to reconsider our earlier beliefs. In fact, it has since then been one of the very grounds for my teaching that practically everybody can learn a significant amount of mathematics, given the right conditions. This fits well with Vygotsky's view on learning. Both cognitive and affective factors must be considered in order to achieve good teaching results. Novak (1998, p. 24) writes:

> Feelings, or what psychologists call affect, are always a concomitant to any learning experience and can enhance or impair learning.

Interest and motivation, attitudes, self-reliance and feeling of success and the social climate in the classroom are much more important than I had believed. This new knowledge changed my views on teaching considerably, and made me understand in a much deeper way what happens in a classroom. Furthermore, it has given me tools to improve my work in many aspects.

THE TEACHER AS RESEARCHER

The study started, as mentioned above, as a teachers' development project. Our perspective was that of the practitioner and our main concern was how to improve our students' learning of algebra and our own teaching practice. At that time we had no real thoughts about calling it 'research' and, at least in the beginning of the project, we also could not meet the essential criteria for it to be defined as such. Hart (1998, p. 411) presents a list of minimum criteria for when a 'disciplined inquiry' can be said to be 'research':

1. There is a problem.
2. There is evidence/data.
3. The work can be replicated.
4. The work is reported.
5. There is a theory.

Our project fulfilled some of these demands at that time, but not all of them, for example a coherent theoretical background. However, our perspective was broadened beyond our own classrooms after further research and consultation of the relevant literature. Together with the fact that we had established good contacts with and support from the university, above all professor Grevholm, the study could be raised to a higher standard. Our reports were published and our own competence sufficient for the methods we used and the theories we based our study on. The type of research was 'action research'. Crawford and Adler (1996, p. 1194) describes it as:

> …investigation and inquiry processes undertaken with an intent to change professional practice or social institutions through the active and transformative participation of those working within a particular setting in the research processes.

A traditional form of action research is when a researcher from a university cooperates with teachers in the field. This often proves successful, but sometimes there are problems, for example if there is too much imbalance in the relations between university and schools. But there is a solution to this, described by Grevholm (2001). The gap between theory and practice can be bridged over if the teacher, the practitioner himself, becomes a researcher and the one who asks the questions and performs the investigations, which have relevance for his practice. However, Grevholm strongly emphasises that the criteria above must be fulfilled, and also that the results must be critically reviewed and discussed.

There are a number of benefits for the teacher as researcher. Boero, Dapueto, and Parenti (1996, p. 1112) have described some of them:

> …teachers overcome the individualistic, 'isolationist' idea of their profession and learn to cooperate with one another. They learn to use both research tools and results to plan, observe and evaluate their classroom work. They learn to take some distance from their classroom experience and to profit from other people's experience.

Through guided action research the teacher also learns to interpret and understand what happens in the classroom. Crawford and Adler (1996, p. 1201) write about the teachers:

> Only through active engagement with problems and questions that are personally meaningful to them will they develop a rationale for action. Only through understanding their own learning through research, inquiry, investigation, and analysis will they come to understand such processes among students in their care.

They also compare with traditional forms of teacher education in pre-service or in-service courses. The difference is that learning through research results in knowledge that is actionable. It can be used as a basis for professional action, which it indeed was in our own research. Much of our findings and experiences resulted in changes of my practice and of my views on knowledge and learning. But they also shed a light on existing theories and showed their strengths and weaknesses. My opinion of some theories significantly changed during the research process. In some cases they did not fit very well with our own findings, and in some cases new research either reinforced or questioned them.

Are there then no problems with being a teacher and a researcher at the same time? Hatch and Shiu (1998) discuss some of the prerequisites for quality in the research you must observe. One obvious problem is objectivity. When you are in the middle of a teaching situation everything you observe is coloured by your role as a teacher. You must 'take a step back' to be able to watch the processes with unaffected eyes. Hatch and Shiu call this 'distancing'. It is necessary to document in different ways what happens with notes, portfolios, recorded interviews or videos. Then it is possible to handle the material with more objective methods.

In some situations there might be a conflict between the role as a teacher and that of a researcher. If a student has great difficulty with a problem, which he/she cannot solve, should you then leave him/her to struggle with it in order to observe what happens (as a researcher), or should you interfere with an explaining discussion (as a teacher)? My personal experience and opinion is rather clear. In this situation you give priority to the acute needs of the student, of course, and put your role as a researcher a bit on the side. It is however vital that you are observant of what happened if you later want to analyse this particular process of problem solving.

The tests, interviews and other parts of the research take time, of course for the teacher but also for the students. Sometimes it can even seem to be a little too much for them. It is necessary to discuss this with the students in order to motivate them for participating in the research. There are however important benefits for all involved with focussing on the improvement of teaching and learning on a metacognitive level. Both teacher and students become aware of the factors that provide success in the learning process, as well as those which are serious obstacles. This leads to more effective learning situations in general and also to a more relaxed relation between teacher and students. The paradoxical thing, that might happen, is that you as a researcher-teacher actually feel that you have more time available for your students instead of less.

CONCLUSION AND SUMMARY

The study covers a period of six years and a great amount of material was collected. From this material, the five main factors that influence students' learning of algebra were extracted, along with more general guidelines for teaching mathematics. The results and conclusions of the research will not be presented here, but the interested reader will find them in my licentiate thesis, 'Bokstavliga svårigheter – faktorer som påverkar gymnasieelevers algebralärande' ('Difficulties with Letters – Factors influencing the teaching and learning of algebra for upper secondary students') (Persson, 2005).

The major part of the research findings focus on the close relations between teacher and students. They originate from the teacher's questions about how to improve his own teaching, but the conclusions could not have been possible to draw without a solid background in theories and research methods. This background can be achieved if the teacher is given the opportunity and the time to go through a proper training as a researcher. Both universities and schools must in such cases take responsibility for this important professional development. The substantial benefits for all involved, schools, teachers, researchers and, of course the most important, for the students, should be made evident for all involved. With the teacher also being a researcher, the gap between research and practice in mathematics might be bridged.

The research findings, together with my own long experience as a teacher, were also used in my thesis to give some important recommendations for teaching algebra (and mathematics in general). These implications were, briefly:

- Start from what the students know.
- Count on and with different meanings of letters and expressions.
- Work with algebra from several perspectives.

- Let the students cooperate.
- Learning must be allowed to take time.
- Believe in the students' possibilities.
- Cooperate with mathematics teachers across the boundaries.

My intentions with these recommendations are to point at some crucial issues for both pre-service and in-service teachers to consider in their professional work. I seriously emphasize that all mathematics teachers must be aware of them and form their own, well-grounded opinion about how they apply to their classroom practices. With this, my original purpose to improve my own teaching has transformed into a general interest in how mathematics teaching and learning can be improved for all. This had important personal consequences for me and, in fact, finally resulted in a change of profession.

In the didactical project, led by professor Grevholm, I was privileged to, for the first time in my professional life, be able to discuss mathematics teaching and learning with teachers from all school levels. It was an amazing experience and I soon found that this type of contact is most important if we want to develop and improve mathematics learning for both the individual student and for the group. Much of the problems which my students at the upper secondary level had could be explained in the light of their former experience of mathematics. At the same time, it was valuable for the teachers from the primary and lower secondary levels to be able to see the whole 'picture', what mathematics a student will meet through the whole school system and, for example, why a specific concept is so important and what consequences it will have if it is not properly trained and understood.

Based on my belief in the importance of cooperation across the borders, I took the initiative to and led a project in my own commune, involving teachers from all levels of school from pre-school up to upper secondary level. The aim was to create a forum for discussion of mathematics teaching and for improving it in a variety of ways. One specific aim was also to make professional development in the mathematical field possible for all participants. The results of the project have been presented in different ways (see for example Persson, 2004).

My experiences from this developmental project, together with my new knowledge of theories and of the results of my research, then led one step further. I started to consider the possibility that I could work with this full time. When an opportunity opened for this in 2003, I applied for and got an employment as a teacher educator at Malmö University. So, in my present work I am now able to use and relate to the collected experience from my 26 years as a mathematics teacher, as well as to the research I have done on the teaching and learning of mathematics. It is my true hope and belief that my student teachers really benefit from this, and that they can use much of my results in their future teaching.

Finally, I will once more underline how much fun it has been in my research to work together with all interested and competent colleagues, with all positive and capable students and with all new acquaintances, researchers and others, being a newcomer in the field of mathematics didactics. It has been a fantastic experience!

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MATHEMATICS LEARNING WITHOUT UNDERSTANDING – COGNITIVE AND AFFECTIVE BACKGROUND AND CONSEQUENCES FOR MATHEMATICS EDUCATION

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Mathematics learning without understanding and learning by rote is a widespread phenomenon. But learning by rote contradicts the view that learning – and especially mathematics learning – necessitates understanding. The present paper tries to illuminate this phenomenon from the cognitive as well as the affective viewpoints. Recent results from neuroscience are used to understand the process of such learning. These results suggest that this kind of learning is based more on the form of the description of a problem than on its content; and therefore that this "knowledge" is very inflexible and not transferable to other problems. Consequently – if acquiring competencies that are useful outside school, too, is the goal of mathematics learning at school – efforts in the classroom towards developing concepts and using demanding problems ought to be intensified.

PRELIMINARY REMARKS

Judging by comments made with regards to my abstract, there appears to be a need to clarify the position adopted by the present paper. (I thank the referees who drew my attention to this.) This paper is concerned with theory, and may be classed as *basic research* according to the mathematics education research (MER) categorization scheme – MER consists of basic research, strategic research and applied or developmental research – introduced by Margret Brown in CERME4 and reported by Ole Björkquist (2005).

> The first category, basic research, is concerned with the development of theoretical knowledge. It is potentially influential on practice, but not immediately. In mathematics education research the outcomes of such research include some impressive theoretical structures (theories of conceptual fields, mathematical objects and processes, affect in mathematics education, etc.) as well as single, not necessarily minor, contributions to the theory by individual researchers. (p. 1)

2007. In C. Bergsten, B. Grevholm, H. S. Måsøval, & F. Rønning (Eds.),
Relating Practice and Research in Mathematics Education. Proceedings of Norma 05, Fourth Nordic Conference on Mathematics Education (pp. 441–453). Trondheim: Tapir Academic Press

Research into mathematics education has produced many interesting empirical papers in the last decade. Acceptance of qualitative empirical research methods in particular has led to the consideration of a large number of case studies. However, while all research fields require empirical justification and empirical identification of phenomena, a theoretical basis for the phenomena is also necessary. Science needs theoretical concepts in order to discern order in results, to understand what one is looking for in empirical research, and to interpret empirical results, as well as to develop further the theoretical basis. All hypotheses in empirical studies, as well as the interpretations of these studies, are pre-empted by theoretical background concepts. Thus, in the last few years, there has been a trend towards clarifying theoretical concepts (see Leder, Pehkonen, & Törner, 2002; Di Martino & Zan, 2001). Even if these research directions do not lead to a unified concept, such clarification processes are very important for the development of a research field. Particularly in research fields such as mathematics education research that straddle numerous scientific areas, research problems tend to be highly complex and conclusions from several scientific areas can enrich the research. Uncovering the relevance of concepts from other scientific areas is therefore important to extending the theoretical basis of mathematics education research.

All of our research questions are connected with the learning and teaching of mathematics. Therefore mathematics must be part of the core of our research. Questions such as the following are important: What is mathematics? What does doing mathematics mean? What is the function of mathematics in our society? Semiotics has important applicability here, because signs are central to the doing of mathematics, and consequently, to mathematics learning (Hoffmann, Lenhard, & Seeger, 2005).

It is not only in mathematics education that learning is an object of research: cognitive processes and learning are studied in psychology, pedagogical psychology and sociology, too. Therefore, concepts from these sciences should be applicable to MER. Furthermore, learning is not just a cognitive process; the learning process is embedded in an affective background comprised of moods, motivation, anxiety, joy and so on. Therefore MER considers the learning process in a more complex way, as a process influenced by intertwined cognitive and affective conditions. While these two factors are studied in psychology, the last two decades have also seen important relevant results from neuroscience. New research methods have helped neuroscientists study cognitive and affective processes within the brain. The neuroscience of memory, in particular, can help us understand our research questions better.

The goal of this paper is to develop theoretical knowledge of a special problem. My starting point is the fact that not all of our students understand mathematics sufficiently well to be able to solve mathematical tasks. The empirical basis of this problem consists of my own experience as a mathematics teacher; discussions with other teachers; interviews with adults (see section 2); and the results of studies (see section 2) which reinforce the view that non-understanding of mathematical concepts starts in an early stage of mathematics learning. The aim of this paper is to gain a better understanding of the following questions: Why do learners fail to understand mathematical concepts? What is the impact of non-understanding on learners' further learning strategies? What can they do to pass exams? What are the reactions of teachers to this situation?

To obtain (mostly preliminary) answers to these questions, I use interviews with adults that serve to elucidate learners' belief of their non-understanding of mathematics, and which reinforce the view that the origin of this belief can arise in a very early stage of school learning. To discuss non-understanding, I have to discuss understanding first, and only after that I consider the consequences of non-understanding. To do this I use theoretical concepts from a number of fields. These concepts should help us in two ways: On the one hand, they should provide insights into the situation; on the other hand, they should extend the theoretical basis of concepts already used in mathematics education research.

DESCRIPTION OF THE PROBLEM

Introduction

Mathematics teachers (at least in Austria) often say that weak learners are unable to understand mathematics. In a previous study, when we asked adults about mathematics learning at school, many of them reported that they always received bad marks in mathematics and were unable to understand mathematics at all. The following quotations underscore the situation. They are taken from interviews with adults who were changing the direction of their career and were compelled by circumstances to learn mathematics; interviewees were asked to talk about their experience with mathematics learning in school.

> Regina had problems with mathematics right from her first year in primary school. The problems were exacerbated during compulsory secondary schooling. She sees the main cause of her problems with mathematics as being difficulties "with logical thinking". These appeared, for instance, when solving mathematical exercises formulated in words, which had "a lot of numbers and you don't know how and what you should calculate anymore." (Jungwirth, Maasz, & Schlöglmann, 1995, Part E, p. 30, author's translation)

> Mathematics was always Richard's weak point. Once he had to repeat a year because of mathematics. He particularly disliked algebra and equations, "which did not interest me, and which I have never understood." (Jungwirth, Maasz, & Schlöglmann, 1995, Part E, p. 32, author's translation)

Teachers and learners often agreed that the only way to learn mathematics and pass tests was to rote learn rules and algorithms, and to practice applying these to mathematical exercises.

> One adult (Mr. P) replied as follows to the question of why he didn't like mathematics in school: "Well, it is because I never found the right approach. I actually did well in mathematics at school but I never really learned anything. I rote learned everything…" (Stroop, 1998, p. 173, author's translation)

Rote learning is not an uncommon strategy when teaching weak learners. For example, the Kumon school, which is active worldwide, uses this learning concept to train students (Der Spiegel, 48/2003, p. 74). Malle (1993) speaks of an "ideology of stereotype exercise". Teachers complain that the knowledge acquired in such a learning process is very inflexible. That means, for instance, that when just the signs of the variables are changed, the learner cannot solve the exercise anymore.

More on the problem

To elucidate the view that non-understanding can originate in an early stage of school learning, I refer to a study by Stern (1997, 1998) of German first grade students. In the following, the percentage in brackets refers to correct answers.

1. Maria has 3 marbles. Then Hans gives her 5 marbles.
   How many marbles does Maria have now? (89 %)
2. Maria has 9 marbles. She has 4 marbles more than Hans.
   How many marbles does Hans have? (16 %)
3. Maria has 4 marbles. She has 3 marbles less than Hans.
   How many marbles does Hans have? (22 %)
4. Maria has 3 marbles. Hans has 4 marbles more than Maria.
   How many marbles does Hans have? (53 %)
5. Maria has 5 marbles. Hans has 3 marbles less than Maria.
   How many marbles does Hans have? (58 %)

(Stern, 1997, p. 407; Stern, 1998, p. 89, author's translation)

Thinking briefly about each of the above tasks, we can see that in each case, the model necessary to solve the problem is different. One can model Task 1 in an operative way. The child can imagine the process of giving a marble and can see in his or her mind how many marbles are left. For Task 2, however, it is necessary to consider a relationship between Maria's set of marbles and the unknown set of marbles belonging to Hans. Stern (1997) argues that this task needs an abstract problem model (a part-whole scheme): the student must first decide what the whole is, and what the part is; furthermore, the word "more" does not refer to the operation of addition, but implies a relation between two sets. The concept of relation is also required in Task 3. Both tasks necessitate a new understanding of numbers. Numbers are not only elements for calculations, they have attributes. If a student does not understand the concept of the ordering of the natural numbers, he or she cannot interpret the task correctly. In Tasks 4 and 5, students can succeed by working with "key-word strategies" (Schoenfeld, 1985; Stern, 1997) and interpreting "more" to mean addition and "less" to mean subtraction.

Briefly, the problem is not that the students were unable to read a text in a linguistically correct manner; while they could calculate some tasks correctly, their problem is that they did not have a proper understanding of the mathematical meaning of the text. That means that their problem is a mathematical problem (Stern, 1997): mathematisation of a problem means seeing it from a mathematical point of view, interpreting a situation in terms of mathematical concepts. In some of the above tasks, the operations of addition and subtraction defined an order on the set of natural numbers, and could not simply be interpreted as increasing or decreasing the cardinality of a set. But if students do not possess this concept, how can they understand explanations that implicitly use this concept – and how should they solve mathematical exercises? To obtain more insight into this phenomenon, we need to consider the following question: What does learning without understanding mean from the cognitive and from the affective perspectives?

REMARKS ON UNDERSTANDING MATHEMATICS

A crucial point in problem solving is that students must possess knowledge of concepts and terms that are necessary to model the situation described in the task. That means they must properly understand the problem from the perspective of mathematics. Understanding mathematics and the construction of meaning are the central problems of mathematics learning and have therefore been the subject of intense study in mathematics education research (see e.g. Sierpinska, 1994). On the one hand, understanding is an individual student's problem; however, it has a social component, too. Maier (Maier & Steinbring, 1998) gives the following characterization of understanding with regard to mathematical knowledge:

> Understanding is an initiated and intentional mental process, initiated mostly by linguistic expressions.
>
> Understanding is an individual process of construction of meaning and knowledge, in which the person who has an understanding of the work constructs knowledge in an autonomous and individual manner.
>
> Understanding is a mental process in which the cognitive process is inseparably linked to affect as well as to non-cognitive personality factors.
>
> Understanding is a sequence of processes that always use already understood knowledge, and that, on this basis, produce new and changed knowledge.

> Understanding is a process in that we have to divorce description of the process from the result of the process. (p. 295–297, author's translation)

The complexity of this characterization of understanding shows that understanding cannot be a by-product, especially since the aim of the mathematics learning process is not just any kind of understanding, but a specific type of understanding that is suitable for problem solving. Therefore, organization of the learning process and interaction in the classroom are central elements that must be considered if the individual and autonomous learning process is to lead to the result strived for. We have to keep in mind, however, that the offer of meaning in the classroom is used by different learners in different ways, and this can lead to at least two different forms of understanding: a *reductive* understanding that uses the offer of meaning in some qualitatively or quantitatively curtailed or reduced manner; and an *abductive* understanding that uses the offer of meaning in its full sense, and giving it a subjective colour by adding elements from the individual's own knowledge (Maier & Steinbring, 1998).

While understanding is the result of a learner's autonomous process, this process is in most cases interactive, and is stimulated by teacher activities as well as comments from other learners in the class. Nevertheless, we ought to discuss this autonomous individual process of knowledge construction. For Dörfler (1988), the mathematical objects required in the learning process are comprised of subjective knowledge units that emerge through certain actions. To construct mathematical objects it is necessary to have a cognitive distance between action and mental operation (Dörfler, 1986) in order to make an abstraction process possible. Of importance to the development of this process are not only concrete actions but also their description through so-called protocols (Dörfler, 2002a). This description of one's concrete actions (real or concretely imagined) by means of "verbal, iconic, diagrammatic or symbolic algebraic *inscriptions* (which form the backbone of the protocol), encompasses and represents the essential assumptions, conditions, phases, segments, consequences and results of the respective actions" (Dörfler, 2002a, p. 90). The description acquires its meaning in the context of the material or social situation. On the one hand, by acting with symbols and diagrams, the learning process instigates the development of the symbolic-diagrammatic level; on the other hand, by perceiving the inscriptions in a certain manner, the learning process instigates the development of abstract elements.

The emergence of an abstract and general form of symbolic expression is also important for the learning process. Through diagrammatic representation, the abstract-general form of expression is objectified, and this can become a new starting point for actions (Dörfler, 2002a, p. 103). One central problem in mathematics learning is that properties of mathematical objects such as numbers, functions, spaces etc. are often left implicit, embedded in the linguistic form, and we have no reference objects for them. To overcome this problem, we use elements from semiotics such as inscriptions and diagrams. Perhaps most importantly, these facilitate learning because they generate finite prototypes (Dörfler, 2002b).

To grasp the important relationship between object/reference context and sign/symbol in the development of mathematical concepts, Steinbring (2005) uses the "epistemological triangle":

> The epistemological triangle represents a theoretical instrument for tackling the problem that while mathematical knowledge requires signs and symbols, these signs and symbols in themselves are not mathematical knowledge. Mathematical knowledge cannot be reduced to signs and symbols. The connection between the signs that code the knowledge and the reference contexts that establish the meaning of this knowledge can be represented in the epistemological triangle. (p. 93)

In the interplay between sign-symbol systems and reference contexts, the learner actively constructs mathematical concepts. "In the progressive development of knowledge will modified respectively generalized correspondingly the interpretations of sign systems and the corresponding reference context" (Steinbring, 2000, p. 34).

This brief discussion of the development of mathematical terms and concepts gives us an idea of the complexity of the process. We know from many investigations (see e.g. Stern, 1997, 1998, 2001) that many learners do not achieve the kind of concept development required to solve mathematical problems. Next, we discuss what the consequences are if learners do not have at their disposal an adequate grasp of the concepts required for solving problems.

NOT UNDERSTANDING – THE AFFECTIVE SIDE OF THE PROBLEM

We ought to consider the situation of a student who cannot solve a mathematical task. Each student has mental concepts that he or she uses to solve tasks. These concepts are consequences of prior learning processes and were often successful in earlier problem-solving tasks. In Stern's study of German first grade students the percentage of correct answers are quite different depending on the type of problem. Such a situation – in which conceptualizations that work in one context don't work in another, even though the verbal formulation is almost exactly the same – can be very confusing for a child trying to learn. This is particularly so if the child uses recipes taught in the mathematics classroom, such as "'more' or 'together' mean 'use addition', and 'less' means 'use subtraction'". Many teachers use such keyword recipes in the classroom (Schoenfeld, 1982; Stern, 1997) and students invoke them to solve exercises. When students see that their strategies work successfully in some situations but not in all situations, it is a confusing state of affairs and they want an explanation. Teachers should show them that they have to extend their conceptual knowledge. In mathematics education, there has been extensive discussion of the problem of "epistemological obstacles", which refers to the fact that in the concept-building process, the ideas and the power of imagination acquired at an earlier stage of the process can become an obstacle to further progress (Sierpinska, 1994).

If such a confusing situation did not result in an adequate concept-building process, from then on the learner will have concentrated more and more on characteristics of exercises that they were able to do before. These characteristics are mostly superficial; and the problem-solving strategy now is just to use key-words and the algorithm they learned before, inserting all the numbers given in the exercise. This strategy is exacerbated by performance-oriented classroom technique favoured by many teachers (Stern & Straub, 2000)

If we think that it is important for learners to understand teachers' explanations, then we ought also to acknowledge that it is necessary for the learner to have the right level of conceptual development. Students who do not have this conceptual sophistication cannot understand explanations. A learner's cognitive analysis of this situation can be the starting point of negative affects (McLeod, 1992), and can also initiate the belief that he or she cannot understand mathematics.

Another kind of situation students find themselves in is being able to neither understand explanations nor solve problems, resulting in the need to handle unpleasant experiences. To describe the handling of emotional situations, Goldin (2002) introduced the concept of "meta-affect". In the mathematics classroom, meta-affect has the important function of enabling the handling of unpleasant situations. Many students do not learn mathematics well. Many learners find that frequent negative learning experiences are difficult to cope with. This can lead to a loss of self-confidence or sense of self-worth. The main reason for the development of affect in evolution was to help humans cope with pain. Humans are therefore able to develop concepts of self that help them hold onto their sense of self-worth. The strategies that can be used also depend on cultural factors. For example, in German speaking countries it is possible to say, "I'm unable to learn mathematics", without loss of prestige. Many adults use this strategy and avoid nearly all use of mathematics in everyday life (Goldin, 2002; Schlöglmann, 2000). Students at school are unable to avoid mathematics but they often have other strategies for handling unpleasant situations:

A very important controlling authority for all verbal statements is the 'feeling of one's own worth'. We therefore try to give our emotional remembrances a meaning which does not destroy the feeling of one's own worth. For example, in interviews people have a tendency to trivialize their own weaknesses. This trivialization can, for instance, take place through the expression of opinions that particular mathematical content is unimportant if the person is unable to cope with these kinds of problems. (Schlöglmann, 2002).

NOT UNDERSTANDING – THE COGNITIVE SIDE OF THE PROBLEM

We have discussed strategies, used by learners in the mathematics classroom for handling unpleasant situations; these involve the learner's concept of self. But learners also have to cope with the situation that although they feel unable to understand mathematics, they nevertheless have to pass tests in mathematics. If you speak to teachers about learning in the classroom, they often describe the situation in the following way: "I have students who are only able to handle routine problems. They are unable to understand mathematics. Therefore I train them to solve routine problems. But their knowledge is so inflexible. If I change the formulation of a problem, they are unable to solve it."

This description leads to the following question: Is learning without understanding possible? And if so, what are the characteristics of this "knowledge"?

To discuss these questions, we must take a closer look at memories. In recent decades, neuroscientific research has made great advances in understanding the various memory systems, particularly the long-term memory. Most researchers today use the concept of "multiple memory systems", according to which two or more systems exist that are "characterized by fundamentally different rules of operation" (Sherry & Schacter, 1987, p. 440). Two basic types of memory are used: 1) the explicit, or declarative, memory; and 2) the implicit, or non-declarative – or processual – memory. The former is concomitant with consciousness and linguistic reportability, whereas this is not the case with the latter (Roth, 2001, p. 152). The explicit, or declarative, memory contains the episodic memory (stores our personal facts) and the semantic memory (stores our knowledge). In the implicit (nondeclarative or processual) memory, skills and habits are distinguished, and this type of memory forms the basis of priming, categorical learning, classical conditioning and non-associative learning (Roth, 2001).

The explicit memory, particularly the semantic memory, is always necessary for understanding. This leads to the question, "Is learning without using the semantic memory possible?" Neuroscientific research has shown that no learning at the semantic level is possible by people with impaired explicit memory (Schacter, 1999) but it is possible for them to learn with the nondeclarative memory. To explain this, Tulving and Schacter (1990) postulated a perceptual representation system (PRS) "which exists separately from, but interacts closely with, other memory systems" (p. 302). Implicit learning, such as priming, is based on the PRS:

> Priming appears to be a presemantic phenomenon, in the sense that (a) it occurs whether or not one performs semantic encoding operations; and, (b) it is quite sensitive to target information. Explicit memory, on the other hand, is generally dependent on, and greatly enhanced by, semantic encoding operations and is less sensitive to changes in perceptual properties of target information. (Schacter, 1990, p. 545)

I should emphasize that sensitivity to perceptual properties of target information is specific to the perceptual representation system: "…access to information in PRS is hyperspecific, probably because, unlike other cognitive memory systems, it contains no abstract focal traces" (Tulving & Schacter, 1990, p. 302).

In a nutshell, the perceptual representation system allows us to identify everyday objects as well as well-known words on printed pages (and a PRS exists for each of the other senses, too). Although the PRS is specialized to process the form and structure of words and objects, it "knows" nothing about the meaning of the words and the use of the objects. The PRS and semantic memory systems usually collaborate closely, but in cases of brain damage, the semantic memory can be disturbed whereas the PRS works relatively correctly. Consider the meta-affective situation of a student who is convinced that they cannot understand mathematics. This belief is the consequence of many negative experiences in mathematics learning processes. Besides having to find a strategy to handle this situation without lost of self-worth, this student also has to manage classroom situations and tests. In many situations, the student can repeat word for word what the teacher said or wrote, even though they haven't understood it. In the light of the functions of the perceptual representation system, we may suppose that this system is utilized in this process. Furthermore, we may assume that students also use the perceptual representation system for handling mathematical tasks. But we know from the characteristics of the PRS that the retrieval process is very strongly dependant on perceptual properties. We can therefore understand why changing the letters used to represent variables, or altering the word order of word problems, leads to inability to solve the "new" task: it is a consequence of the hyperspecifity of PRS.

CONSEQUENCES FOR MATHEMATICS EDUCATION

Considering the classroom situation, many teachers know the problems their students have. They try to help them by training them to solve routine problems with nearly no changes in the way the problems are represented. Students get the feeling that if they look at the form of the task and of the steps of the solution process, they can succeed at least in routine situations. This success leads to a strengthening of the meta-affect and beliefs. In mathematics courses for adults, learners often demand that teachers train them to solve routine tasks and not explain the mathematical background. It is like a feedback loop out of control: this method of teaching and learning leads to a certain small measure of success in special situations; this success strengthens the meta-affect; and the meta-affect leads to demands for this teaching method (and consequently to the propagation of this learning method). The main problem here is that this knowledge is unhelpful outside school – it only helps the student to pass tests (Schlöglmann, 2006).

If we consider the process that leads to both the strategy described above and the student's believing that he or she cannot understand mathematics, we find that the crucial points are the stages of conceptual development. This shows that these stages are central to the learning process, in the present as well as in the future. Therefore conceptual development ought to be carried out very carefully. Teachers ought to be aware that their students possess concepts that can act as epistemological obstacles that hinder the learning process. Therefore teachers ought to carefully evaluate their students' concept images. Furthermore, it is necessary to develop concepts properly before training the learners to use algorithms. (Steinbring (1989) mentions a common strategy amongst primary school teachers in which pupils are trained in calculation methods before conceptual development occurs. This can lead to the consequence that students acquire the belief that only algorithms are important, because these – and not concepts – are necessary to solve tasks.)

Both for enriching the learning process and for identifying the stage of conceptual development that pupils are at, it is important that students be given the chance to apply the concepts they know to demanding problems. Only by solving demanding problems can learners recognize that concepts are necessary. In a number of studies Stern (1998) has called attention to the observation that weak learners, too, can profit from trying demanding problems, and that teachers are only able to recognize inadequate mathematical conceptualisations amongst students if no training-acquired affects mask the students' difficulties.

A very crucial problem is to change students' beliefs because beliefs are very stable (Leder, Pehkonen, & Törner, 2002). To change the belief of non-understanding it is necessary to open students' mind to the importance of their own thought in mathematics learning. This can only be done by using new, didactically rich tasks that give students the feeling that they are able to understand mathematics and that mathematics requires not only learning formulas and algorithms but also independent thought. To do this it is helpful not to use typical school tasks. Real life tasks with a real life situation as a starting point may be suitable to open students' minds to a new type of learning because the usual classroom situations activate the belief of non-understanding and all its associated counterproductive learning and thinking strategies.

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RESEARCHING POTENTIALS FOR CHANGE: THE CASE OF THE KAPPABEL COMPETITION

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This paper reports on the design and methodology of an ongoing study in belief research. The study addresses the question of the potential and perceived influence of the KappAbel competition on the mathematical attitudes and practices of the participating teachers and students. We shall outline our main methodological considerations and also discuss, and to some extent challenge, what we consider dominant approaches to research on teachers' beliefs. We address three interconnected methodological difficulties: (1) the problem of not using conceptual frameworks that are well grounded empirically; (2) the question of overemphasising teachers' views of mathematics for their educational decision making; (3) the problem that no terminology carries unequivocal meanings, and more than an indication of agreement or disagreement with the rhetoric of reform is needed to outline teachers' and students' school mathematical priorities.

INTRODUCTION

In this paper we shall discuss the main methodological problems of a study that we are currently conducting. The intention of the study is to develop an understanding of if and how the KappAbel mathematics competition (www.kappabel.com) has an influence on the beliefs and attitudes and the teaching/learning practices of the participating teachers and students. This is a question that is closely related to the more general one of the role of external sources of influence on teaching/learning processes: What may initiate and sustain change in mathematics classroom practices and in students' and teachers' beliefs and what – if any – is the relationship between the two sets of changes? As the point of departure in this study, we claim that people's relationship with mathematics is developed in different communities of practice (family, school, working life, etc.). However, the practice of the mathematics classroom is the most significant of these communities (see Wedege, 1999).

It is not our intention to discuss any substantive findings of the study, as it is still ongoing. Instead we shall outline our methodological approach and in particular elaborate on our attempt to address what we consider three significant problems of main stream belief research. These are the problems of the lack of empirical grounding of the involved constructs of mathematics; the one of a possible over-

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emphasis on mathematics in most research on teachers' beliefs; and the one of ambiguities of the terminology used. In addition to discussing how we have dealt with these problems, we shall discuss some general difficulties related to surveys and qualitative interviews. However, we shall start off with a short presentation of the KappAbel competition.

THE KAPPABEL COMPETITION

KappAbel is a Nordic mathematics competition for pupils in lower secondary school. The overall aims of the competition are (1) to influence the students' beliefs and attitudes towards mathematics and (2) to influence the development of school mathematics. KappAbel is based on collaborative work in whole classes: the class counts as *one* participant. The competition begins with two web-based qualifying rounds of joint problem solving activity. The problems are to be solved by the whole class in 100 minutes. In Norway one class from each county continues to the semi-final. Before meeting for the semi-final, these classes do a project on a given theme, for example mathematics and music. The project is presented in a report, a log book and at an exhibition at the semi-finals. Each class is represented by a group of four students (two boys and two girls) who are to present the project work orally and to solve and explain a number of investigative tasks. The groups representing the three best classes at the semi-final meet for the national finals, where they are to do more non-routine, investigative tasks.

KappAbel, then, focuses on investigations and project work and sends a signal that mathematics does not consist merely of closed lists of concepts and procedures with which to address routine tasks. Also, the emphasis on collaboration in whole classes suggests that there is more to mathematical activity than individuals engaging in development or use of such concepts and procedures. Hence, KappAbel seems to be in line with international reform efforts.

The study reported on in this paper seeks to contribute to an understanding of the extent to which KappAbel meets the aims mentioned above.

CHALLENGES TO RESEARCH ON BELIEFS AND ATTITUDES

Research on teachers' beliefs has developed into a significant field of study, attempting to improve present understandings of (i) the character of teachers' beliefs, (ii) how these beliefs develop, and (iii) their possible connection to the classroom practices. Mainstream belief research claims that (student) teachers' beliefs are fairly resistant to change. However, it also suggests that they may develop in line with the reform, if pre- and in-service teacher education programmes model the types of teaching envisaged in reform documents and involve the participants in long-term collaborative efforts to develop the practices of their own classrooms through continued reflection. Also, there seems to be some consensus that teachers' beliefs play a considerable role for the learning opportunities unfolding in the classroom. Making and applauding this last point, Wilson and Cooney (2002) claimed that there has been a tradition of basing

> research on teachers' beliefs […] on the assumption that what teachers believe is a significant determiner of what gets taught, how it gets taught, and what gets learned in the classroom. (p. 128)

For the larger part of belief research, there seems to be an expectation of a positive correlation between beliefs and practice, with the former determining or significantly influencing the latter.

However, belief research has also been subject to a combination of methodological and substantial criticism in recent years. This criticism has challenged its implicit or explicit premises. Lester (2002) raised a radical methodological criticism. He pointed to the risk of research on belief-practice relationships becoming a self-fulfilling prophecy, because of its circular argument of claiming that certain observed mathematical practices are due to beliefs, while at the same time inferring mathematical beliefs from the very same practices. The problem Lester refers to seems to stem from an agreement in much belief research that espoused versions do not have a privileged position as an entry point to understanding beliefs. Beliefs are often conceived as propensities to engage in certain practices in particular ways under certain conditions (e.g. Cooney, 2001, p. 21). Consequently they are often inferred from observations of the practices in question. Lester's point, then, relates to a situation in which the call for different methods in belief research is confounded with a type of methodological triangulation that assumes identity between objects researched with different methods. We shall return to this issue later.

Lester made the above point in relation to students of mathematics. If reinterpreted so as to relate to teachers it is in line with the first author's criticism of mainstream belief research (e.g. Skott, 2005). Skott questioned the tendency to use teachers' beliefs as an explanatory principle for practice. The general affirmative answer to the question of a possible positive correlation between teachers' beliefs and the classroom practices appears to be a premise rather than a result of belief research. This is so in spite of occasional calls to look into another and less researched question of a possible opposite relation between practice and beliefs (e.g. Guskey, 1986). Skott's criticism challenges the tradition in belief research that Wilson and Cooney (2002) later described (cf. the quotation above).

Also, a rather more substantive criticism of belief research has been raised, especially of the highly individual approach normally adopted. Lerman (2001, 2002) has argued that although there may be 'family resemblance' between the views of school mathematics expressed in research interviews and those held in the mathematics classroom, they are qualitatively different entities. Skott (2001) argued that a social perspective is needed that acknowledges that the objects and motives of the teacher's activity emerge from the interactions with specific students in the specific classroom. Further developing the argument, he pointed to the teacher's involvement in multiple, simultaneous communities of practice each of which frame certain aspects of the teacher's activity, and claimed that beliefs of mathematics and its teaching and learning played variable roles in different contexts dominated by adherence to each of these communities (Skott, 2002).

These two sets of criticisms – one primarily methodological, the other primarily substantial – are interconnected and share a concern for the risk of creating or assuming the existence of an unambiguous object of study (students' or teachers' beliefs) that may be alien to the students and teachers in question and the significance of which is presumed rather than investigated. We shall elaborate on more specific aspects of these criticisms below and describe ways in which we have tried to take them into consideration within the time frame available for the KappAbel study.

METHODOLOGICAL DIFFICULTIES IN THE KAPPABEL STUDY

It is apparent from the introductory description of KappAbel that the competition seeks to influence the immediate teaching-learning practices of the participating students and teachers as well as their beliefs and attitudes towards mathematics. It does so by suggesting reformist classroom processes that are not necessarily in line with those that normally dominate the classrooms in question or with the teacher's general school mathematical priorities. This is different from attempts to develop teachers' beliefs for instance through pre- and in-service teacher education. In the latter cases, the expectation is often that a change in an independent mental construct of beliefs will subsequently inform teaching-learning practices, as the teacher 'carries' her reformed attitudes to school mathematics into the different setting of the mathematics classroom. In contrast to this, KappAbel seeks to have an immediate impact by inserting a different set of teaching-learning practices into that setting by structuring the prevailing type of collaboration and by determining the types of task to be used. This is the first of KappAbel's intentions as described in the second section of this paper. The other intention may be rephrased as an attempt to capitalise on the part of the reciprocal belief-practice relationship that has so far been researched the least, that is the one from practice to beliefs. In these terms the intention is to influence teachers' and students beliefs and attitudes through an imposed set of changes in collaborative structure and types of task.

One aim of the present study is to shed some light on the extent to which KappAbel is successful in this: Does it succeed in enrolling teachers and students in new mathematical practices through the provision of resources for those practices in the form of the competition and the tasks? And if that is the case, does enrolment in such practices – initiated by outsiders to the specific classroom – pave the way for changes in the way teachers and students conceive of mathematics and its teaching and learning?

The design of the study seeks to take into account the critical remarks made above about mainstream belief research. In the initial project description (Wedege,

2004) we questioned an assumed straight-forward relationship between teachers' beliefs and the classroom practices. Inspired by Pehkonen and Törner (2004), we also pointed to the need to distinguish between teachers' ideal and real teaching of mathematics in questionnaires and interviews. Finally, and referring to Wedege and Henningsen (2003) we pointed to a possible conflict between the teachers' and students' beliefs and attitudes towards mathematics expressed in their own words and the researchers' words expressed in a questionnaire. Later in the process of designing the study, we considered in particular three specific, interconnected methodological problems that may be seen as instances of the general criticisms mentioned in the previous section.

First, there has been a tendency in belief research to build on Ernest's (1989, 1991) distinction between three educationally relevant views of mathematics. Ernest claimed that the subject may be seen as (1) a set of unrelated facts and procedures, a toolkit that is useful for purposes external to the subject itself; (2) a Platonic and objectively existing body of knowledge to be discovered; and (3) a problem-driven and process oriented dynamic field, an ever-expanding human creation. Pehkonen and Törner (2004) is one example of a use of Ernest's scheme. Phrasing the scheme as a toolbox view, a systems view and a process view of mathematics, they asked teachers to locate their real and ideal teaching practices on an equilateral triangle with the three views placed at the vertices.

Ernest's trichotomy was made with reference to the apparent philosophical relevance of the three views and to what he terms their "occurrence in the teaching of mathematics" (Ernest, 1989, p. 250). Further, Ernest links the three views of mathematics immediately to corresponding views of the role of the teacher and the student in mathematics classrooms.

Notwithstanding the significance of Ernest's scheme as a way of structuring the educational discourse on mathematics, it may be questioned whether the three views and the corresponding connections to teaching and learning are well grounded empirically. If this is not the case, it is a problem to the extent that the scheme determines the types of questions asked and the types of answers obtained in research. In other terms, it may be argued that Ernest's scheme does not necessarily capture significant aspects of the teachers' school mathematical priorities, but impose the trichotomy (toolbox/system/process) on the teachers or students in question, if it is used as an analytical tool.

Second, it is not obvious how important the teachers' views of mathematics are, even if the tripartite scheme does capture significant aspects of it. Even if the mathematical priorities of a teacher may be seen to be in line with one or a combination of the three views of mathematics mentioned, it does not imply that such a view is of particular significance neither to the way the teacher views him- or herself as a teacher of mathematics, nor to the way he or she contributes to the interactions of the mathematics classroom. Consider, for instance, a teacher with a strongly child-centred view of education. It may be that she also considers mathematics to be associated with a certain combination of the three elements of the scheme and that she is therefore able to position herself in the equilateral triangle suggested by

Pehkonen and Törner (2004). In and by itself, however, this does not indicate how significant her mathematical priorities are for her contributions to the interactions in the classroom.

This problem relates to the well-known question of the central or peripheral character of beliefs. However, in contrast to most studies in mathematics education it contextualises this question by acknowledging that the objects and motives that dominate the teacher's activity in a mathematics classroom may not be the same as the ones that frame her activity when being an object of study. In other terms, structuring a questionnaire, an interview protocol or an observation schedule along the lines of the Ernest scheme may reify the view of mathematics and attach a much greater significance to it than what is warranted if a more grounded approach to research on teachers' and students' educational priorities and activity is used.

Third, no terminology carries unequivocal meanings. This is so also for the rhetoric of the reform in mathematics education. More specifically, an emphasis on notions of problem solving and project work, or claims that school mathematics should be directed at working with the students' real world problems do not necessarily mean the same for all. This is banal, but it challenges the use of fairly closed questionnaires or interview protocols as a means of accessing teachers' or students' school mathematical priorities. At the same time it may explain the prominence of the approach that Lester (2002) described (cf. the previous section) of becoming involved in a circular argument: We do need more than espoused views of mathematics and its teaching and learning in order to warrant a claim about the existence of beliefs. From this perspective Lester's point may be rephrased to say that although several approaches may be needed in the study of beliefs, these approaches do not necessarily shed light on the same object – a context-independent mental construct called beliefs. In spite of that, multiple methods have proved to be of relevance also to overcome the problem of the multiple meanings of the terminology used. For instance, classroom observations with follow-up interviews stimulated by recordings of interactions in the classroom in question have been used successfully. The strength of this combination, however, is not that it draws a more accurate picture of a stable and decontextualised construct of school-related mathematical beliefs. Rather, it is that it may situate teachers' and students' reflections about mathematics and its teaching and learning within the context of a relevant classroom and contribute to an understanding of how the teacher interprets the interaction in question from the perspectives that the teacher herself finds the most significant.

One further aspect of our methodological considerations – related to all the three problems discussed above – is related to Bourdieu's notion of symbolic violence. In his last major research project, *La misére du monde* ("The Misery of the World"), Bourdieu and his team (1993) collected and analysed testimonies from hundreds of respondents about their lives. In a retrospective methodological chapter called *Comprendre* ("To understand"), Bourdieu presents the underlying epistemological assumptions of the diverse operations of the inquiry: selection of interviewers, transcription and analysis of the interviews, and so on. The interview, he claims, is a social relation with effects on the results obtained. This is so although the research interview differs from most ordinary dialogues in that it aims at pure cognition and by definition of scientific inquiry excludes the intention of using any form of symbolic violence that may influence the responses. However all kinds of distortions are inscribed in the mere relation of an inquiry, for instance in an interview.

It is, for instance, the interviewer who sets the rules of the game and who initiates it. This creates an asymmetry between interviewer and interviewee, the significance of which is doubled, if there is a hierarchical relationship between them in terms of cultural capital. In any piece of research, this inserts a power relationship – an element of symbolic violence – that necessarily influences the results.

A DISCUSSION OF THE DESIGN OF THE STUDY

The methodological challenges and difficulties mentioned above implied that we had to use different empirical approaches to the problem field. Dealing with Norwegian experiences, the KappAbel research project includes five types of empirical data – quantitative as well as qualitative. These are

- LS1: A questionnaire administered to the teachers of 2856 grade 9 mathematics classes in the 2004–2005 academic year.
- LS2: (a) A short questionnaire sent by email to 351 teachers whose classes took part in the two introductory rounds of KappAbel. (b) A questionnaire administered to 15 of these teachers whose classes intended to continue with the project work, whether their classes progressed to the semi-finals or not;
- Interviews conducted with six teachers and three groups of students at the national semi-finals; further interviews with two teachers and two groups of students are planned.
- The reports and process log books of five classes on the project work of *Mathematics and the body*. (Students from these five classes were among those interviewed or observed.)
- Observations of 10–15 lessons in 3–4 classes all of which progressed to the national semi-finals or were one of the five best in the county (planned).

The intentions behind LS1 are to get some understanding of who the participating teachers are, both with regard to their objective characteristics (education, teaching experience, gender, etc.) and to whether they consider themselves in line with the rhetoric of the current reform. The latter of these intentions is to contribute with knowledge about the extent to which KappAbel is confirmative rather than formative. The question, then, is whether the participating teachers consider themselves more in line with the overall intentions of the competition than their non-participating colleagues. Considering the methodological problems mentioned above, the aim is not to point to any substantial 'objective' correspondence between the KappAbel intentions and those of the participating teachers, let alone to imply that the teaching-learning practices of the participating students and teachers are more or less in line with current reform initiatives than those of their peers. LS1 merely attempts to find if and how the terminology of the KappAbel project resonates with the priorities of the participating teachers to a greater extent than with those who do not participate. Furthermore, the statistical data from the large quantitative data set LS1 will also be used to see the qualitative data from a limited and heterogeneous sample (LS2 and the interviews) in a proper perspective: to what extent are the participants in the qualitative study different from the population in general in terms of their objective characteristics. For this reason the factual information obtained in the two questionnaires (the teacher's background and experience) is identical.

The second questionnaire, LS 2, was administered to the teachers whose classes took part in the project work on *Mathematics and the body* following the two introductory rounds of KappAbel. The items of LS2 are to some extent informed by a preliminary analysis of the responses to LS1. For example we used factor analysis in LS1 as a way of reducing the dimensionality of the problem (What should be the main focus of school mathematics?) and find latent structures in the data. These structures were created solely from the data. The following dimensions were constructed on the basis of a factor analysis with three factors: (1) Everyday mathematics and basic skills; (2) Investigations and new mathematical questions; (3) Logic and structure. The theoretical framework (toolbox/system/process) was not involved in the analysis at this stage. However, from the outset our questions were constructed on the basis of these three theoretical categories.

In LS2 some of the items are closed, but the limited number of respondents allowed us also to use more open and qualitative response items. Some of these are fairly traditional, and for instance ask the respondent to reflect on good and bad experiences with teaching mathematics. Other items require the respondent to comment on statements made by (imaginary) colleagues about key aspects of KappAbel (collaboration, project work, gender). One example reflecting a teacher's view of boys' and girls' work in the mathematics classroom reads like this:

"Boys and girls have different kinds of strengths and weaknesses in mathematics, which they should be allowed to go for. That's why boys should work together with boys and girls with girls."

The intentions behind this dual format of the open items may seem self-contradictory. On the one hand they aim to allow the respondent to focus on what he or she sees as the most significant aspects of mathematics teaching and learning. This is opposed to a situation in which they are exclusively to respond to items framed for instance by the scheme (toolbox/system/process). On the other hand, the other set of items invite the teacher to respond to a set of very specific statements from 'their colleagues'. These may provoke a concrete response, but are meant to contextualise the teacher's reaction by involving him or her in a virtual dialogue that allows any response whatsoever. In spite of the concrete wording of these items, then, they intend to allow the teacher not only to signal for instance degree of agreement or disagreement, but to reject the agenda inherent in an item altogether.

Semi-structured interviews were conducted with both teachers and students during the national semi-finals. The intention was on the one hand to let the teachers and students talk freely about what they conceive as significant and valuable teaching/learning experiences without initially framing their answer within a pre-conceived scheme. On the other hand we wanted to get an impression of how they perceived KappAbel and their own participation in it in relation to their everyday teaching. This may be seen as an attempt to strike a typical balance in semi-structured approaches to research between too tight and too lose a structure. For our purposes, and to minimize the symbolic violence in the interviews, we have tended to move towards the lose end of the continuum. The Norwegian research assistant, who is also a mathematics teacher, conducts the interviews, which may help in reducing the significance of a hierarchical relationship between researches and teachers.

Observations of 10–15 lessons in 3–4 classes are planned. We intend using the interpretations of the individual teacher's response to LS2 and to the questionnaire as well as the students' comments in the interviews as an interpretive device in relation to practice. This is not an attempt to make 'consistency' or 'inconsistency' descriptions of the teachers in question, but in some sense to triangulate our previous understandings (based on interviews and questionnaires) of what they prioritise in mathematics education and to get some indication of how prominent a role these priorities play in practice under the institutional constraints of classroom teaching. However, we do not consider this a case of traditional methodological triangulation. The problem is that methodological triangulation is normally considered an attempt to adopt a variety of different perspectives on the object under study. This is part of what we wish to achieve. However, some of the substantive criticism raised above of the lack of social orientation of mainstream belief research exactly questions the extent to which it is indeed the same object that is being studied in for instance a research interview and in classroom observations. It may be that teacher's beliefs play a part for the classroom interactions, but the call for a more social approach in belief research is exactly built on the understanding that the teaching-learning practices unfolding in the mathematics classroom are not the teacher's practices in the possessive sense of that term. However, having teachers comment on specific interactions in follow-up interviews may provide increased insight into their school mathematical priorities, including what their potential role is in relation to unfolding classroom practices.

CONCLUSIONS

Research on beliefs and attitudes is a difficult field, methodologically speaking. On the one hand, one needs a variety of different approaches to capture significant aspects of teachers' and students' beliefs and attitudes towards mathematics. On the other hand one should not expect different methods to shed light on the same object of beliefs, and consequently the idea of triangulation takes on a different meaning than one of locating a single point or object in a space of two or more dimensions by using a variety of such methods.

These methodological difficulties lead to some rather more conceptual ones: Does the constructs of belief and attitudes point to an object that is as stable and decontextualised as one is sometimes led to believe(!)? More specifically, are the claimed manifestations of teachers' and students' mathematical beliefs sufficiently grounded empirically to claim that they are part of the way in which the person in question conceives the world and him or herself within it? Does the significance attached to students' and teachers' beliefs about mathematics resemble their own priorities, for instance in comparison with their general view of children, of school, of education and of mathematics in schools – or does that significance to a grater extent impose a mathematical perspective on the teachers and students, a perspective that may be much less important to them than to the researcher?

These questions may be summed up as one of the extent to which the researcher's imposition of the construct of beliefs on the students and teachers in question is an exertion of symbolic violence in the sense of Bourdieu. And this feeds back into the methodological considerations: Is there any way of reducing the significance of that violence by exerting it differently?

In the KappAbel study we use a well-known methodological cocktail of large scale questionnaires, interviews, observations and analyses of students' work. Within the timeframe of the present study we have only managed to a very limited extent to come up with suggestions for methods that break with or supplement the ones that are used in most other studies in the field. One may then rightly ask if and to what extent we fall prey to the same types of criticism that we raised against other studies in the field.

To some extent this is undoubtedly so. However, our answer to that question has two more elements. First, we do use well-known methodological tools, but we do so in ways that are not quite so well-known. For instance in LS 2, we seek to establish a context for teacher reflection by simulating an interaction between the respondent and a fellow-teacher. This intends to limit the extent to which the teacher conceives of the situation as a confrontation with an agenda imposed on him or her by the researcher. The intention is, then, to initiate a type of virtual dialogue, by inviting the teacher to react to a statement made by a 'colleague'. We have also reasons to believe that the technical dimension of the methodological construction is important when it comes to evaluating the findings. As a consequence, the interviewer is a research assistant who is a mathematics teacher herself: young Norwegian mathematics teacher interviews young Norwegian mathematics teachers. This reduces the asymmetry inherent in the social relation of the dialogue. This was not the case with the students who obviously felt ill at ease in the situation. As an experiment we plan to invite one of the students who participated at the KappAbel semi-final to interview the rest of her group, although – as pointed to by Bourdieu (1993) – a social and cultural resemblance between interviewer and interviewee may create other problems.

Second, and more importantly, we try to make sense of the data collected with those tools in ways that acknowledge the significance of the types of criticisms listed. This last element, then, has more to do with the analyses and interpretations of the data than with the character of the data-collecting tools themselves. For example, a teacher's response to the first questionnaire is not interpreted as an indication of 'real' compatibility between his or her thinking about school mathematics and the intentions of KappAbel. Rather, it is seen merely as an indication of the teacher's possible compliance with the rhetoric of the reform, and therefore as an indication of the extent to which participants in KappAbel to a greater extent than other teachers consider themselves in line with the dominant reform discourse. Also, the interpretations do acknowledge that the results are not grounded in an empirical study and that whatever answer we obtain in terms of teachers' priorities of school mathematics should be interpreted with this in mind: we have not managed to set up a design that build on the teachers' own thinking. What we may be able to do is to shed some light on how the teachers, given the alternatives inherent in the Ernest scheme, prioritise aspects of mathematics. In relation to the aims of the study, we can claim that if it is fair to suggest that the priorities of the KappAbel competition include a vision of mathematics as at least encompassing the process perspective, the study may shed some light on the extent to which teachers at the level of rhetoric orient themselves in the same direction, when faced with a limited range of alternatives.

Note

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Short communications and workshop

PEER-ASSESSMENT IN MATHEMATICS AT THE NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY

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Maugesten and Lauvås reported on spectacular improvements in student performance after the introduction of a peer-assessment methodology in a class of approximately 100 mathematics student teachers at Østfold University College in 2003. We wanted to investigate whether similar gains could be made at the Norwegian University of Science and Technology and also consider the effect of peer-assessment in a more controlled study. Here we report on a pilot study conducted during the Spring Term 2005, in which peer-assessment was tried out in one section (consisting of student teachers) of the mathematics course Multidimensional Analysis.

INTRODUCTION

Maugesten (2005) asked, "How can one reduce the percentage of students flunking at a mathematics exam in teacher education from more than 50 % to 15 %, and without teachers and students being stressed, and without lowering the standards?" Let us review what led to substantial improvements in student performance reported by Maugesten and Lauvås (2004) in a class of approximately 100 mathematics student teachers at Østfold University College in 2003. First, the obligatory mathematics course for student teachers was divided into three modules. More important, the national Quality Reform[1] with its focus on closer student guidance and new forms of assessment inspired Maugesten, Lauvås and their colleagues to introduce some special measures: Level differentiation and peer-assessment. Level differentiation involved forming four tutorial groups at three different levels based on a test taken during the first class. Peer-assessment meant that the students them selves were involved in marking (for formative purposes only) the written home-work solutions of their peers. One out of six hours of weekly class-time was allocated for this kind of work. The students had to participate seriously in seven out of nine peer-assessment sessions to be admitted to the exam.

Peer-assessment is certainly not a new idea in tertiary education, and systematic studies on its efficiency have been conducted for at least 30 years according to

[1] See http://odin.dep.no/kd/norsk/tema/utdanning/hoyereutdanning/tema/kvalitetsreformen/bn.html

2007. In C. Bergsten, B. Grevholm, H. S. Måsøval, & F. Rønning (Eds.),
Relating Practice and Research in Mathematics Education. Proceedings of Norma 05, Fourth Nordic Conference on Mathematics Education (pp. 469–473). Trondheim: Tapir Academic Press

Zariski (1996). However, not many of these studies focus on mathematics (cf. Hart, 1999), and some authors have even claimed that the exact character of the subject renders peer-assessment superfluous. Furthermore, many of the studies are from the Anglo-Saxon world where students are typically younger than in our context. As one of the main ideas with peer-assessment is to help students take more responsibility of their own work, this is certainly an issue to be reckoned with.

We wanted to investigate whether gains similar to those at Østfold could be made at the Norwegian University of Science and technology (NTNU) and at the same time consider the effect of peer-assessment in a more controlled study. Here we report on a pilot study conducted during the Spring Term 2005, in which peer-assessment was tried out in one section (consisting of all the student teachers) of the course MA1103, Multidimensional Analysis.

OUR SETUP

The course MA1103 consisted weekly of two two-hour lectures in a class of approximately 100 students, and a two-hour tutorial (problem session) with groups of 20–30 students. The students enrolled in the teacher-education program (LUR) formed a single tutorial group, in which peer-assessment was used. The other tutorial groups followed the normal format, where teaching assistants corrected the homework. For all students, 8 out of 12 homework assignments had to be approved.

A typical LUR tutorial group started by dividing the students into pairs. The pairs swapped their written solutions of the home-work problems and corrected them, using a suggested solutions handout, the text-book and the teacher for help. The students were asked to give hints and feedback on the solutions they corrected. This lasted one class hour. In the second class hour, the teacher explained unclear problems on the blackboard.

As a basis for our analysis we collected the final grades of all students who completed the course. We also surveyed student opinion on peer-assessment in the LUR tutorial group on two occasions, one in February, early on in the course, and the second in the study period in May after the end of the lectures.

EXAM RESULTS

Out of the 100 students in MA1103, 68 also participated in the class MA1202, Linear Algebra and Applications, where all students had regular problem sessions. In this sample of 68 students the final grades on the exams were distributed as shown in Table 1.

	MA1103						MA1202					
Grades in %	A	B	C	D	E	F	A	B	C	D	E	F
LUR (n = 21)	24	29	10	5	19	14	14	0	14	29	24	19
Others (n = 47)	19	30	23	15	6	6	9	2	23	13	34	19

Table 1: Grades in percent for LUR-students and other students

The averages based on the passing grades A = 5, B = 4, C = 3, D = 2, E = 1 are as follows. For MA1103 the LUR-group gets 3.39 and the other students 3.43. For MA1202 the averages are 2.41 and 2.24, respectively.

It is clear that there is no difference in average grades between the LUR-group and the other students. There appears to be a larger variance for the LUR students, especially in the MA1103 class, but a two-tail F-test of pair-wise differences in variances shows that these are not statistically significant (smallest p is about 0.1).

SURVEY RESULTS

The February survey was answered by 19 of 21 LUR students. The complete survey, along with the results, is shown in Appendix A.

The May questionnaire was handled electronically and 11 of the 21 students answered. The questionnaire had many of the same statements as the first one. It is interesting to notice that the answer distribution to the statement "I read the teachers' and tutors' feedback carefully" in the May survey is 10-1-0-0-0 (i.e. 10 students saying "definitely yes", and so on, see the Appendix for more details). In the second part of the questionnaire we noted that the students agreed most to the statement "Student teachers should have at least one tutorial like this one" (0-1-2-1-7) and meant that "Other students than the student teachers should have at least part of a tutorial like this one" (0-1-2-6-2). However, to the statement "The description of the tutorial presented in the beginning of the term was adequate", the answers were 0-4-2-4-1.

DISCUSSION

Our initial findings, although based on a rather small sample, are not in line with the large improvements reported by Maugesten and Lauvås (2004). Furthermore, the experience in Østfold was that peer-assessment was most beneficial for the weakest students. In contrast to this, we found indications of the opposite phenomenon, that is that peer-assessment would have had a polarizing effect. One possible explanation for this is the difficulty of the course: Maugesten and Lauvås reported on the first two modules of their mathematics curriculum. On the exam in the third module with more new and difficult material the students had a high percentage of failure (Maugesten, personal communication, 2005).

Peer-assessment presumably also has qualities not measured in an exam, like preparing students to give adequate feedback. In our case it is clear that we should have explained and discussed the whole setup of peer-assessment more closely in the beginning. Two of the LUR students sum up the situation quite well in a comment on the May questionnaire: "This way to arrange the problem sessions has a potential for success. But in the situation when the participants are poorly prepared for the task it is rather inefficient."

In the next stage of the project we will try out the peer-assessment (somewhat revised based on the feedback from the students) in a larger class of engineering students as well as with LUR students in MA1103 next spring.

Acknowledgement

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APPENDIX A – THE FEBRUARY SURVEY

How do the following statements relate to you **last term**?	A	B	C	D	E
I worked on homework problems with my classmates also out-of-class	14	1	2	1	1
I always tried to work on all the homework problems	8	8	2	1	0
I focused on the most important homework problems	11	5	2	1	0
I often compared my solutions with other students' before handing it in	10	3	4	2	0
I read teachers' and tutors' feedback carefully	7	4	2	0	6

A – This is what I did. B – I would have liked to do this, but couldn't.
C – I did not think this was necessary. D – I did not think this was possible.
E – I'll do it next time/later/some other time.

Please check <u>one</u> box that best describes your opinion on MA1103 now:

Do you agree with the following statements?	1	2	3	4	5
I enjoy working with mathematical problems	0	0	3	11	5
I am among the strongest mathematically in the tutorial group	5	5	7	2	0
I enjoy explaining mathematical ideas to others	0	0	6	10	3
I spend a lot of time on mathematics homework # hours per week: <u>6.4 ± 2.9</u>					
I would spend more time on homework, if the benefit was greater	0	5	9	2	3
Since other students see my work, I will work harder	0	2	5	10	2
I understand the motivation for this type of tutorial	0	0	3	12	4
It is a good idea to have a tutorial like this	0	0	3	9	7
There should be more tutorials like this one	0	0	7	6	6
Tutorials like this make it more difficult to have both fast and slow students in the same group	3	10	2	3	1
Working in groups is good for learning mathematics	0	0	1	11	7
Working in groups is good for communicating and teaching mathematical ideas	0	0	2	6	11
Working in groups is more fun and inspiring	0	0	3	6	10
Assessing other students' solutions deepens my understanding of the material	0	0	5	9	5
Assessing other students' solutions helps me write my own solutions clearer	0	0	5	9	5

1 – definitely not 2 – no 3 – maybe, not applicable 4 – yes 5 – definitely

ALGEBRA – GEOMETRY RELATION IN TEACHING FUNCTIONS AND CALCULUS

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Algebraic calculations, formal applying of formulas very often dominates in mathematics classes. Such teaching distorts the essence of mathematics and forms bad opinion on the subject in society. The situation can be corrected by including much more geometry into the algebra and calculus. This stimulates deeper understanding of mathematics. A few examples showing harmony between algebra and geometry are presented in the article. Emphasis is done on direct-inverse functions relation, their geometrical interpretation and its applications. One problem from Vilnius University students' examination papers is discussed.

PROBLEM

Algebra, functions, differential and integral calculus are the main topics in the last grades of high school. Mathematical analysis (Calculus) is the heart of the first year universities curriculum. A lot of material about difficulties in learning calculus can be found in the Proceedings of the ICME-7 in Quebec (see Artigue & Ervynk, 1992). I think Michael Atiyah (2001) precisely pointed out one aspect of bad performance in algebra and calculus. The quote from the chapter "Geometry versus algebra" in this article:

> As you know, Faust in Goethe's story was offered whatever he wanted by the devil in return for selling his soul. Algebra is the offer made by the devil to the mathematician. The devil says: "I will give you this powerful machine, and it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine." Of course we like to have things both ways: we would probably cheat on the devil, pretend we are selling our soul, and not to give it away. Nevertheless the danger to our soul is there, because when you pass over into algebraic calculation, essentially you stop thinking; you stop thinking geometrically, you stop thinking on the meaning.

2007. In C. Bergsten, B. Grevholm, H. S. Måsøval, & F. Rønning (Eds.),
Relating Practice and Research in Mathematics Education. Proceedings of Norma 05, Fourth Nordic Conference on Mathematics Education (pp. 475–479). Trondheim: Tapir Academic Press

Most likely, Atiyah had in mind advanced algebra and geometry but his metaphor fits for elementary level as well. I am sure that teaching of algebra (elementary functions included) and calculus (mathematical analysis) sometimes is too formal without geometrical interpretation or adequate geometrical language. Both secondary and university students very often are not able to interpret algebraic formulas or word statements geometrically in the right way.

I deliver a one semester course on methods of mathematics teaching for future mathematics teachers. Almost all the time is spent on creating (or restoring) harmony between algebra and geometry in a variety of topics, like derivative-tangent relation, inverse function, elementary functions, convex functions, limit notion, parametric curves, portrait of a function $f(x, y)$ and others.

In this short presentation we shall not discuss any theoretical problems of geometrical thinking itself. We shall concentrate on a few examples concerning understanding of the graph of direct-inverse function.

EXAMPLES

Derivative-tangent relation

I suggest to start the calculus course in high school by Descartes's method of finding tangents. This method allows to find equations of tangents for graphs of power functions $y = x^n$, $n \in Z$. The tangent of a curve is a geometrical object and much more perceivable than the derivative. After finding equations of tangents of graphs of all power functions with integer powers we can find tangents of inverse functions. We must change the traditional way of understanding and geometrical interpretation of inverse functions. We perceive graphs of direct function $y = x^n$ and its inverse $x = \sqrt[n]{y}$ as one curve. This point of view on inverse functions was widely presented by Kudžma (2003, 2005) in the Tallinn and Liepaja conferences.

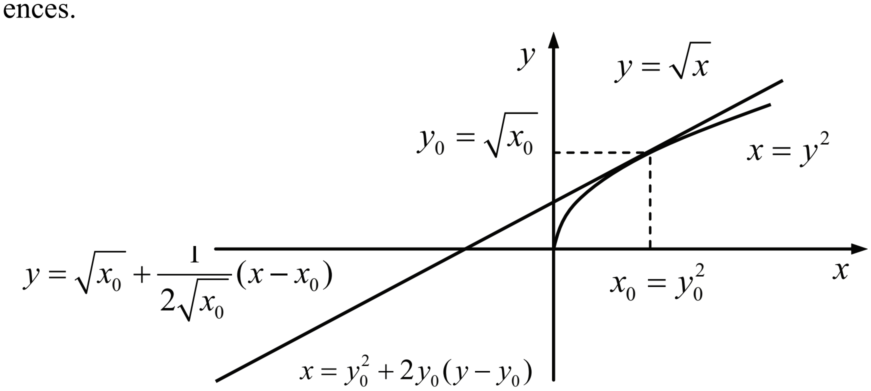

Figure 1

In the simplest case, using the relation $x_0 = y_0^2, y_0 > 0$ we transform parabola's $x = y^2$ tangent equation $x = y_0^2 + 2y_0(y - y_0)$ into

$$y = \sqrt{x_0} + \frac{1}{2\sqrt{x_0}}(x - x_0) \ .$$

This is the equation of the same line, the tangent of the parabola $x = y^2$ or tangent of the graph of the function $y = \sqrt{x}$

Later on we introduce the derivative as the limit and develop standard calculus.

Convex functions

This topic is very suitable for the algebra-geometry relation. In standard calculus courses one can find the problem about convexity of inverse function. Very often students are asked to apply the second derivative test for that. Fig. 2 shows how the right interpretation of direct-inverse functions graph allows us to solve this problem without any calculations (a function is convex if its graph is below the secant).

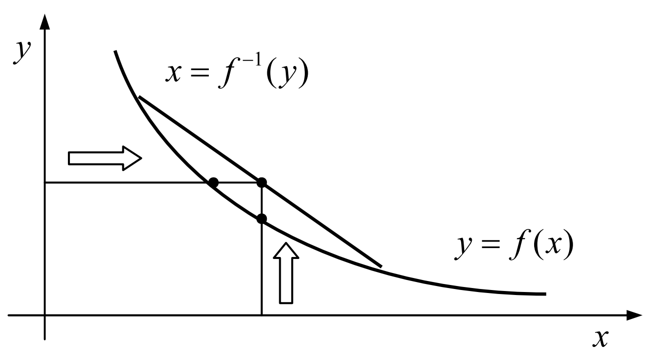

Figure 2

EXAMINATION QUESTION

We shall consider one examination problem of the third year mathematics student teachers of Vilnius University on methods of mathematics teaching.

Problem

This excerpt is from the 11th grade textbook:

> Let us remember how to simplify the expression $\sqrt{a^2}$. Look carefully at two evident equalities: $\sqrt{2^2} = 2$, $\sqrt{(-2)^2} = \sqrt{4} = 2$. It was sufficient to cancel the exponent and root signs in the first case and to change the sign of the power basis, in addition, in the second case. In general, $\sqrt{a^2} = a$, if $a \geq 0$, and $\sqrt{a^2} = -a$, if $a < 0$. Briefly, it is possible to write as $\sqrt{a^2} = |a|$.

a) Represent geometrically (in graphs of functions) in four different pictures the first four equalities of the excerpt.
b) Comment the text.

The problem was a typical one in the form – to represent geometrically the written mathematical text. But it was an unexpected and a difficult one in content. The inverse function is not mentioned explicitly in the text. Figure 3 is the right geometrical interpretation of the third and fourth equalities. For solving this problem it was necessary to perceive that on the parabola $y = x^2$ it is possible to explain geometrically actions of both two functions $y = f(x) = x^2$, $x = g(y) = \sqrt{y}$ and their composition $g(f(x))$.

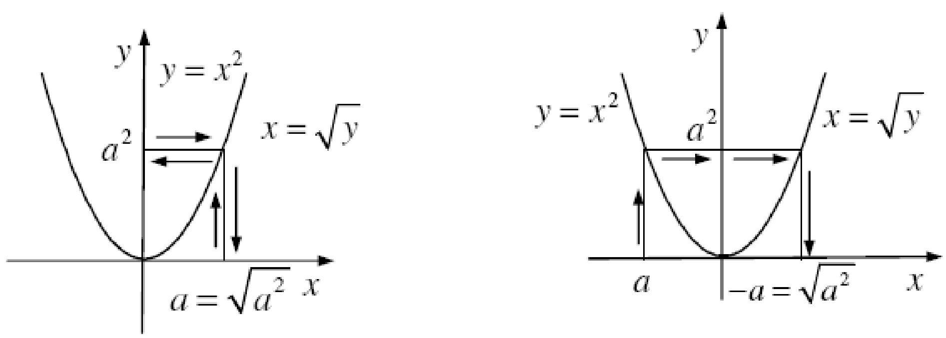

Figure 3

I did not expect good performance on this problem by the students. But one student (out of 26) surprised me. He drew all four pictures completely correct. His comment is also worth citing. Aleksej: “It’s a pity there is no figure to this text. Of course, if it is done and the inverse function has been taught correctly (not the way it being taught now in the school).”

CONCLUSION

In this short presentation I wanted to give a few examples, which demonstrate benefits of understanding the direct and inverse functions graphs as one curve (limited space of article does not allow more). It permits us to maintain full harmony between algebra and geometry. I think that traditional teaching of graphs of direct and inverse functions as symmetric with respect to the line $y = x$ is misleading. Then it is impossible (or would be very complicated) to interpret geometrically equalities $f(f^{-1}(y)) = y$ or $f^{-1}(f(x)) = x$, which define the inverse function.

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BUILDING ADDITION AND SUBTRACTION STRATEGIES IN EARLY PRIMARY SCHOOL MATHEMATICS

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1st and 2nd grade pupils' thinking strategies in addition and subtraction tasks are researched in this multiple case study. Using a phenomenographical approach I interviewed 6 first graders and 11 second graders three times during the school year 2003–2004. I found four categories in counting.

How to build efficient conceptual structures for children starting school in solving addition and subtraction tasks is a most interesting challenge. According to Fuson (1992) children's first cardinal meanings of number words are perceptual quantities of objects. Then follows increasing integration of sequence, count and cardinal meanings of number words. In conceptual structures for addition and subtraction this means that first children can solve problems that require a set of entities to present first an addend or a sum at a given time. On the next level most children spontaneously invent sequence-counting procedures, in which number words present both addends embedded within the sum. According to Fuson counting down is more difficult and error-prone than counting up. The next level is when the abbreviated sequence procedures are chunked into derived fact procedures. For example, instead of solving 12–5 by counting down 5 words from 12, five is chunked to 2 and 3 and the problem is solved as 12–2=10 and 10–3=7. Neuman (1987) called this level "structuring", and according to her, pupils counted in a way they named "Measuring by tens". I think achieving this level is not as simple as we seem to think. Cumming and Elkins (1999) report the lack of development of thinking strategies and automaticity in many children even at high-grade levels. It seems to me that a good ground for counting can be made in the grades 1 and 2, but in teaching this demands a lot more than we are used to think. First we should know better how children think when they do addition or subtraction tasks.

As a teacher-researcher my starting point in teaching mathematics was that the textbook itself is not sufficient. I gave each pupil 20 Unifix cubes always to be at hand, and I used all kinds of other manipulatives, like Montessori material and plastic ten base models to concretize numbers and tasks (ones, tens, hundred grids and thousand blocks). Pupils practiced often in pairs or in small groups for example playing mathematical games. At the same time I was interested to see how chil-

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dren's addition and subtraction strategies were developing, which strategies pupils had and how the strategies were changing.

To find an answer to my research question I interviewed pupils three times during the school year 2003–2004, in September, January and May. There were 6 first-graders and 11 second-graders in my own combined 1–2 class. I asked children, if possible, to explain how they solved addition and subtraction tasks. The focus in a test was between 0–20 with the 1st graders and between 0–100 with the 2nd graders.

My approach in this study is phenomenographical, as in Dagmar Neuman's study (Neuman, 1987). According to Neuman (1999, p. 102) in phenomenographic research the intention is to map the variation of ways a phenomenon can appear to us while in phenomenology the aim is to search its essence. Analyzing the results I found an outcome of four different ways to use strategies in addition and subtraction tasks.

The school starters who cannot yet count other than in extremely simple tasks presented the first category of strategies. They can only count with numbers less than 5, and they even cannot count numbers down from 20 to 0. They have not yet developed the ability to do double counting to keep track of how many number words they have gone forward. These pupils perform significantly worse compared to those who already when coming to school have developed better counting strategies. Here is an example of a first-grade pupil's inadequate strategy:

Interviewer:	7+7
Emre:	(counts with cubes) 1…7, (and continues) 8, 9, 10, 11, 12.
Interviewer:	Did you count 7 more? How do you know that you have counted 7 more?
Emre:	(laughing) I don't know.
Interviewer:	So?
Emre:	I just guessed, I don't know really.

The second category of counting strategies is the category of counting with number words. This is also a very basic strategy. It means that double counting has developed, and commutativity soon becomes familiar. Children can use concrete objects or fingers to help counting, or they only say number words. To this strategy belongs that they really say the number words, not just whisper or say them in their minds. For some children subtraction is also considerably more difficult than addition. The strategy is risky, for example one can count 15–7 saying numbers 15, 14, 13, 12, 11, 10, 9 tracking with fingers and then pick 9 as the. In the next episode a first-grade pupil counts 12–5:

Interviewer:	12–5
Martin:	(counting in his mind) 8
Interviewer:	How did you count?
Martin:	Kind of with these cubes, I looked 12, here is that 10, here is 12, and then you minus 5, so it is then 8!

With big numbers the strategy is exhausting and many mistakes can occur, for example a task like 15+15 is very difficult to those who use this strategy. And then there is always a risk to pick up a wrong number for the answer. On this level also number facts are used, but many mistakes occur. For example someone can remember that 8+8=16 and then make a wrong decision that 7+7=15.

The third category in my findings was the “self learned strategies“. If strategies are not taught in mathematics or pupils do not internalize some known strategies they can develop their own. This kind of strategy is when 55–9 is counted by “making ten“ first counting 50–9=41 and then 41+5=46. Another example is 15–7 where first 10–7=3 is made, and then 3+5=8. Also 8+8 can be counted first as 10+8=18, then 9+8=17 and finally 8+8=16. This is quite an effective strategy with small numbers, but with numbers larger than 20 it is more difficult and time consuming. It does not support children’s number sense either. Tasks such as 75–20 or 15+15 can be difficult to those who use this kind of strategy. That is because their counting is still at the end based on counting with number words. Here is an example of a second-grade pupil’s way of solving 12–5:

Interviewer:	12–5
Alice:	7
Interviewer:	Why?
Alice:	Well, I made 12 to a ten and took 5 of it and then I made kind of 2 more, 2 more also, because it’s 12; I made kind of 10, (10 fingers up) then I counted 5 (5 fingers down) and then 2 more is taken (5+2 fingers up).
Interviewer:	More? Say it again.
Alice:	I put 12 to a ten (shows with fingers), took minus 5, then I know that 12 was so (2 fingers up in right hand) so I put 2 more to this (to 5).
Interviewer:	You first counted 10–5 and then put 2 more, did you?
Alice:	Yes.

The fourth and most effective strategy is when someone is counting “via ten“. Such as when 82–4 is counted 82–2=80 and 80–2=78 or 55–9 is counted 55–5=50 and then 50–4=46. To this group of pupils 15+15 is also easy, and 75–20 is quite obvious. On the one hand, these pupils do not even need to remember number facts. For example 8+8, if not remembered directly, can be solved as 8+2+6 = 16, splitting the number 8 in parts, or it can be deduced from the fact that 7+7=14. On the other hand, familiar number facts are often quite obvious to those who use this strategy. For example a first-grade pupil solved 20–15 easily “by fives“ because he realized that 20 contains four fives.

Interviewer:	20–15
Alex:	It is 5
Interviewer:	Why?
Alex:	Well, with five numbers only.
Interviewer:	Aha!
Alex:	One (points at 15). If you take all the15 away, then it stays only one five.

Very essential for this strategy is an ability to partition all needed numbers and find the solution. The counting is not based any more on counting with number words. This kind of strategy can be taught and I think it should be the aim in addition and subtraction strategy use.

Compared to what is already known about efficient counting strategies and basic arithmetic skills I found that known number facts seem to be more useful for children only in tasks with numbers under 20. With bigger numbers between 20 and 100 they do not seem to be very helpful. Those who really also can count with bigger numbers usually count "via ten". For them numbers between 0 and 100 seem to be visible, they can move on the number line 0–100. I regard this ability as a good number sense. To it also belongs the ability to split numbers effectively in parts and find an essential knowledge in the task. In my class three pupils from the second grade were on this level already in September, two first-graders and two second-graders achieved it quite soon and finally two more second-graders were on this level in May. That means 2/3 of the second-graders and 1/3 of the first-graders had efficient counting strategies. The rest had more problems, but were moving in the right direction. Anyway, if we do not pay attention to the strategies pupils use it may happen that many of them stay on the level of counting with number words.

I also found that children could move directly from the second category to the fourth category. That means that they learn number partitioning effectively with small numbers and then they successfully continue with bigger numbers. Between them stay those who need more practicing and who count in a way described in the third category. In a class where no one asks pupils how they count and what they think when they count, it is possible that these problems never will be recognized. When coming to higher classes these pupils may start to dislike mathematics. It has been argued that in mathematics and science teaching a modelling approach based on previous research compared to just a constructivist approach can better serve a meaningful learning (e.g. Keeves, 2002, p. 338). I agree with this. We need a good model for our ten based number system, since we have to make tasks concrete and visible with objects. And furthermore we must start to teach children counting strategies.

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ON TEACHING PROBLEM SOLVING AND SOLVING THE PROBLEMS OF TEACHING

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George Polya's first principle in teaching is what he defined as active learning: "let the students discover by themselves as much as feasible under the given circumstances". I have found that by adopting this principle in a specific way, it is possible to get students to change their attitudes towards mathematics and become more involved in the subject. Besides from the more immediately feedback and reactions in the classroom, it also resulted in more students passing exams and even selecting mathematics as a continuation course in their teacher program. So even if it is difficult to engage students during a 90 minutes lecture, the benefits have proved to be substantial.

INTRODUCTION

Problem solving inevitably means quite different things to different people and for different students. For some students, it includes an approach, an attitude or predisposition toward investigation and inquiry as well as the actual processes by which individuals attempt to gains knowledge. Other students sense despair and disapproving emotions when someone is approaching them with problem solving. There are several reasons that college students sometimes fail to reach a satisfactory level of proficiency in problem solving. They often suffer from fears and anxieties; especially fear of failure, which hampers their efforts to solve problems. Particular learning styles may very well make it harder to learn to solve problems.

Some creative teaching strategies may allow us teachers to address the emotional, psychological, and cognitive barriers to problem solving simultaneously. For example, on the first day of class, I try to accomplish a reasonably open discussion about the nature of the course material, encouraging students to voice their fears and concerns about it. This approach has shown to assist in creating a possibly more comfortable learning environment, a classroom in which students are encouraged to question and take risks without fearing negative consequences. Continuing this kind of open dialogue in the classroom throughout the course will strengthen a positive didactical contract between teacher and students and provide many opportunities for students to discuss different ideas and approaches to solving problems. Studies suggest that active involvement is critical in developing problem solving skills, so

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using student learning groups to promote active experimentation with problems is a reliable teacher strategy.

If mathematics teachers allow group work, discussion, and information gathering in libraries and over the Internet, and also want students to learn more mathematics in collaborative work, then they face great demands on what types of problems they should pose. Silver and Kilpatrick (1989) argue for the use of open-ended problems in the assessment of mathematical problem solving, thereby moving from facts and procedures to concepts and structures. A relevant problem should encourage students to make various assumptions and use various strategies in which technology can serve as an aid but never as a goal. The problems teachers choose also need to provide the students with opportunities to express what they have learned in the course and in previous courses.

Hypothesis

To teach problem solving in a specific way, will help students to appreciate and enjoy mathematics and eliminate the emotional, psychological, and cognitive barriers to problem solving.

TEACHING

Almost every course in mathematics I teach, every lecture I give, starts with the introduction of a problem and a discussion about how to solve it. In my perspective, mathematical problem solving involves much more than the routine use of algorithms. In the best of situations, problem solving should engage students in rich and complex tasks that require them to think, reason, communicate, and apply their understanding of previous mathematical knowledge such as number concepts, operations, algebra, basic geometry, and so forth. In general I try to introduce the students to fundamental ideas: the importance of mathematical reasoning and proof, to search for patterns and try to generalize, and of sustained mathematical investigations where my problem functions as starting points for serious explorations, rather than tasks to be completed directly. Sometimes I start with problems that may be solved in different ways, in order for the students to understand the richness of alternative approaches when solving mathematical problems. One strong reason for my way of teaching is that I want the students to be active.

George Polya's first principle in teaching is what he defined as active learning: "let the students *discover by themselves as much as feasible* under the given circumstances" (Polya, 1981, p. 104).

> What the teacher says in the classroom is not unimportant, but what the students think is a thousand times more important. The ideas should be born in the students' minds and the teacher should only act as midwife. (Polya, p. 104)

There are several ways to get more active students and to involve them more in the problem solving process; to challenge them to solve problems in different ways, to encourage them to put together solutions that might be presented to the whole class, to ask them to invent similar problems which address the same mathematical areas, and so forth. Consider the following problem:

> *Angela has 2 one liter bottles. Bottle A contains 1 liter pure orange juice, bottle B is empty. She pours some of the orange juice from bottle A into the empty bottle B. Thereafter she fills up bottle B with water, so that bottle B is full and shakes it so the liquid is well mixed. Finally she fills up bottle A with the mixture from bottle B, until bottle A is full again. Calculate the smallest possible amount of orange juice in bottle A after this process.*

The intuitive solution

This is a problem where it might be useful and helpful to make some kind of sketch or drawing of the bottles and the mixing. When doing that, one might start to think about the most extreme possibilities – like for example "what happens if I pour *all* the orange from *A* into *B* or if I pour *no* orange juice from *A* into *B* at all? In both cases, I will end up with 100 % orange juice in bottle *A* afterwards."

Then it is likely that the largest amount of orange juice that we lose is in the middle between pouring *all* or *nothing* – when pouring half of the orange in bottle *A* to bottle *B*. After mixing the orange juice in bottle *B* with water, we will get back at least a quarter of a liter orange juice (half of the half liter that is poured back) and the logical conclusion is that we always ends up with minimum 75 % orange juice in bottle *A* afterwards.

The formal solution

Most students are quite unsatisfied with this reasoning; they seem to long for a more formal solution – sometimes expressed as "we want real mathematics involved." Such reasoning could be developed along the following kind of a more algebraic discussion. Presume that Angela pours over x liter orange juice to B from A. Then bottle B contains x liter orange juice and bottle A contains $1 - x$ liter orange juice. After mixing the orange with water in bottle B, every part of the mixture in bottle B will contain ax liter of the orange juice from A. Angela pours back as much fluid as she took (the amount x liter mixture) which means that $a = x$ and that bottle A therefore contains an amount of orange juice equal to $y = 1 - x + x^2$ which can be written $y = (x - 1/2)^2 + 3/4$ yielding that y is at least ¾ or 75 %.

The strict solution

Next step could be to use derivatives to ensure the minimum value amount that way. Since $y' = -1 + 2x \Rightarrow y' = 0 \Leftrightarrow x = 1/2$. This is a further development and refining of the formal solution, and some students are attracted to use sophisticated tools like derivatives when possible. They seem to consider this formal approach more trustworthy, as if they do not really accept the informal solution as true.

Extension

Some students took the problem further and discussed whether the size of the bottles and the bottle volume matters? What happens with the solutions if the bottles are of different size?

TEACHING AND LEARNING

Nevertheless, in my class there were enough students who favoured solutions in all these three categories to get a good discussion about the problem solving process, and about the importance of being able to acknowledge both intuitive and formal approaches to a problem. To be able to switch between an intuitive and formal approach as the students with the orange juice problem did, forms a good example of how students structures their mathematical knowledge. Fischbein has acknowledged the vast importance of students learning to master the interplay between formal mathematics and intuitive ideas.

> One of the fundamental tasks of mathematical education – as has been frequently emphasized in the present work – is to develop in students the capacity to distinguish between intuitive feelings, intuitive beliefs and formally supported convictions. In mathematics, the formal proof is decisive and one always has to resort to it because intuitions may be misleading. This is an idea which the student has to accept theoretically but that he has also to learn to practice consistently in his mathematical reasoning. (Fischbein, 1987, p. 209)

Benefits and obstacles

It is of course difficult to engage students during a 90 minutes lecture, to get them organized in active discussions, working in groups with solving a mathematical problem, and at the end of the lecture get everyone back in class to focus on a mutual subject, on different approaches and solutions, and to discuss experiences and benefit of this experience. Compared to the amount of concepts, methods and topics that could be covered within an "ordinary lecture" it sometimes honestly seems like a wasted 90 minutes time slot.

Nevertheless, the benefits have proved to be substantial, and in some cases surprising, as shown by the following example. In a group of 20 elementary teachers who experienced this kind of teaching during the academic year of 2004/2005, five prospective elementary teachers decided to take another semester of more advanced mathematics as an extra working load in their program, probably the best evidence anyone could get that one's teaching strategy works. Other students concluded that when they defeated their own fear of failure, the problem solving process became both engaging and rewarding. My hypothesis proved to hold.

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A STRATEGIES APPROACH TO MENTAL COMPUTATION FROM YEARS 1 TO 10

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Many countries are moving from an over-emphasis on written computation to a greater emphasis, particularly in the early stages, on developing mental computation through a strategies approach. For the past five years the author has been working with schools in Australia on a project funded by the federal government and two states to develop an organised approach to mental computation from the basic addition and subtraction facts to decimals, fractions and percents. This paper describes the background and gives a brief account of the research and curriculum development aspects of the project.

INTRODUCTION

In schools in Australia, as in most schools worldwide, written computation has traditionally held a more important and more central place than mental computation in the primary school curriculum. As Hope (1986) remarked, "computational facility is [traditionally] understood to mean proficiency with the conventional paper-and-pencil algorithms" (p. 47).

Thirty years ago, in almost any primary school classroom, the great majority of the mathematics time was spent on teaching children written arithmetic, preceded by a short session of 'mental arithmetic', consisting of tests of speed and accuracy, mainly of multiplication tables and basic facts, which one needed in order to perform written computation efficiently (Biggs, 1967).

However, as early as 1908 Branford wrote of the 'incomparable superiority of mental over written calculations'. 'Written arithmetic' he said, 'is not to be introduced until the pupil's mind sees the necessity for it in the difficulty of registering more than a certain amount in the memory' (Branford 1908). In more recent times the prior importance of mental over written computation has been asserted more frequently and more widely (for example, McIntosh, 1990; Plunkett, 1979; Reys, 1984; Wandt & Brown, 1957).

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CHANGES

The impetus for the changes in outlook comes from a number of different quarters. First, research into the computational practices of adults in their everyday lives suggests that at least three-quarters of all calculations done by adults are done mentally (Northcote & McIntosh, 1999; Wandt & Brown, 1957).

Second, the availability of cheap electronic calculators actually increases our need for mental calculations, because of the necessity of checking calculator results.

Third, there is plenty of evidence (see, for example, Kamii & Dominick, 1997) that many children simply do not trust or understand formal written algorithms.

The fourth reason for the change in emphasis in relation to mental computation follows from a general move towards a constructivist approach to teaching, informed by much more interest in, and efforts to understand, children's self-devised strategies for calculating mentally. McIntosh, Reys, and Reys (1992), Markovitz and Sowder (1994), and others have singled out mental computation as both an important ingredient of number sense and an important indicator of its presence.

OFFICIAL STATEMENTS

These changing currents of opinion and the evidence of research are increasingly reflected in the national and state documents of many countries. In the United States *The Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics, 1989), and in the United Kingdom *Mathematics in the National Curriculum* (Department of Education and Science, 1991) both stressed the equal importance of ability with the calculator and with mental and written computation.

In Australia mental computation is emphasised in current state and federal curriculum documents both as a critical component of functional numeracy, and as an effective means of developing number sense in students (see for example Australian Education Council, 1991).

In Norway, the draft 2006 mathematics curriculum states that after Year 2 the students should be able to 'develop and use a variety of calculation strategies for addition and subtraction of two-digit numbers'. After Year 7 they should be able to 'develop and use methods for mental calculation, estimation and written calculation'.

These are however all very general statements. Even within the detail of the succeeding paragraphs, very little guidance is given, and in some official documents no guidance at all is given, as to the range of computations that students can be expected to calculate mentally, or how the teaching of mental computation is to be approached. In general, little research has been carried out into the mental computation ability of children in Australia or elsewhere, beyond the basic number facts.

A NEW APPROACH TO TEACHING MENTAL COMPUTATION

Since the mid 1970s McIntosh, along with others, has been exploring the teaching of mental computation in the context of a number-sense-based approach to computation. This led to the development of some 'key formats' for new kinds of mental computation sessions (McIntosh, De Nardi, & Swan, 1984) that emphasised discussion of strategies used by children and validated a variety of possible methods. A succession of books (McIntosh, Reys, Reys, & Hope, 1996; McIntosh, Reys, & Reys 1996a, 1996b, 1996c) also detailed lessons designed to develop number sense through involving students of all ages in choice and discussion, with an emphasis on mental computation.

These and other materials provided teachers with a variety of ideas and formats for one-off lessons incorporating a more flexible approach to mental computation. However they did not provide support for a coherent developmental approach to mental computation through the years of compulsory schooling.

From 2001 to 2003 the author has led a team at the University of Tasmania that has worked with project schools to produce developmental and sequenced teaching materials for teachers to use to develop mental computation with whole numbers, fractions, decimals, ratio and percents. The project, *Assessing and Improving the Mental Computation of School-age Children,* which was run in collaboration with the Department of Education, Tasmania, the Australian Capital Territory Department of Education and Training, and the Catholic Education Office, Hobart, received federal funding under the SPIRT (Strategic Partnership with Industry – Research and Training) scheme. The project had two main aims – a research aim and a curriculum development aim.

THE RESEARCH

The research aim was 'to describe levels of achievement in mental computation'. Tests were administered to 3035 students in Years 3 to 10. Four separate but overlapping tests were administered, for Years 3/4, 5/6, 7/8, 9/10. For uniformity the tests were administered orally only on a CD by the class teachers, and answers only were recorded by the students. All responses were placed on a single scale using Rasch modelling techniques. An eight-level scale of mental computation development was identified, and student achievement across the grades was mapped onto these levels. More details can be found in Callaghan and McIntosh (2001).

CURRICULUM MATERIALS

The curriculum development aim was 'to provide sequential modules of activities for improving the mental computation of students in grades 3 to 10'. These materials were developed and trialled in six schools over three years. The final publication consisted of six modules, covering respectively introductory material, basic addition and subtraction facts, basic multiplication and division facts, two-digit computations, fractions and decimals, and ratio and percent. The materials are published by the Tasmanian Education Department.

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MATH CLUBS
– A DOOR OPENER TO MATHEMATICS

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At the Norwegian Center for Mathematics Education we have an ongoing project developing math clubs and looking into several aspects of the effects of participation among the children and youngsters. Two master studies with different focuses on the math clubs will be presented briefly in the workshop. The math clubs have been designed for pupils 5, 9 and 14–15 years old. In the workshop we will focus on the target groups for the master studies, 5 year olds and 14–15 year olds. For the presentation we will set the stage of the math clubs, the theory behind the studies, as well as the main results. We will show pictures and give examples of the activities that were designed for the club. The audience will take part in the activities, and we will challenge them to try to characterize the tasks and challenges the pupils are given.

MATH CLUBS – WHAT AND WHY

The Norwegian Center for Mathematics Education has during the academic year 2003–2004 arranged math clubs for:

- Five year olds
- 5. grade (10 year olds)
- Junior High (14 year olds)

In this presentation we will look at the five year olds and the fourteen year olds. These groups were subjects for the research carried out by Settemsdal and Sivertsen.

When participating in a math club a pupil will experience mathematics in another setting, another situation and context than in regular school mathematics. By giving them this opportunity, our hypothesis was that the children would develop a positive attitude towards mathematics, become more aware of the mathematics around them, and see the relevance of mathematics in their own lives. The activities were designed and carried out on the pupils' premises.

2007. In C. Bergsten, B. Grevholm, H. S. Måsøval, & F. Rønning (Eds.),
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The clubs had several purposes. We wanted the pupils to experience the diversity in mathematics, to find their own strengths and become aware of what they are good at. We wanted them to feel pleasure and experience success, and stimulate their willingness to learn mathematics, also within the school.

The selection of children and the atmosphere at the clubs

The five year olds are children from a pre-school close to the university. There is no reason to believe that they are a special group of children, but represent an average group of five year olds in Norway. They came to the clubs in a math room at the centre approximately once a month together with one to three teachers. The teachers were asked not to help the children with the tasks.

About 30 pupils from two different schools came to the math club after school once a month. They came on a voluntary basis, and from this one might conclude that it was a special group of pupils, very interested in mathematics. From interviews we know that this was not the case. Some of them came because their parents wanted them to go, some because they thought they would be better at mathematics, some because they wanted to meet friends, and some because mathematics was their special interest. They spent two hours at the club each time. No teachers followed these pupils.

For both groups we had a meal during the gathering, to create a nice atmosphere and let them relax for a while. The same children came every time, so they got to know each other and felt like a group. Three or four researchers and teachers from the centre were present each time. We list some important principles that were followed:

- The adults showed clearly that they enjoyed working with the children/pupils
- The children/pupils were taken seriously and were given the impression that the adults were genuinely interested in hearing about their mathematical ideas
- The activities were challenging for all, but easy to approach
- The children/students learned to believe in themselves

Focus for the research projects

The study of the five year olds (Sivertsen):

The central purpose of the study has been to find out which factors in communication make it easier to uncover the learning potential of each child, and how the mathematical communication competence and the mathematical language develops through interaction between the children and adults and among the children in the math clubs. The math club became a good arena for looking at different people with various mathematical backgrounds who communicated mathematics with the children. This included people with a doctor's degree in mathematics to people with no education above college level, who said that they always had thought of mathematics as a "hard" subject.

The study of the fourteen year olds (Settemsdal):
Many students lack motivation to learn mathematics, and many of them experience the transition from junior to senior high school as problematic. Why is it so? In an attempt to find an answer, theories on motivation and beliefs were studied, and theories about student active teaching and learning, and problem solving.

The focus of this research was:

- Does participation in math clubs give the students better motivation to learn mathematics at school?
- Will the math club experiences affect the students' attitudes towards mathematics as a subject?
- Can math club activities be used in the mathematics teaching and learning at school?

At the clubs we have student active teaching and learning. That means we want the students to explore and wonder. We focus on the mathematical dialog, questions that make the students think. It is important never to give them the answer, but ask "What if….?" And "Why…?"

Before the clubs started the students were asked to write down the first association they get when someone say mathematics. Their answers were sorted in positive, negative and neutral statements. Answers like "calculation", "numbers", "boring" and so on were received, and only one of them was positive. After 10 meetings at the club they were asked: "What will you answer when someone asks you what mathematics is?" 8 out of 23 answered, "Mathematics is everything!" This is very positive, and we think this indicates that they have experienced mathematics as more than numbers and a school subject.

Questions to reflect upon

During the workshop the participants were engaged in some activities, to experience what the clubs were like. We encouraged them to reflect upon the following questions while working with the tasks:

What makes the children/students enjoy working with the mathematics problems at the clubs? Did it have to do with the atmosphere? – the adults? – the tasks? – other factors?

Activities

The activities presented at the workshop were:

- Today our number is 100
- Time
- Symmetry and kites
- Probability
- Hexa-flexagon

TEACHERS' AND STUDENTS' PERCEPTIONS ABOUT THE NEWLY DEVELOPED MATHEMATICS CURRICULUM: A CASE FROM TURKEY

Gülçin Tan
Middle East Technical University, Turkey

The purpose of the study is twofold: (1) to find out how the 5th grade classroom teachers and the students perceive the newly developed mathematics curriculum (NDMC), and (2) to investigate how the NDMC is put into practice. The study was conducted with a sample of sixty-five 5th grade students and two 5th grade classroom teachers from one of the pilot schools in Ankara, Turkey. The data were collected both qualitatively and quantitatively. The results indicated that most of the students perceived learning mathematics as enjoyable and easy in the NDMC. For the teachers, the NDMC reflects a constructivistic approach and provides opportunities for the students to learn mathematics with understanding. Generally, both the teachers and the students are satisfied with the NDMC.

INTRODUCTION

It is obvious that there have been continuous changes all over the world in both the economical and social sense. These changes influence almost everything around us, our culture, economy, values as well as the education system. The needs of the information age and the influences of globalization force nations to improve their education systems, which is regarded as one of the most influential changing agents for development. Since the foundation years in 1923, education has been regarded as a means to catch up with the modern civilization in Turkey. The general aim of the Turkish educational system is to raise students as highly skilful individuals who have improved thinking, perceiving and problem solving abilities, are open to universal values and new ideas, have a strong national culture, personal feelings of responsibility and social sensitivity, and are inclined to production of technology and science (MONE, 2004). Nowadays, there are various attempts in Turkey to improve the educational system and to fulfil the society's needs and expectations. The newly developed curricula for the first level of primary education can be seen as one of these efforts. At the end of the curriculum development activities have been carried out according to national needs and values, taking contemporary scientific-technical data and also the differing interests, wishes, and capabilities

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of the students into account. The new curricula were pilot-tested in 9 provinces of Turkey during the 2004–2005 academic year. The new curricula are based on a constructivistic philosophy and this view is also reflected in the newly developed mathematics curriculum (NDMC, 2005). The previous mathematics curriculum reflects a behavioristic approach and teacher-centered instruction. Because of the emphasis on procedural understanding, a rote learning environment was created. This curriculum had a characteristic where the students' needs, interests, differences, and real life experiences were not taken into account and all students were expected to perform in the same manner. The students were also not expected to make use of and develop their critical thinking, reasoning, or decision making skills. On the other hand, the NDMC gives special attention to the construction of conceptual bases of mathematical knowledge and to connections between conceptual and procedural knowledge. The content, the learning opportunities, the instructional strategies and assessment methods are organized in line with the student-centered approach. It provides opportunities for students to construct their own knowledge based on what they know and experience. Further, the development of students' problem solving, reasoning, and critical thinking skills are highly emphasized issues in the NDMC. The connection between mathematics and daily life is also taken into consideration in order to make mathematics more meaningful for the students.

Methodology

In Ankara, there are 25 pilot schools in 22 different provinces. Among them, one of the pilot schools was selected purposefully. One of the reasons to select this particular pilot school was that it is designed for the implementation of student-centered education and is the first public school that serves this purpose in Turkey. It opened its doors for the students in the 2004–2005 academic year. Another reason was that this pilot school has more opportunities than the other public schools in terms of instructional materials and technological devices. In addition, all of the teachers and the principals in this school were selected by the committee from the Board of Education and Discipline through individual interviews. In this school, there are two 5th grade classrooms and the total number of 5th grade students is 70. Among them, sixty-five 5th grade students (32 female and 33 male) and two 5th grade classroom teachers (1 female and 1 male) participated in this study voluntarily. Both of the teachers have a 4-year bachelor's degree from The Faculty of Education, and they have 7 and 15 years of teaching experience respectively.

The data were collected through interviews with the two 5th grade classroom teachers. Further, detailed classroom observations were made through the use of an observation form by the researcher. In order to check reliability, two lessons were observed by two researchers and inter-scorer reliability was .89. The Mathematics Attitude Scale developed by Aşkar (1986) was also used to determine students' attitudes toward mathematics. In order to find out about students' opinions about the implementation of the NDMC, they were administered a questionnaire developed by the researcher. Further, the data derived from interviews and observations were transcribed verbatim and then subjected to qualitative data analysis techniques where codes and categories were formed. For the quantitative data derived from the attitude scale and the questionnaire, the statistical analysis was carried out by using the SPSS-PC software program.

RESULTS

The results of the interviews with the teachers indicated that the NDMC has the characteristics of student-centered instruction and a constructivistic approach. According to them, the content of the NDMC is organized in such a way that it meets the needs and the interests of the students and it is appropriate to the students' cognitive level. The principles of learning such as from simple to complex, from concrete to abstract are also taken into account. The teachers thought that the mathematical concepts are well connected with other courses and with real life, and also that inappropriate topics that exceed the cognitive level of students are excluded from the NDMC. For the objectives, the teachers reported that the objectives were determined by considering the complexity of mathematics, needs, interests and the cognitive levels of the students. When the teachers were asked how they transfer the constructivistic approach into their mathematics lessons, both of them said that the transformation is really difficult and takes too much time for mathematics lessons. Since the textbooks reflecting the constructivistic approach have not been published yet, one of the most used materials is the teacher-prepared activity sheets and also projects that provide opportunities for the students to construct their own knowledge. In order to enrich the learning environment, they use concrete materials, models and technology such as OHP, video, and computers. One of the teachers said: "We celebrated one of my students' birthday in math lesson and I taught fractions using birthday cake. While learning the concepts of litre, we made lemonade in our laboratory class."

Further, the teachers reported that evaluation became an integral part of instruction in the NDMC. However, the evaluation and assessment instruments suggested in the curriculum guide were not practical and useful, because of limited time and crowded classrooms. They reported that classroom observations provide useful information about students' progress. In addition, they use portfolios, projects, and pen-paper tests. Further, the results of the observations indicated that the role of the teacher is facilitator and organizer. Both of the teachers tried to ask open-ended, probing questions that encourage the students to share their knowledge and experiences with other members of the class. It was observed that discovery learning, investigation, cooperative learning and discussion were the most frequently used instructional methods and that the students were active, happy, willing to participate in the activities and to share their ideas during the math lessons. According to the results of the attitude scale, the students have positive attitudes toward mathematics. The total scores of students ranged from 51 to 100; the average score was found to be 83. A majority of the students (93 %) described their mathematics lesson as very enjoyable, easy to understand and well connected with the real life. They reported that during the mathematics lessons, they made different kinds of activities such as designing holiday villages by using geometric shapes, making kites, lemonade and playing games. The majority of them (88 %) believed that by doing such activities, they not only can learn mathematics better, but also will like mathematics better. Most students (84 %) stated that they did not like mathematics lessons in their previous schools, as they were too boring. According to them, these lessons were based on copying the teachers' writings in their notebooks and the students tried to memorize all formulas and concepts from their textbooks, rather than learning by understanding. One of the students said: "In my previous school, we learned mathematics just from the textbooks and we were told to write formulas and then memorize them, but now, we are learning mathematics with worksheets, computers, playing games etc."

When the students were asked to describe the mathematics lessons in their dreams, most of them (91 %) stated that the math lesson in their dreams is like in the current ones. However, they complained about the time devoted to the lessons which is insufficient to complete the task or to discuss the topics and some of them would like to make more exercises and to use textbooks.

DISCUSSION AND CONCLUSION

Like in other countries, there are various attempts in Turkey to improve the educational system and consequently to fulfil the society's needs. It can be said that the newly developed curricula for the first level of primary education is one of the most important attempts in this area. By considering its importance, this study aims not only to find out how the 5th grade classroom teachers and the students from one of the pilot schools perceive the NDMC but also to investigate how the NDMC is implemented. At this point, the results are limited, but the findings indicate that both the teachers and the students in this school are satisfied with the NDMC and that ideas of student-centered instruction and the constructivistic philosophy are implemented successfully. The following suggestions can be drawn from the study: a) the evaluation and assessment methods and instruments should be revised in terms of their practicality and usability, b) the time devoted to the lessons should be increased, c) the textbooks reflecting constructivism should be published, d) the continuous in-service training programs should be offered for the teachers.

In spite of some weaknesses where there is a need for improvement, the NDMC is an obvious attempt to help young children learn mathematics with understanding and appreciation of its role.

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SCIENTIFIC PROGRAMME

Friday 2nd September

15.00-17.00 Registration. Coffee and tea available

17.00-17.30 Opening Session:
Music
Welcome addresses by
– Torunn Klemp, Vice Chancellor of Sør-Trøndelag University College
– Barbro Grevholm, chair of the international programme committee

17.30-19.00 **Plenary lecture.** Mogens Niss: *The Concept and Role of Theory in Mathematics Education*

Saturday 3rd September

09.00-10.30 **Plenary lecture.** Simon Goodchild: *Students' Goals in Mathematics Classroom Activity*

10.30-11.00 Coffee break

11.00-12.00 **Paper presentation 1.** A: Erfjord and Hundeland
B: Kristinsdóttir, C: Kleve, D: Skott and Wedege

12.00-13.30 Lunch

13.30-14.30 **Paper presentation 2.** A: Fuglestad, B: Pálsdóttir, C: Gade,
D: Schlöglmann

14.45-15.45 **Paper presentation 3.** A: Berg, B: Mosvold, C: Persson,
D: Knudtzon

15.45-16.15 Coffee break

16.15-17.15 **Paper presentation 4.** A: Hundeland, Erfjord, Grevholm, and Breiteig, B: Iversen, C: Bjarnadóttir

Sunday 4th September

09.00-10.30 **Plenary lecture.** Birgit Pepin: *Making Connections: An Exploration of Connections Made in and 'around' Mathematics Textbooks in England, France, and Germany*

10.30-11.00 Coffee break

11.00-12.00 **Paper presentation 5.** A: Kislenko and Grevholm, B: Carlsen, C: Måsøval, D: Fyhn

12.00-13.30 Lunch

13.30-15.00 **Short communications (3x20 min).**
A1: Ambrus, Hortobágyi, Näätänen, Liira, and Salmela,
A2: Tan, A3: McIntosh
B1: Taflin, B2: Lingefjärd, B3: Hag
C2: Kudzma, C3: Lakka
Workshop (60 min).
D: Settemsdal, Sivertsen, and Stedøy

Monday 5th September

09.00-10.30 **Plenary lecture.** Markku Hannula: *Understanding Affect towards Mathematics in Practice*

10.30-11.00 Coffee break

11.00-12.00 **Paper presentation 6.** B: Hansson, C: Braathe, D: Wæge

12.00-13.30 Lunch

13.30-14.30 **Paper presentation 7.** A: Askevold, B: Juter and Grevholm, D: Hansen

14.45-15.45 **Workshop on classroom research.** Simon Goodchild, Heidi S. Måsøval, and Barbara Jaworski: *Learning from students' talk in Mathematics classroom*

15.45-16.15 Coffee break

16.15-18.00 **Workshop on classroom research** continues

Tuesday 6th September

09.00-10.30 **Plenary lecture.** Barbara Jaworski: *Learning Communities in Mathematics (LCM): Research and Development in Mathematics Teaching and Learning*

10.30-11.00 Coffee break

11.00-12.00 **Paper presentation 8.** A: Bjuland, B: Lauritzen and Dysvik, C: Høines and Lode

12.00-13.30 Lunch

13.30-14.30 **Paper presentation 9.** A: Hedrén, B: Kristjánsdóttir

14.30-15.00 Closing session

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