Challenges in Teaching Mathematics

Becoming special for all

NORSMA[5]

Conference proceedings from
the fifth Nordic Research network on
Special Needs Education in Mathematics

Editors: Hafdís Guðjónsdóttir, Jónína Vala Kristinsdóttir and Edda Óskarsdóttir · Design: Hringbrot
 · ISBN 978-9935-9025-0-4 · University of Iceland - School of Education · Reykjavík 2010

Conference proceedings from the fifth Nordic Research network on Special Needs Education in Mathematics, NORSMA5: Challenges in teaching mathematics: Becoming special for all.

Editors: Hafdís Guðjónsdóttir, Jónína Vala Kristinsdóttir og Edda Óskarsdóttir
Design: Hringbrot

ISBN 978-9935-9025-0-4

University of Iceland - School of Education

Reykjavík 2010

Table of Contents

Preface

The Nordic Research network on Special Needs Education in Mathematics, NORSMA, was established in 2003 by professionals working within the fields of mathematics and special needs education. The network has held conferences every other year in the Nordic countries focusing on research on teaching students with difficulties in learning mathematics. The fifth NORSMA conference was held at the University of Iceland, School of Education in Reykjavík, October 14th to 16th 2009.

The theme of the conference was: *Challenges in teaching mathematics: Becoming special for all,* and was chosen to help us focus on the challenges we meet while educating teachers to teach students in inclusive schools. It is widely recognized that the greatest challenge teachers meet today is responding effectively to the increasing number of diverse learners.

The conference was divided into four themes:

- Creating a learning environment
- Teacher Education
- From theory to practice
- Becoming special

For each theme there was a plenary session and a number of paper presentations. The themes were discussed from different sites of interests' fields and an effort was made to get viewpoints from various fields of special interests such as psychology, special and mathematics education.

The opening plenary session: *There is nothing as practical as a good theory* was held by Dr. Robert Siegler professor at the Carnegie Mellon University in the United States. He discussed theoretical analyses of the development of numerical representations that suggest that playing linear number board games which ground learning in sensorimotor activity enhance young children's numerical knowledge.

The second plenary session: *Concerns about the students in the "gray zone" of mathematics learning* was held by Professor Anna Kristjánsdóttir at the University of Iceland and the University of Agder. Her focus was on what results from general mathematics education research can offer students meeting difficulties in their mathematics learning.

The third plenary sessinon was held by Dr. Barbara Jaworski, professor of Mathematics Education in the Mathematics Education Centre at Loughborough University. In her lecture: *Special for all, special for one: developing an inquiry culture in mathematics teaching* she discussed ways in which teachers' creation of an inquiry culture in the classroom can lead to a celebration of diversity and a recognition of difference.

The final plenary session was held by Dr. Hafdís Guðjónsdóttir, associate professor at the University of Iceland. In her lecture: *Preparing and supporting mathematic teachers to become special for all* she focused on teacher education and discussed how teacher educators can support mathematic teachers to teach mathematics to a diverse group of students.

There were 61 participants at the conference, 22 presentations and four posters reflected different matters and opinions from various perspectives. Teacher workshops, with 72 teachers, were held before the conference in collaboration with some of the participants of the conference.

This publication is a collection of 11 papers from the conference. We received 35 preliminary proposals of which 30 were accepted for presentations and the abstracts published in the conference program. We also offered all presenters to turn their paper in for publication in the conference proceedings on the Web: http://stofnanir.hi.is/norsma/

The peer reviewing process

The editorial process for the proceedings for the conference occurred in two stages:

1. A double blind review process was used for short preliminary proposals, and feedback provided to authors. Reviewers evaluated and scored the submissions based on a review criteria and submission requirements. Reviewers were expected to provide substantive comments as well as scores.

2. Authors whose proposals were accepted for presentation at the conference were then invited to submit a longer paper for inclusion in the proceedings. A second double blind review process was used to assess submitted papers and guide the development.

Special for all, special for one: developing an inquiry culture in mathematics teaching and its development

Barbara Jaworski
Loughborough University, UK and University of Agder, Norway

This paper addresses the achievement of a mathematical classroom learning environment that provides for each individual student within a pedagogy for all, and works overtly to accommodate difference. It begins with three narratives from research which set the scene for the kind of activity that is envisaged and goes on to describe a research project which had as its aim the development of such activity in classroom settings. The research described is located within a sociocultural perspective in which *communities of practice* in school and university settings form a basis for the constitution of a *community of inquiry* between teachers in eight schools and didacticians at a university in Norway. The concept of inquiry at three levels is interpreted in the project to achieve desired activity in classrooms. Outcomes indicate the importance of developing relationships between teachers and didacticians that include all within a community of inquiry and accommodate difference in ways that result in learning for all.

1. Introduction

1.1 Special for all, special for one

My title says, deliberately, "Special for all, special for one", with the *all* coming before the *one*. I contrast this with an alternative, "special for one, special for all", which might be seen as a more common phrase. Putting the *all* first is extremely important in designing mathematics teaching which creates opportunity for all students at any level, whatever their needs, to engage successfully with mathematics. Creating such opportunity in classrooms is a challenge for a mathematics teacher and for the development of teaching that seeks such provision.

This paper has developed from an invited presentation at the NORSMA[1] conference in Iceland. I began this presentation with a sequence of four short video clips showing different teaching-learning settings in mathematics. They were *different* in a number of ways. One was from an English setting and three were from Norwegian settings. The settings were a kindergarten, a primary school, a lower secondary school and an upper secondary school. In two cases we saw a teacher working with a small group of students, in two the teacher worked with the whole class. In all, despite these differences, there was opportunity for students to engage in dialogue with the teacher and other students to express mathematical ideas relative to the level of education. All of them showed the teacher creating opportunity for all the students in her or his class from which individuals could achieve.

Thus, in the matter of *achievement for the one* within *provision for the all* the idea of difference is important, for example the differences that we see in individual learners:

- *different learners have different ways of learning mathematics – different cognitive styles;*
- *different learners have different learning needs;*
- *different learners have different physical and emotional needs that affect learning.*

Such differences contribute to a diverse mathematics classroom setting in which it is important to address the individual within the all. Two ways of seeing this relationship are:

- *Special for each one: we attend to the special needs of each student, and therefore make the situation special for all.*
- *Special for all: we attend to the special needs of the whole group and hence make the situation special – provide opportunity – for each one in the group.*

[1] NORSMA is the Nordic Research network on Special Needs Education in Mathematics. Its 5th conference was held in Reykjavik from 14th to 16th October, 2009. http://stofnanir.hi.is/norsma/

The second of these suggests a pedagogy for all, that deals with difference and diversity. Research into neurodiversity in education – including differences like dyslexia, dyspraxia, dyscalculia, autism and ADHD– raises issues about how education is made inclusive for all of these 'special' needs. Pollack (2009, p. 7) writes " are we not all neurodiverse?" He writes further

> *If the goal of inclusion is to be attained, it will only be [attained] by considering the specifics of need as well as a pedagogy for all.*
> *(Pollak, p.7, citing Powell 2003, p. 6)*

A key question here concerns how we make the distinction; *how* we provide for difference within a pedagogy for all.

For example, differences in students' cognitive style may be seen in Chinn & Ashcroft's inchworms and grasshoppers: *inchworms* focus on parts and details, they separate ideas; *grasshoppers* tend to overview, to the holistic, putting ideas together.

> *You can usually go a long way to finding out how a child solves a problem by asking the simple question 'How did you do that?' This interest, based on awareness rather than a judgment, will be a major source of help for many students, especially when combined with an awareness of what the child brings to the question*
> *(Chinn & Ashcroft, 1998, p. 23-4)*

It is well known that many students at all levels experience difficulties with mathematics (Cockcroft, 1982) and that these difficulties create anxiety for the student (Tobias, 1993), perhaps from seeing the difficulties as their own fault, due to their own problems with the subject, problems not experienced by other students and not recognised by their teacher. The following quotation comes from a student, Jane, recorded by my colleague Clare Trott[2] (with Jane's permission) as part of her research on special needs experienced by students who came to her for help.

> *The feeling that I had when I was learning GCSE maths was very difficult. I couldn't understand the concept of the numbers between nought and one; I couldn't understand the fractions. It may seem very simple to people who are looking at this now, but for me it was very difficult and it caused me a lot of anxiety. I was trying to learn the maths and the anxiety got involved … it hampered my learning. I had to control the anxiety as well as learn the maths. … I felt a lot of frustration. Its unfair that I can't understand these basic maths, I should be able to but I just couldn't do it.*

How can a teacher at any level provide the kind of environment and opportunities that enable someone like Jane to learn mathematics, and how might this be done as part of a pedagogy for all?

1.2 Inclusion and diversity

Starting from a Vygotskian position that all learning is social and that individuals internalize from their engagement in sociohistorically rooted settings, the difficulties that learners experience in mathematics can be seen to relate to ways in which society, schooling, and mathematics have taken on their current characteristics (Daniels, 2001; Vygotsky, 1978). For example, schools in England organize students into ranked sets based on their mathematical achievement (Boaler and Wiliam, 2001); it is common for people to acknowledge that mathematics was their worst subject at school and that they found it boring and intractable (Cockcroft, 1982). Setting achieves a broad separation of students according to their ability to work within the system and, as Boaler and Wiliam show, results in alienation from mathematics for many. The quotation from Jane above shows just one example. Nardi and Steward (2003) report from typical English classrooms that classroom mathematics is T.I.R.E.D; their study shows that students who are 'quietly' disaffected exhibit characteristics of *t*edium, *i*solation, *r*ote learning, perceptions of mathematics as *e*litist and *d*epersonalization from mathematics. Thus the systems of setting and approaches to teaching mathematics that are common in English classrooms are ignoring diverse needs and alienating a wide range of students.

---

[2] Clare Trott directs the *Eureka Centre for Mathematical Confidence* at Loughborough University in which she works with additional needs students including those diagnosed as dyslexic or dyscalculic (e.g., Trott, 2008).

The literature suggests that inclusive approaches to learning and teaching mathematics need to respect and celebrate diversity, recognize and value difference, maintain awareness of anxiety, and challenge students appropriately (e.g., Chin & Ashcroft, 1998; Tobias, 1993). Ollerton & Watson (2001) draw attention to the 1999 National Curriculum for England in Mathematics which sets out “three principles for the development of an inclusive curriculum: setting suitable challenges, responding to diverse needs and overcoming potential barriers to learning and assessment” (p. 3). Ollerton and Watson go on to say, “Given that a mathematics qualification is an important passport to higher education and further social and economic opportunity, it is especially important that mathematics teachers do not limit the possibilities for their students.”

Skovsmose and Säljö (2008) refer to “an exercise paradigm” as dominant in the culture of mathematics classrooms widely and limiting opportunity (p. 40). They write:

> *This [the exercise paradigm] implies that the activities engaged in the classroom to a large extent involve struggling with pre-formulated exercises that get their meaning through what the teacher has just lectured about. An exercise traditionally has one, and only one, correct answer, and finding this answer will steer the whole cycle of classroom activities and the obligations of the partners involved ... (p. 40).*

In contrast to the exercise paradigm they propose that a focus on mathematical inquiry open up possibilities:

> *The ambition of promoting mathematical inquiry can be seen as a general expression of the idea that there are many educational possibilities to be explored beyond the exercise paradigm (p. 40).*

For educators, teachers and school organizers, therefore, it seems important to address what it means to include, to recognize and value, to respect and celebrate, to challenge all students at appropriate levels and to approach mathematics through inquiry. How is a teacher in any setting at any level to achieve such provision? The next section includes three examples, three narratives drawn from studies of classroom data, that illuminate these questions and suggest a basis for a pedagogy for all that celebrates diversity.

1.3 Examples of addressing diversity in mathematics classrooms

1.3.1 Turning “I can’t” into “I can and I did”

This narrative comes from a project conducted jointly between the Open University and the Mathematical Association in the UK in the early 1980s: *Working Mathematically with Low Attainers*. Mathematics educators from the Open University worked with teachers in a number of schools to focus on ways of teaching low attaining students to enable mathematical achievement. A published videotape (Open University, 1985) resulting from the project had the title “Turning I can’t into I can and I did”.

In this video compilation we see a teacher, working with a class of students on the problem “If a number of circles intersect in a plane, how many regions can be created?”. The teacher had taken into the classroom a set of “hoola hoops” which she and the students used to represent circles in real space. Some students used the hoops, others drew circles on the board or in their books, arranging their circles to try to find the maximum number of regions for a given number of circles. In each case they counted regions and noted down their results: one circle, one region; two circles, three regions, three circles, seven regions, …

The case of 4 hoops is shown. Regions inside the hoops may be counted to reveal 13. An important part of the mathematics here is to justify that this is the maximum number of regions for 4 circles and to relate this to the number of circles more generally. Most students had addressed such questions and come up with convincing explanations, and the teacher had encouraged them to express their findings algebraically.

Table 1: Numbers of hoops and region

Hoops	Max. Regions
1	1
2	3
3	7
4	13
5	?
…	…

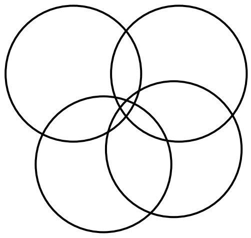


Figure 1 – Maximum number of regions with 4 circles

Then a student, Mary, approached, and showed the teacher the work in her book. She had drawn the following diagram (Figure 2)

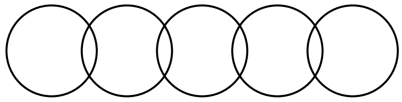


Figure 2: Mary's arrangement of circles

And her table was as follows:

Table 2: Mary's results

Hoops	Max. Regions
1	1
2	3
3	5
4	7
5	9
…	…

The teacher asked Mary to explain what she had done and what she had found, and seemed satisfied with Mary's response.
Then finally the teacher said:

"You're doing something different from everyone else, Mary. Don't worry, that's fine. Just ignore what everyone else is doing".

She asked Mary to write an algebraic formulation of her own case.

All students here had the opportunity to decide for themselves how to tackle the problem, with the teacher encouraging and challenging them to seek mathematical generalization.

This situation for all allowed Mary to do things differently. The teacher had to balance a wish to respect and value Mary's work with a wish for Mary to see a broader generality in the situation. The investigative situation allowed for such differences of approach and direction, but also raised questions about overall achievement in terms of the required curriculum and desired mathematical achievement.

1.3.2 Investigating mathematics teaching – a lesson on fractions

The second narrative comes from my own research (Jaworski, 1994, p. 92). A teacher Clare, was working with her whole class on *fractions*. She pointed to ½ (written on the board) and asked one girl, Katy, "What is one divided by 2?". Katy said "two". Clare asked Katy to work out on her calculator, "one divided by 2". Katy said "Nought point five". Clare asked, "Surprised?" "If you have one thing shared between two people, how much does each get?" Katy looked blank.

Clare then said to the whole class: "Can we have a hands-down-think. I did ½. I want you to think what you might do next." Then she went to talk with Katy.

Later in the lesson she was working with the whole class on 1/7 – dividing 1 by 7 to get a decimal representation – some students were finding this problematic. To the others in the class she said "Anyone who's ahead of this, try to think how to explain repetition in 1/7"
[1/7 = 0.142857142857142…].

The class was a mixed ability class in Year 8 (ages 12-13). Katy was one of a number of students who were struggling with the work on fractions, although there were other students in the class who could tackle more challenging questions. We see here a teacher differentiating according to the needs of different groups of students. The emphasis on *thinking* was typical of her situation; *all* students were required to *think* (before putting up their hands to give an answer). Those who understood the more basic questions were given more challenging problems to think about; *explaining* repetition in 1/7 is seriously challenging for students of this age. Students had to take some responsibility for which group to be in, whether to work further on the basic ideas or to tackle the more difficult problem. Thus the teacher encouraged students to think not only about their mathematics but about their degree of understanding. They learned to make choices within what was offered in the classroom. A challenge for the teacher was to maintain levels of achievement commensurate with students abilities and needs.

1.3.3. What shape is it?

This narrative comes again from my own research (Jaworski, 1988, p. 287). Look at the drawing in Figure 3. What is it? What shape is it?

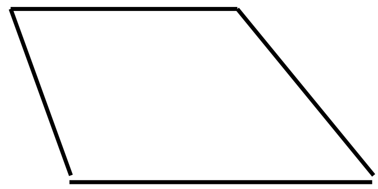


Figure 3: The teacher's original drawing

A Year 8 class had been asked by their teacher to name the shape, which he had drawn on the board. Someone said that it was a trapezium. Some students agreed with this, but not all. The teacher said, 'If you think it's not a trapezium then what is it?' Michael said, tentatively, 'It's a square …'. There were murmurings, giggles, 'a square'?! … But Michael went on '… sort of flat.'

The teacher looked puzzled, as if he could not see a square either. He invited Michael to come out to the board and explain his square. Michael did. He indicated that you had to be looking down on the square – as if it were on your book, only tilted. He moved his hand to illustrate. 'Oh' said the teacher. 'Oh, I think I see what you mean … does anyone else see what he means?' There were more murmurings, puzzled looks, tentative nods.

Then the teacher drew onto the original shape as in Figure 4.

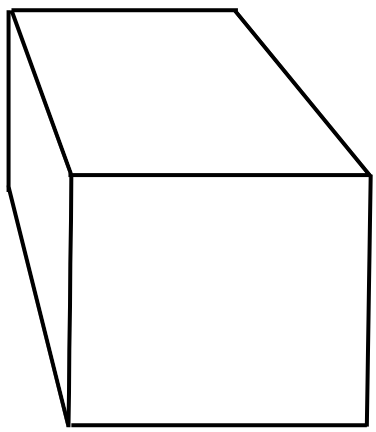


Figure 4: The teacher's modified drawing

Oooooh yes (!) said the students and there were nods around the class.

Here the teacher had to be prepared to suspend his initial plan for the lesson to accommodate Michael's special viewpoint. The result showed an enhanced vision for the whole class, special encouragement for Michael, and a contribution to an ethos of listening to and respect for others.

1.3.4 Tasks for all

Each of these teachers had put care and thought into their design of tasks for students. In each case, the nature of the task allowed everyone to make a start, diverse directions and ways of thinking, fluidity and flexibility in activity and serious mathematical thinking and outcome. The teachers' actions encouraged all students to participate, supported individuals who do things differently, provided extra support where it was needed and challenged all students mathematically at appropriate levels. We saw in each case a certain degree of *contingency* (Rowland, Huckstep & Thwaites, 2005), in which the teachers needed to respond and make decisions in the moment as to how to act. My observations in each classroom over a period of time suggested that these were not just serendipitous moments, but were a result of careful ethos building over considerable time. Environments in which such contingencies arise do not happen by chance or overnight; they need to be worked at overtly and nurtured by the teacher. Teachers also need visions of mathematics as an open and flexible subject in which all students can participate and in which challenges can be offered to deal with widespread needs. An inclusive classroom environment can enable student choice and responsibility and allow for the teacher to recognize and respond to needs as they arise and to work with them over time.

1.3.5 Demands on a teacher

The examples highlight the complex demands on a teacher of creation of an environment that is inclusive and respects diversity. This complexity includes:

- *a teacher's own knowledge, confidence and love of mathematics*
- *a teacher's design of tasks that encourage participation, connection and understanding in mathematics*
- *a teacher's use of resources in ways that support learning*
- *a teacher's knowledge of students and their particular needs*

Teachers have to be knowledgeable and experienced in mathematics, having a vision of where what they teach is going: for example, teaching pattern spotting with algebra in mind; teaching fractions with rational numbers in mind; teaching 2-dimensional geometry with three dimensions in mind. This requires them to act in didactic mode – that is in a mode of converting their own mathematical understanding into tasks for students in which student can have opportunity to reach mathematical understanding.

In addition teachers have to have a vision of classroom interaction in mathematics which allows difference and diversity to flourish. This requires pedagogic understanding and a knowledge and vision of strategies that can engage students and encourage participation in and understanding of mathematics.

Thus, what is needed is a bringing together of the mathematics, the didactics and the pedagogy in a way that respects and celebrates diversity, recognizes and values difference, and includes everyone. How do we go about achieving these very serious demands? How does this knowledge and these qualities develop? Is this demanding too much of teachers? In the next section, I will introduce a developmental research project, with *inquiry* as a central concept, which aimed to develop activity of the kinds demonstrated above.

2. An inquiry approach: theoretical background and methodology

2.1 Inquiry and its roots

Inquiry involves questioning, investigating, exploring, wondering, seeking out, conjecturing and looking critically at whatever we are inquiring into. In a recent 4-year research project in Norway[3] (Jaworski, Fuglestand, Bjuland, Breiteig, Goodchild & Grevholm, 2007), *Learning Communities in Mathematics* (LCM), we explored inquiry in three layers or levels:

A. *Inquiry in students' mathematical activity in the classroom*
B. *Inquiry in teachers' exploration of classroom approaches*

[3] The author was employed at the University of Agder in Norway during the time of this project.

C. *Inquiry in addressing questions and issues to do with teaching and how it can develop to promote mathematics learning.*

Briefly, these can be expressed as *inquiry in mathematics* (A), *inquiry in mathematics teaching*(B), and *inquiry in research into learning and teaching mathematics* (C). The LCM project was a collaboration between B and C, with a focus on A. It involved teachers in classrooms in 8 schools from lower primary to upper secondary (working largely in level B) and didacticians[4] from a university mathematics education group (working largely in level C). There was a complex overlap between activity at levels B and C (Jaworski, 2008a). The main collaborative aim was to improve opportunity for students to engage with mathematics with understanding and skill, and to research the processes involved. Moreover we saw the research process itself as an important developmental tool and sought to create an *inquiry community* through which collaborative inquiry would lead to development.

This project was rooted in a sociocultural perspective on learning and teaching deriving from Vygotskian theory that all learning is socially rooted, that learning precedes development and that interaction with more experienced others can enable learners to develop their potential more effectively (e.g., Vygotsky, 1978). We conceptualised our established practices in school and university as *communities of practice*, drawing on Wenger's (1998) concepts of *belonging* to a community of practice as requiring *engagement*, *imagination* and *alignment*. We learn through engagement in practice, using imagination to interpret our own roles in practice and aligning with the established norms and expectation of the practice. In the project we sought to develop an inquiry community in which inquiry is emphasised as "a willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them" (Wells 1999, p. 122). Wells emphasizes the importance of *dialogue* to the inquiry process in which questioning, exploring, investigating, and researching are key activities or roles of teachers and didacticians (and ultimately, we hope, students). These activities can be discerned through the analysis of dialogue in interactions within the community. So, developing inquiry as a way of being involves *becoming, or taking the role of,* an inquirer; becoming a person who questions, explores, investigates and researches within everyday, normal practice. As a community of inquirers we aspire to develop an inquiry way of being, an inquiry identity, in our practice as a whole.

So, whether the practice was learning mathematics in a classroom, being a teacher in a school and designing the teaching of mathematics in classrooms, or being a university academic, working with teachers to promote developments in teaching, inquiry would enable us to look critically at our practice while engaging with it. Whatever the practice, in order to engage effectively, one has to fit in, align with the norms and expectations; however, it is possible also to question what we are doing and why. We proposed a concept of "critical alignment" in which, through inquiry, we might become more knowledgeable about practice and therefore more able to engage in alternative ways of being. We suggested that inquiry would start as a tool for alternative engagement and that, through collaborative interaction in inquiry, we would move towards an inquiry way of being, an inquiry community (Jaworski, 2006, 2008b). A key feature of the inquiry at levels B and C was to design inquiry ways of working with mathematics for students in classrooms to foster inclusion and diversity in mathematical learning and understanding.

2.2 Inquiry in the project – three levels of activity

Central to the project was the creation of opportunity for students to engage with mathematics more effectively – that is to enable better understanding and skill with mathematics. We sought to introduce inquiry in classroom tasks, designed by teachers with didacticians' support. Our belief was that engaging in mathematical tasks that are inquiry-based allows multiple directions of inquiry, differing degrees of challenge, mutual engagement and support, harmony in balancing sensitivity and challenge[5], and acceptance of and respect for difference. Designing such classroom activity was a

---

[4] Didacticians are academics who focus on theory and practice in the teaching of a subject to enhance learning opportunities for students.

[5] This refers to concepts of *sensitivity to students* and *mathematical challenge* as two important dimensions of teaching mathematics (Jaworski, 1994). The idea of *harmony* is that the degrees of mathematical challenge must be well matched with sensitivity in both affective and cognitive domains (Potari & Jaworski, 2002).

central developmental focus which involved teachers-as-inquirers exploring the kinds of tasks that engage students and promote mathematical inquiry; ways of organizing the classroom that enable inquiry activity with access for ALL students; and the many issues and tensions that arise related to the classroom, school, parents, educational system, society and politics.

Inquiry in research into learning and teaching mathematics involves teachers and didacticians *as researchers* undertaking research into

Responding to the mathematics curriculum

Task design

How students respond to tasks

Learning processes and outcomes

Issues in social relationships within classroom and school

Issues in power and responsibility

Clearly teachers are bound to attend to the national curriculum, and the ways of doing this within an inquiry process are an important focus of study, as are the design of tasks and their use by students. It is important also to study the outcomes for students in terms of how we judge learning and compare it with outcomes in the past and those reported in research more widely (for example, in international studies such as TIMMS and local variants in Norway such as the KIM study: e.g., Mullis, Martin, Beaton, Gonzalez, Kelly, & Smith, 1998; Brekke, 1995). Development cannot be separated from the social settings of which it is a part, so the study also paid attention to forces within the established communities which impacted overtly on the developmental approach, and especially power relationships within a community and between communities.

As well as the students for whose benefit the project was conceived, key players in research and inquiry were teachers and didacticians. It was the project's aim that these groups would work together in partnership, each group bringing their own special knowledge and expertise and drawing on each other's strengths. Didacticians should not act as external experts but as co-learners, co-inquirers, seeking to create a knowledge balance. To analyse data within the project, we used an *activity theory model* in which concepts of *community*, *rules* and *division of labour* enabled us to characterize activity (Engeström, 1999; Jaworski & Goodchild, 2006). *Division of labour* between teachers and didacticians proved to be a key outcome of learning within the project. The *rules* within which we operated were complex in their relationships to two established communities (of practice) and a fledgling project community (of inquiry). I will say more about these ideas later.

2.3 Operationalisation of activity in the project

The project ran during four years in which there were three school-year phases of field work. This work involved two central areas of activity: regular workshops in the university (16 over three years), involving all teachers (~30) and didacticians (~12) in the project, and school teams of project teachers working in their schools (8 schools) designing for the classroom and undertaking innovation in the classroom. Each school had an associated team of didacticians (3 per school) who were available to provide help if requested, and who had responsibility for collecting data from school activity. The workshops, each of length 3 hours, included plenary sessions led by a didactician, a teacher or a group of teachers; small group tasks in which teachers and didacticians together focused on mathematics and related this to classroom didactics and pedagogy, and plenary feedback sessions to share outcomes for small groups. Data included field notes, audio and video recordings and associated documents. The workshops were recorded in their entirety, either by audio or by video means.

A requirement on schools for participation in the project was that at least three teachers from the school would be involved and that the school leadership would support the project, allowing teachers to attend workshops and encouraging project work in the school. The project was supported by the Research Council of Norway[6] and there was a small amount of funding for each school. Schools

[6] The LCM project was funded by the Research Council of Norway (RCN) as part of its KUL (Kunnskap, Utdanning og Læring) programme. Project number 157949/S20. LCM publications can be found at http://fag.hia.no/lcm/papers.htm

volunteered to participate and teachers in the project were selected by their school. All showed willingness to be a part of the project and most engaged with commitment and enthusiasm. The project was designed by didacticians who sought the funding. Thus, there was an initial power imbalance regarding ownership of the project. This was mediated during the project as I shall explain.

Workshops, in the main were designed by didacticians. Teachers were invited to take part in design but largely declined; ostensibly because of the demands of the school day and lack of time, but also because, at least in the early days, such activity was seen by teachers as the province of didacticians. One teacher is recorded, in a focus group interview, during the second phase of fieldwork, as saying:

> *Agnes: At the beginning [of the project] I struggled, had a bit of a problem with this because then I thought very much about you [didacticians] coming and telling us how we should run mathematics teaching. That was what I thought, you are the great teachers … but now I see that my view has gradually changed because I see that you are participants in this as much as we are, even though you are the ones organizing this. Nevertheless I see that you are participating and are just as interested as we are in solving tasks on our level and finding possibilities, finding tasks, that may be appropriate for the students, and that I think is very nice. So I have changed my view during this time. And I think it's much better now, I feel much more comfortable, because now I feel that we are more equal than we were in the beginning, from my point of view (Bjuland and Jaworski, 2009).*

Thus we saw that early perceptions were strongly related to established community norms and expectations. This was also true in the didactician community, evidenced by the expectations didacticians had of the activity that might take place in schools. For example, it was suggested that teachers would plan for teaching in their project groups in school, individually operationalize plans in their own classrooms and observe each other teaching when that was possible. Over the first year of the project it became clear that project teachers, who were in different year groups within the school, found it impossibly difficult to meet within the school day, and observing each other in the classroom was not even contemplated. Such recognition formed a central plank of learning for didacticians. Over a year, didacticians and teachers had to negotiate ground. Didacticians started with the initial power in design and planning, but it was clear that teachers and schools wielded power in controlling what could happen within a school. In the second year of the project, workshops provided time for planning for the classroom between teachers from different schools who taught students of approximately the same age. The realisation that such a form of activity was necessary developed over the first year, and came to a head in a project-wide special meeting between first and second years of field work. We characterised this as a case of "expansive learning" (Engeström, 1999) in which perturbations within existing systems trigger for individuals or groups new ways of seeing and doing. Over time, through interactions between individuals and groups, these new ways expand to permeate the systems as a whole and we see what is described as *expansive development*.

3. Outcomes from developmental research: findings and discussion

3.1 Inquiry in practice in workshops and schools

Planning for the project cast workshops as the place where ideas about mathematics learning and teaching would be generated and discussed along with related possibilities for the classroom. Didacticians, in early consultation with a small group of teachers, decided to make the mathematical task (or problem) a key *tool* for generating inquiry, discussion and collaboration. In lengthy meetings at the university, didacticians considered a wide range of possible tasks, selected tasks for a workshop, and designed the ways in which workshop participants would work together on the tasks. A central purpose of workshop tasks was to bring teachers and didacticians together to open up possibilities for discussion of issues relating to didactics and pedagogy. Their focus on mathematics in which we all had a stake was deliberate, and served as well to enable teachers to work on mathematics in a way they were unlikely to do in school. Although the persons designing the task would, potentially, be at an advantage in knowing more about the mathematics of the task, only a small number of didacticians, differing for each workshop, were part of this design. For others the task was as 'new' as it was for the teachers. Thus all could be drawn into activity on an unfamiliar task. As an example, one task "the mirror task" asked the question: "*how tall a mirror must you buy if you want to be able to see your full vertical image?*" (Jaworski, Goodchild, Daland and Eriksen, in press). The didacticians designing the task (Daland and Eriksen), in plenary, engaged the whole community in working with small mirrors and short columns of multilink, small interlocking cubes, to try the task practically and initiate activity and thinking. Following plenary activity, small groups were formed to take ideas further and mathematize the situation. After this, one teacher used a modified version of the task with students in her school, and reported on its outcomes at a subsequent workshop. It was clear that the experience of taking and redesigning a task for her students, and a comparison between her vision of what was possible and the actual classroom outcomes, created for her an important learning experience, and not exactly a comfortable one.

Although didacticians designed tasks for workshops, and not for classrooms, many teachers saw the workshop tasks as presenting classroom opportunity. Data from schools shows that tasks became transformed and that teachers and students worked with variants of the tasks. Didacticians had suggested that teachers, gaining insight into tasks and activity in workshops would design tasks themselves relating to their own curriculum. While there is a small number of examples of this (e.g., Hundeland, Erfjord, Grevholm & Breiteig, 2007; Jaworski, 2007, Jørgensen & Goodchild, 2007), in the main, teachers took the ideas from the workshops and worked further with them in schools. Using activity theory, we conceptualize this in terms of the established school communities and their norms (or rules). Teachers are perhaps more familiar with working with ideas from a range of sources, rather than designing their own materials. Such design, as well as being less familiar, requires a different kind of activity and a different way of seeing the teaching role. It requires different kinds of vision and confidence. Through the three phases, data shows such vision and confidence growing for a number of the teachers involved. However, didacticians also had to change their visions. They had to become aware of teachers' worlds, of teachers' activity and values, of school structures and demands, and of the ways in which being a teacher fitted into teachers' personal lives.

An important principle for didacticians, from the beginning, was not to act as external experts telling or advising teachers; rather to create opportunity for joint engagement and collaborative endeavour (Wenger, 1998). The focus on "inquiry", however, was not negotiable, and considerable effort and emphasis was placed on inquiry throughout the project. The tasks were a tool or a medium in which inquiry action could take place and inquiry activity be exemplified. The Norwegian language has no word that translates *inquiry* exactly, so there was much debate as to which Norwegian words to use; gradually the word *inquiry* itself entered into Norwegian discourse. The project members were characterized, firstly by didacticians, and gradually by all participants as "an inquiry community".

However, practice in schools could not be so characterized except in tiny pockets. Didacticians were challenged overtly by certain groups of teachers as to the intentions and outcomes of the project. One group of higher secondary teachers, highly focused on their delivery of the curriculum and the shortage of available time, challenged the inquiry nature of workshop tasks and their relation to work in classrooms (Goodchild & Jaworski, 2005). Teachers generally were not happy to work in cross level groups – they preferred to be with teachers working with the same age of students. This related to the level of mathematics required, and some teachers' preference to be located within their mathematical comfort zone. It went against didacticians' views that much could be learned from teachers working with students at different levels of schooling. Compromises were made to accommodate with expressed preferences, so that operationalization looked rather different from that envisaged in the design of the project[7]. We see, here, accommodation to difference in apparently opposing ways. Each group had to appreciate the differing perceptions of the other, and a compromise had to be made to allow a productive way forwards. However, this sacrificed the (theoretically-based) expressed intention of didacticians to enable teachers to learn from their colleagues at different levels of schooling.

These findings speak to the third level (C) of inquiry within the project, i.e.

[7] Copies of initial and revised proposals to the Research Council of Norway can be found at http://fag.hia.no/lcm/

C: Inquiry in addressing questions and issues to do with teaching and how it can develop to promote mathematics learning.

They focused in an overarching way on considerations in levels A and B.

A: Inquiry in students' mathematical activity in the classroom
B: Inquiry in teachers' exploration of classroom approaches

Teachers' activity in the project focused essentially onto designing inquiry-based activity for their classrooms often based on tasks from workshops and building on workshop discussions. They called variously on didacticians for help and support in preparing for the classroom and to video-record classroom outcomes of their planning (Daland, 2007; Erfjord, 2008; Hundeland et al, 2007). Teachers offered presentation at workshops in which they reported on classroom activity supported by video extracts. Inquiry in levels A and B was closely linked in an inquiry cycle of plan, act and observe, reflect and analyze, and feedback to planning. Reporting in workshops provided an important dissemination purpose as well as supporting thinking and feedback. We all learned from what teachers has done and their insights into the inquiry and developmental processes involved. Our unit of analysis here was on teachers' thinking and development and the associated developmental activity between teachers and didacticians. Further details of this work can be found in articles listed at http://fag.hia.no/lcm/.and in two doctoral theses (Erfjord, 2008; Hundeland, 2009).

3.2 Developing inquiry at the three levels

The LCM project focused on teachers' activity in eight schools, and through this the activity of their students, on development across the eight schools and on relationships between teachers and didacticians. The scope of the project did not allow a systematic study of any one of the classroom settings (as in Goodchild, 2001, for example). The nature of classroom data is therefore a collection of snapshots from classrooms in which teachers interpreted an inquiry approach to mathematics. Through these snapshots we see the tasks teachers used, the ways these were designed or re-designed for students, the nature and detail of the activity showing engagement of students with a task and with the mathematics of the task. Generally tasks and associated activity fulfilled dimensions set out above (everyone able to make a start, diverse directions and ways of thinking, fluidity and flexibility in activity, serious mathematical thinking and outcome). A resource from the project is a large set of video recordings of classroom activity which can be used for professional development purposes to exemplify inquiry in mathematics classrooms. We see students engaging in different ways with the material of the tasks (as in narrative 1 above), students contributing different images and visions of mathematics in the tasks (as in narrative 3) and students experiencing differing levels of difficulty (as in narrative 2). Interviews and recorded conversations with teachers provide insights into the thinking of the teachers underpinning the classroom activity (e.g., Bjuland & Jaworski, 2009; Erfjord, 2009; Hundeland, 2010). Overwhelmingly, what was seen to be possible in classrooms was tempered by school norms and establishment demands. *Time* was universally a critical factor. Working with colleagues not in the project created issues – sometimes constraining what was possible, at others providing opportunity to persuade colleagues to work in alternative ways. Where whole school ethos was in accord with project goals one school used the project to achieve it own goals and another adapted its goals to capitalize on the project (Bjuland & Jaworski, 2009). One school dropped out of the project because project goals could not be achieved alongside a range of other interests and involvements. One school was silent for a long period, creating dilemmas for didacticians not sure of how best to respond (Erfjord, 2009). When didacticians eventually seized the initiative, the school realigned with the project and has since become a leading school in another project developing from LCM (The TBM Project – http://prosjekt.hia.no/tbm/).

One outcome is that we can add extensively to the kind of narratives included above and this is done to an extent in the publications referenced, often in the Norwegian language. Each teacher has a story about their work in developing inquiry-based activity with their students, sometimes an individual story, sometime a school story, sometimes both. For all teachers, this activity was just the beginning of developing as a teacher through inquiry at levels A and B. The activity raised many issues. For some teachers, mathematics itself was a problem and support was needed in enabling further learning of mathematics. For some teachers, dealing with diversity in their classroom was an issue. In Norway schools do not put students into sets, so all classes are 'mixed ability'. Teachers were used to organizing students into small homogeneous groups for classroom mathematics and needed to think about how inquiry-based activities could foster more diverse groupings. While teachers liked inquiry-based tasks, both for themselves in workshops and for their students, it was not clear how such tasks fitted into their curriculum and it was an issue as to how they might ‘cover’ a demanding curriculum and incorporate inquiry-based activity. The ways in which inquiry-based activity could form a basis for curriculum work was an area for further inquiry. Teachers undertaking inquiry-based activity with students were sometimes confronted by new challenges to their practice which current experience did not equip them to handle. Thus, it became necessary to deal with feelings of lack of achievement which suggested a return to the safer ground of traditional methods.

I highlight these issues and challenges to show that the developmental path was neither easy nor smooth. The major achievement of the LCM project was the learning that took place between teachers and didacticians with regard to the meaning of inquiry-based activity (at A, B and C levels) and its implications in both the established communities of university and schools and the inquiry community of the project. Each teacher was dealing with issues of inclusion and diversity either in their own classroom or in their school more generally, or both. These issues were brought to project workshops and shared as part of the inquiry community. In this community we negotiated differences across the schools and between didacticians and teachers in our attempts to *be* a community and to develop *inquiry ways of being* across the community. Resolution of issues led to expansive learning as indicated above. Outcomes were not always what we desired or anticipated, but we can look back and recognize new knowledge and awareness in the joint enterprise that was only possible through the inquiry of the project.

I have tried to show here a glimpse of the nature of activity and issues arising from dealing with diversity between teachers from eight schools and didacticians from the university research group. What was achieved here provided a starting point for further developmental research. The respect, trust and understanding that had developed between teachers and didacticians in LCM formed the basis for two further projects working in conjunction:

The TBM Project (Teaching Better Mathematics) organized by didacticians

The LBM Project (Learning Better Mathematics) organized by teachers and school leaders.

Each project attracted its own funding and meetings between leaders in the two groups set an agenda and initiated activity, building on LCM. Some schools from LCM continued into TBM/LBM others not. The schools that did continue played a leadership role in the new projects with teachers making a strong input to the activity in the new projects. Kindergartens (barnehager) joined the new projects taking the range from early years to upper secondary.

Thus, diversity increased. The inclusive nature of the projects was maintained with constant effort towards mutual understanding and joint tackling of issues. A conjecture at this stage is that to achieve the kinds of activity we desire in classrooms, as set out in the narratives above, requires teachers to be a leading part of the developmental process along with didacticians who bring other levels of expertise to the joint focus. Findings from TBM/LBM are starting to emerge, and the process is ongoing (e.g., Fuglestad & Goodchild, 2008).

References

Bjuland, R. & Jaworski B. (2009). Teachers’ perspectives on collaboration with didacticians to create an inquiry community. *Research In Mathematics Education.* Volume 11 Number 1, pp 21-38.

Boaler, J. & Wiliam, D. (2001). ‘We’ve still got to learn!’ Student’s perspectives on ability grouping and mathematical achievement. In P.Gates (ed.) *Issues in Mathematics Teaching* (pp. 77-92). . London: Routledge.

Brekke, G. (1995). *Kartlegging av matematikkforståelse: Introduksjon til diagnostisk undervisning i matematikk* (Charting of mathematical understanding: introduction to diagnostic teaching in mathematics). Oslo: Nasjonalt læremiddelsenter.

Chin, S. J. & Ashcroft, J. R. (1998). *Mathematics for dyslexics: A teaching handbook* (Second Edition). London: Whurr Publishers Ltd.

Cockcroft, W. (1982). Mathematics counts; Report of the Committee of Inquiry into the Teaching of Mathematics in Schools. London: Her Majesty's Stationery Office.

Daland, E. (2007). School teams in mathematics, what are they good for? In B. Jaworski, A. B. Fuglestad, R. Bjuland, T. Breiteig, S. Goodchild, & B. Grevholm (Eds.), *Learning communities in mathematics* (pp. 161-174). Bergen, Norway: Caspar.

Daniels, H. (2001). *Vygotsky and Pedagogy*. London: Routledge Falmer.

Engeström, Y. (1999). Activity theory and individual and social transformation. In Y. Engeström, R. Miettinen, & R. -L. Punamäki (Eds.), *Perspectives on activity theory* (pp. 19-38). Cambridge, UK: Cambridge University Press.

Erfjord, I (2008). Teachers' implementation and orchestration of Cabri-use in mathematics teaching. *Doctoral Dissertation at University of Agder.* Kristiansand, Norway.

Fuglestad, A. B., & Goodchild, S. (2008). Affordances of inquiry: the case of one teacher. In O.Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepúlveda (Eds.), *Proceedings of the joint meeting of PME 32 and PME-NA XXX* (pp. 3-49-3-56). Mexico: Cinvestav - UMSNH.

Goodchild, S. (2001). Students' goals: A case study of activity in a mathematics classroom. Norway: Caspar Forlag.

Goodchild, S., & Jaworski, B. (2005). Identifying contradictions in a teaching and learning development project. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 41-47). Melbourne, Australia: University of Melbourne.

Hundeland, P. S. (2010). Matematiklærerens kompetanse. En studie om hva lærere på videregående trinn vektlegger i sin matematikkundervisning (Mathematics teacher competence. A study on what teachers at upper secondary level emphasize in their mathematics teaching. Doctoral Dissertation at University of Agder, Kristiansand, Norway.

Hundeland, P. S., Erfjord, I., Grevholm, B., & Breiteig, T. (2007). Teachers and researchers inquiring into mathematics teaching and learning: The case of linear functions. In C. Bergsten, B. Grevholm, H. S. Måsøval, & F. Rønning (Eds.), *Relating practice and research in mathematics education. Proceedings of Norma05, 4th Nordic Conference on Mathematics Education* (pp. 299-310). Trondheim, Norway: Tapir Akademisk Forlag.

Jaworski, B. (1994). Investigating Mathematics Teaching: A Constructivist Enquiry. London: Falmer Press.

Jaworski B. (1998). Mathematics Teacher Research: Process, practice and the development of teaching. *Journal of Mathematics Teacher Educ*ation, 1, 1, 3-31.

Jaworski, B. (2005). Learning communities in mathematics: Creating an inquiry community between teachers and didacticians. In R. Barwell & A. Noyes (Eds.), *Research in Mathematics Education: Papers of the British Society for Research into Learning Mathematics* Vol. 7, pp. 101-119. London: BSRLM.

Jaworski B. (2006). Theory and practice in mathematics teaching development: Critical inquiry as a mode of learning in teaching. *Journal of Mathematics Teacher Education, 9*, 187-211.

Jaworski, B. (2007). Learning communities in mathematics:research and development in mathematics learning and teaching. In C. Bergsten, B. Grevholm, H. S. Måsøval, & F. Rønning (Eds.), *Relating practice and research in mathematics education. Proceedings of Norma05, 4th Nordic Conference on Mathematics Education* (pp. 299-310). Trondheim, Norway: Tapir Akademisk Forlag.

Jaworski, B. (2008a). Development of the Mathematics Teacher Educator and its relation to Teaching Development. In B. Jaworski & T. Wood (Eds.), *The Mathematics Teacher Educator as a Developing Professional. Volume 4 of the International Handbook of Mathematics Teacher Education* (pp. 335-361). The Netherlands: Sense Publishers.

Jaworski, B. (2008b). Building and sustaining inquiry communities in mathematics teaching development: Teachers and didacticians in collaboration. . In K. Krainer & T. Wood (Eds.), *Participants in mathematics teacher education: Individuals, teams, communities and networks. Volume 3 of the International Handbook of Mathematics Teacher Education* (pp. 309-330). The Netherlands: Sense Publishers.

Jaworski, B., Fuglestad, A. B., Bjuland, R., Breiteig, T., Goodchild, S., & Grevholm, B. (Eds.). (2007). *Learning communities in mathematics*. Bergen, Norway: Caspar.

Jaworski, B., & Goodchild, S. (2006). 'Inquiry community in an activity theory frame'. In J. Novotna, H. Moraova, M. Kratka, N. Stelikova (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 353-360). Prague, Czech Republic: Charles University.

Jaworski, B., Goodchild, S., Daland, E. & Eriksen S. (in press). Mediating mathematics teaching development and pupils' mathematics learning: the life cycle of a task. In O. Zaslavsky & P. Sullivan (Eds.): *Constructing knowledge for teaching secondary mathematics: Tasks to enhance prospective and practicing teacher learning.* NY: Springer.

Jørgensen, K.O. & Goodchild, S. (2007). Å utvikle barns forståelse av matematikk. [To develop children's understanding of mathematics.] *Tangenten* 1: 35–40 and 49.

Mullis, I., Martin, M., Beaton, A., Gonzalez, E., Kelly, D., & Smith, D. (Eds.) (1998). *Mathematics and Science Achievement in the Final Year of Secondary School: IEA's Third International Mathematics and Science Study* (TIMSS). Chestnut Hill, MA: Boston College.

Nardi, E. & Steward, S. (2003). Is mathematics T.I.R.E.D.? A profile of quiet disaffection in the secondary mathematics classroom.British Journal of Educational Research, Volume 29, No. 3, pp. 345-368.

Open University (1985). *Working mathematically with low attainers*.(VHS Videotape) Milton Keynes: Centre for Mathematics Education, The Open University.

Pollack, D. (2009). *Neurodiversity in higher education: Positive responses to specific learning differences*. Chichester, UK: Wiley-Blackwell.

Potari, D and Jaworski, B. (2002). Tacking complexity in mathematics teaching development: Using the teaching triad as a tool for reflection and analysis. *Journal of Mathematics Teacher Education* 5, 351-380.

Powell, S, (ed.). (2003). *Special teaching in higher education: Successful strategies for access and inclusion*. London: Kogan Page.

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education,* Volume 8, No. 3, pp. 255-281

Trott, C. (2009).Dyscalculia. In D. Pollack (Ed.) *Neurodiversity in higher education: Positive responses to specific learning differences,* pp. 125-148. Chichester, UK: Wiley-Blackwell

Vygotsky, L. (1978). *Mind in society.* Cambridge, MA: Harvard University Press.

Wells, G. (1999). *Dialogic inquiry: Toward a sociocultural practice and theory of education*. Cambridge, UK: Cambridge University Press.

Wenger, E. (1998). *Communities of practice: Learning, meaning and identity.* Cambridge, UK: Cambridge University Press.

Barbara Jaworski is Professor of Mathematics Education in the Mathematics Education Centre at Loughborough University. She was formerly Professor of Mathematics Education at Agder University in Norway and before that a Reader at the University of Oxford. She has a career that spans mathematics teaching at secondary level and first year university level, teacher education at secondary level, and teaching and supervision of doctoral students. Her research has been mainly into the development of mathematics teaching through research and through partnerships between teachers and teacher educators or didacticians. She was for six years Editor in Chief of the Journal of Mathematics Teacher Education, and for four years President of ERME, the European Society for Research in

Mathematics Education. Her research currently is into the development of mathematics teaching at the university level.

B.Jaworski@lboro.ac.uk

Preparing and supporting teachers of mathematics to become special for all

Hafdís Guðjónsdóttir
University of Iceland – School of Education

The focus in this paper is on teacher education and how teacher educators can support mathematic teachers to teach mathematics to diverse group of students. The paper builds on the development of a course called Mathematics for all and I report on the challenges we have coped with during the reconstruction of the course. In the course teachers have opportunities to integrate their experience and relate theory and practice. We focus on diversity, reflective practice and holistic approaches of teaching. Participants write cases about their experience, analyze and relate to pedagogical theories and discuss their learning online as well as in class. In the paper I use examples from the course to illustrate how we have created opportunities for the teachers built a reflective community of learning.

Introduction

The diversity of students in Iceland mainstream schools continues to expand as students with disabilities enter neighborhood schools and immigration opens pathways for learners with a variety of cultures, languages and expectations of education. In addition, employment and workplaces are changing and the skills people need for the future differ from those required now or in the past. What these changes will be we do not exactly know, but changing social conditions have put increasing pressure on mainstream teachers to modify their practices and take into count the diverse group of learners that form their learning communities. In this paper I will discuss responsive pedagogies in teacher education and illustrate my writing by using data from my self-study in teacher education practice.

Responsive Pedagogies in Teacher Education

Achieving moral purpose in practice requires of teachers the commitment to create interaction and even mutual interest across groups. However the problem is that there are tendencies to keep people different from us at a distance (Fullan, 1999). Commonly schools respond to diversity in ways that divide and separate children into hierarchies of value and perceived aptitude. Evidence of these approaches can be seen in labeling and/or sorting of students by ability and limited consideration of the potential of all learners (Booth, 2010). It is only when teachers open their minds and their classrooms to diverse groups of students that they will be enabled to responsively develop learning communities in which teachers and students engage in a spectrum of different learning needs.

More than a decade ago, Fullan (1999) emphasized that schools *"are still at the beginning of an intellectual burgeoning of the quality and depth of pedagogical knowledge and means of enhancing learning for all. This revolution has barely touched schools"* (p. 47). Now, ten years later, we would hope this has changed and at least expect teacher education to be responsive to the diversity of students and the preparation of teachers to work in diverse classrooms.

Inclusion is one idea that is recognized for its focus on developing educational practices that are responsive to diversity, concerned with and valuing of all students equally. Booth (2005) introduces values that he believes are important for schools as they develop learning communities for all students. These values involve issues of *equality, rights, participation, learning, community, and respect for diversity, trust and sustainability* and also the qualities of *compassion, honesty, courage, and joy*. A commitment to such values prepares teachers and teacher educators to increase the participation of others, overcome discrimination and create an environment that ensures learning for all children. As we understand more about the focus of learning and how people learn, we also understand the critical combination of intellectual and social development and the need to continue learning in face of constant change and societal complexity. The ability to think and present ideas on the one hand and work with others on the other is recognized by education and businesses alike as central to the world's future (Fullan, 1993).

What does this means for teacher education? What kind of teachers do we need in the present and in the future? How should teacher education institutes reform their practice to assist teachers to understand and implement inclusive education?

In a research project with teachers I identified the following diverse roles that professional educators embrace depending on their circumstances and opportunities (Guðjónsdóttir, 2000).

Pedagogues and experts in teaching and learning: Activist teachers share their knowledge and understandings in an ongoing professional dialogue

Reflective and critical problem solvers: Teachers continuously monitor students' progress and learning within the classroom, outside that environment they reflect both as individuals and as communities of practice on their practice and student progress.

Researchers and change agents: In seeking a deeper understanding of their practice, or in seeking to plan for change, teachers use a variety of evaluation and action research techniques to collect and interpret findings, to inform their thinking and decision making

Creators of knowledge and theory builders: In the process of reflective practice and action research teachers develop new understandings of learning, teaching and educational change

Similarly, Cochran-Smith and Lytle (1999) continued their long discussion of teacher research by identifying five critical elements for consideration in future discourse on teacher professionalism: (a) emphasis on the teacher as knower and agent of change, (b) creation of new ways to theorize practice, (c) participation of teachers and colleagues in intellectual discourse about critical issues, (d) linking teaching and curriculum to wider political and social issues, and (e) the creation of inquiry communities that focus on the positive, rather than negative, aspects of what teachers know.

Along these lines teachers working with students who have difficulties in mathematics need wide-ranging competencies. Current research in teacher education suggests that the conceptual and theoretical preparation for mathematics teachers is paramount.

> *The dominant theme in the finding for mathematics studies was that the prospective teachers …had mastered basic skill in school subjects, but lacked a deeper understanding they would later teach. (Floden & Meniketti, 2005, p. 270)*

Simply knowing about mathematics or being able to drill students in algorithms is not enough. To be able to develop opportunities for students to meet the demands of everyday teaching and living is critical and as they teach their students they demonstrate mathematical thinking and problem solving (Hiebert et. al., 1997). All teachers work with students, especially instructions, are based on them knowing students as learners of mathematics and an understanding of their cognition (Kristinsdóttir, 2004).

Self-study of teacher education practice

As a teacher educator I have wondered what kind of teacher educational pedagogy needs to be created to support mathematics teachers to follow their inspiration to respond to all students. My colleagues, Edda Óskarsdóttir and Jónína Vala Kristinsdóttir, and I have focused our research on our efforts to construct and reconstruct an inclusive course that prepares teachers teach diverse group of students and engages students in rich, authentic and purposeful learning.

The goal of the research was to develop a course, built on research findings, that prepares teachers to support all students in learning mathematics in inclusive settings. In addition we sought to understand and improve our work as teacher educators (Russel & Loughran, 2007).

Our research questions are as follows:

- How can we develop a course for mathematic teachers that prepares and supports them
  - to teach mathematics to diverse group of students?
  - to teach students with learning difficulties in mathematics?
  - to assess their own knowledge and attitudes towards mathematics?
  - in their professional development?

Four dimensions of practitioner research presented in self-study made it the most appropriate methodology to this professional inquiry.

> **Self and agency:** The professional identity and action of individuals is intrinsically bound to the creation and renewal of their practice.

Collaborative creation and dissemination of knowledge: Collaborative questioning, dialogic, and action-oriented processes are essential to the development and dissemination of authentic educational knowledge.

Continuous learning and action: the situation of self-study is ever changing and developing, because researchers must give first priority to managing context that is simultaneously being studied and changed.

The emergent and overlapping nature of changes in understanding and practice: Self-study researchers work within the constraints and opportunities of personal histories and organizational cultures as they explore new paradigms and create new ways of working (Bodone et. al. 2004, p. 746-747).

Self-study is one form of practitioner-based study with a strong focus on teacher education. The insider or the teacher educators themselves bring in the research and the researchers themselves are the resource. They identify their own situation of practice and open up for, action - reflection - learning – action, in order to improve their practice. The emphasize in this study is on:

- Intentional data about praxis based learning and teaching by teachers and teacher educators
- A deeper understanding of the different nature of special education and mathematic education, of the different ways people learn mathematics and how we need to teach
- The review and improvement of the researchers' practice as teacher educators

Context
This ongoing study was conducted from the year 2002 and with an intention to continue as long as we are responsible for the course. The focus of the study is a graduate course taught at University of Iceland, School of Education. The participants are teachers (pre -, primary - & secondary schools) and social educators who most often work with students with disabilities. The course is organized as a distant learning subject with campus classes for five whole days and Internet based learning using a program called WebCT. The number of participants attending the course is usually between 20 and 30.

Data gathering
We collected all the material from the courses, e.g., readings, presentations, student tasks, projects and discussions. In addition, in one year we circulated a questionnaire on teacher's knowledge and beliefs to 47 teachers (with 79% response rate). By going through all the data individually and together, looking for trends, patterns and anomalies, plus raising new questions. Critical reflection on this data enabled us extend our understanding of inclusive teaching and establish our practice and research in the following year. In the next section I will report on certain matters we have dealt with or developed in order to support mathematics teachers in their professional development.

Mathematics for all
As we developed the course our purpose and goals, became more transparent so that we could see how the tasks given to the teachers were related to their goals and competences. The *beacon*, *cornerstone*, and *cairn* along the road were diversity, reflective practice, holistic approach to teaching and assessment.

Diversity - Opening up the professional discourse of teaching
Defining diversity is essential and therefore the starting point is to clarify diversity and then to build the work on it. One important factor to recognize is that in the group there are many different perspectives on mathematics and many histories of mathematics learning. We began the course by giving a mathematics problem to the whole group and asking the participants to work in groups of three, solve the problem together and introduce their process to the whole group.

We do not always give the same problem to the participants in the course but our experience of the activity is very similar. The solution strategies vary greatly. Some use trial and error strategies, others use equations and yet others use objects or count.

> *What I found interesting was when we solved the problem in different ways. Then I thought: If we can solve problems in different ways then the children must be able to do it and even in more different ways than we do. They will be more open, because their understanding was respected.'*

Their different ways of working or finding solutions gave us a chance to explore different ways of thinking about the problem and variations in the way people think. Middle school teachers were more likely to use equations to solve problems teachers, just as kindergarten teachers more often chose drawing. Both influenced by their work with their students, the middle school teacher teaching algebra and the kindergarten teacher working with young children who need hands on experience. We also discussed the purpose of solving problems — for whom is this valuable problem and why. The teachers agree that children as young as five years old are able to solve problems if they have objects to work with. They also agree that this is a good problem to give to students that are learning to solve equations with two unknown variables.

These discussions gave us a good start for understanding learning in a diverse classroom. The teachers' use of multiplication strategies to solve problem is often dependent on their previous experience with mathematics. However they all learn something, even though they don't learn the same thing at the same time. They also realize that this is also true for the children, their students.

Building habits of reflective practice

Another aim in the course is to create a community of learners that builds on habits of reflective practice (Dalmau & Guðjónsdóttir, 2002). In doing so we introduced the *Case & Commentary,* which was created for teachers so they can integrate their experience and relate to theory and practice. When they write cases about their experience, generate questions, analyze data and relate to pedagogical theories they are well prepared for rich and informative discourse. They share their emerging understanding online as well as in class. It has turned out to be an effective protocol for the reframing of practice that embodies an authentic view of teaching and learning. We ask the teachers to write cases from their experience with students that have had difficulties with mathematics. They explore the following four dimensions of praxis inquiry:

1. **Practice described:** Questioning and adding more details to their case.
2. **Practice explained.** Interpreting their case or practice and beginning to relate to their own experience and knowledge.
3. **Practice theorized.** Relating their case/practice to theories, professional knowledge and their professional working theory — asking who am I becoming as a mathematics teacher.
4. **Practice changed.** Making decisions and respond to the findings by creating a new plan or a response in order to develop and improve their practice (Kruger & Cherednichenko, 2006; Guðjónsdóttir et al, 2007).

The ideological framework of the Praxis Inquiry was articulated during a review of teacher education completed at Victoria University in Australia in 2004. Kruger and Cherednichenko (2006) report that praxis inquiry based learning and teaching creates a new paradigm for teacher education. Firstly it recognizes that T-Ls' questions about the learning and life outcomes of their students are critical to their learning. Secondly it respects the active creation of educational knowledge within schools and thirdly it offers Teachers the opportunity to develop as informed and competent professionals (Kruger & Cherednichenko, 2006).

Teachers responding to our research, through interviews or questionnaire are reporting their learning.

> *I have changed as a teacher after I went back to school and took some graduate courses. I feel as I have 'renegotiated', become stronger as a professional and that is a positive effect for both myself and my teaching.*

By using the case and commentary and the praxis inquiry protocol teachers are able to reflect on their practice in a systematic way, to relate their work to their knowledge and experience, and to reflect on theory and ethics. This gives them opportunities to respond to their students in more effective way and they can see the difference in their practice.

> *I'm more confident in discussing with the children what kind of questions to ask and when to encourage them to rely on their thinking. When to probe them that they don't need to count if they can calculate mentally. I also feel that I'm much stronger in analyzing the children's solution strategies and understanding of the problems. I am not afraid of changing and to use new teaching strategies or different learning material.*

Teachers make instructional decisions based on their knowledge of individual children's thinking

Holistic and authentic approach in teaching and assessing

Cognitively Guided Instruction (CGI) grew out of inquiry into explicit knowledge about the development of children's mathematical thinking and as a context for the study of teachers' knowledge of students' development (Carpenter, Fennema, Peterson & Carey, 1988: Carpenter, 1985).

Through the years we have introduced mathematics teacher to CGI in order to support teachers to make instructional decisions based on their knowledge of individual children's thinking. In turn these understandings of students' mathematical thinking influence teachers' knowledge, beliefs, and practices and can lead to improved learning for all.

Research into the way teachers used knowledge of students' mathematical thinking and decision-making showed that learning to understand the development of children's mathematical thinking could lead to fundamental changes in teachers' beliefs and practices and that these changes were reflected in students' learning (Carpenter et al. 1989; Fennema, Franke, Carpenter & Carey, 1993; Fennema, Carpenter, Franke, Levi, Jacobs & Empson, 1996). Our experience also suggests that gaining access to knowledge about children's mathematical thinking paves the way for the teacher to understand how children learn mathematics and their capability for finding their own solution strategies.

> When I focused on listening to the children, observed their work and tried to understand their thinking, I learned that they are capable of inventing their own solution strategies. And what was even more interesting was that slower learners were also capable of finding their own ways of solving problems if they were given time and resources to work at their own pace and encouraged to rely on their own thinking.

These experiences replicate the findings of the CGI project on teachers' development in using children's mathematical knowledge in instruction (Carpenter et al., 1999; Fennema et al., 1996). By using the CGI method and the framework teachers received a tool to analyze, interpret, and assess children's understanding of different problem types. With that knowledge and understanding they were better capable of choosing mathematics problems that fit the students abilities and support them with their mathematical learning.

> *I try to teach each student according to his/her level of ability, I create learning and teaching material that fits each student. I make them myself, find them on the internet, buy them here in Iceland or abroad, I translate and adapt. I focus on hands on projects, games, board-games, cubes and so on and try to give the children opportunities to talk about their tasks, explain and think aloud while they work on their problem solving.*

In the reports many of the participants showed a good understanding of the solution strategies the children used and were able to analyze their way of thinking while solving the problems. The solution strategies and the levels of development children go through, explained in *Children's mathematics*, guided the teachers while analyzing the children's thinking.

> *I encouraged the children to think for a while by themselves and then discuss their thoughts in groups of three or four. The group then tried to find a solution that they all could agree on.*
>
> *I don't group students according to their ability but randomly. They like that and find it interesting to be in a group with different students that are at different levels in their mathematic. There is a great progress in the group.*

Some of the participants, though, were not able to follow the children's way of thinking and were too eager to explain to them what to do when the children could not solve a problem immediately. We found that they had difficulties with relying on the children's thinking and giving them time to explain their understanding of the problem. Instead of asking the children questions that would help them clarify the problem, the teachers explained to them their own understanding of the problem and how to solve it. We believe this is reasonable because this is the common practice teachers come from. Participants' expectations of how to solve a problem, e.g. finding the "right" solution strategy, often was an obstacle while analyzing the children's way of thinking.

Anna and Lisa, two teachers who attended the course, in their task they gave the following problem to nine-year-old Paul.

> *Peter headed the ball 4 times in a Soccer game. Charles headed the ball 6 times more than Peter. How many times did Charles head the ball?*

> *Anna and Lisa explained that Paul counted on his fingers, but didn't give any information on how he counted. His answer was 28. When asked to explain his solution he said: "I counted 4, 8, 12, 16, 20, 24, 28." While counting he realized that he had added four, seven times instead of six times and corrected his answer to 24.*

Analyzing his solution strategy Anna and Lisa said: "Paul doesn't realize that times more means that you have to multiply." However we believe something is missing in analyzing this case because we have the feeling that Paul showed a very good understanding of the problem and realized that to find how much six times more than four is, he needed to add four, six times. The example shows that Anna and Lisa expected him to use multiplication tables when hearing the words times more. It looks like that they believe that to be able to solve mathematical problems there is a need to master basic facts and know the words that tell you what to do with the numbers. However, by noting how Paul counted on his fingers you become able to analyze his way of thinking, and that by using addition in this multiplication problem Paul shows that he understands that multiplication is repeated addition.

We are concerned about how to assist the teachers to respect children's way of thinking and to give them the opportunity to develop their own understanding of mathematical concepts. Our experience from the course is that teachers are willing to rely on able students to find their own way of solving problems, but when it comes to the slower learners the teachers tend to believe that it's too difficult for them to find their own solution strategies. Discussing this with the teachers we have learned that they are concerned that these students will waste their time on useless solutions strategies and will soon be left behind the others. This is understandable if you relate this to where teachers learners come from and to their experience and former studies. But our question is how can we support teachers to learn to listen to all their students?

Conclusion: What have we learned?

Defining diversity involves paying attention to both differences and similarities between people and when referring to diversity we should include everyone not only those who seem to be different from the "majority" or us. Inclusive reaction is to welcome diversity and view it as a rich resource rather than a problem to fix (Booth, 2010). The participants in this course are a diverse group of people with experience from different areas of teaching. The opening task gave us an opportunity to explore and discuss the different ways people solved the mathematical problem and that there is not any one right way to solve it but many different right ways. Our experience and what we bring with us plays a big part in how we solve the mathematical problem. This task seems very simple but in spite of that it gives a rich opportunity for defining diversity and for reflections, discussions, and learning.

Introducing the case and commentary method to the teachers opened up opportunities to reflect on own practice and relate to theory and ethics. To begin with participants' experience, their practice and their questions supports them to transfer their learning into practice.

Focusing on research on mathematics learning has positive effect. Still many believe that practicing and mastering traditional algorithms will give children good foundations for being able to perform the calculations they need in their daily lives. It is important to work on beliefs and attitudes towards mathematics and to explore what it means to do mathematics with the teacher learners. The understanding of what it means to do mathematics is sometimes narrow and some are not used to explore different ways to solve math problems. Our work is to create a learning community that is open to diversity, reflective practice and holistic approach of teaching and assessing.

References

Bodone, F., Guðjónsdóttir, H., & Dalmau, M. C. (2004). Revisioning and recreating practice: Collaboration in self-study. In J. J. Loughran & M. L. E. Hamilton & V. LaBoskey & T. Russell (Eds.), *International handbook of self-study of teaching and teacher education practices* (P. 743–784). Boston: Kluwer Academic Publishers.

Booth, T. (2010) *How should we live together? Inclusion as a framework of values for educational development.* Keynote presentation at Dokumentation Internationale Fachtagung, Berlin, June 11.

Booth, T. (2005) *Keeping the future alive: Putting inclusive values into education and society?* Paper presented at North – South Dialogue Conference, Delhi, March

Carpenter, T. P., Fennema, M. L. F., Levi, L. og Empson, S. B. (1999). *Children's mathematics: Cognitively guided instruction.* Portsmouth: Heineman.

Carpenter, T., Fenema, E., Peterson, P. L., & Carey, C. A. (1988). Teachers' pedagogical content knowledge of students' problem solving in elementary arithmetic. *Journal for Research in Mathematics Education 19*, 385–401.

Carpenter, T., Fennema, E., Peterson, P. L., Chiang, C. P., & Franke, M. L. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal, 26*, 499–531.

Carpenter, T. (1985). Learning to add and subtract: An exercise in problem solving. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives*. Hillsdale: Lawrence Erlbaum Associates.

Cochran-Smith, M., & Lytle, S. L. (1999). The teacher research movement: A decade later. *Educational Researcher, 28*, 15–25.

Dalmau, M. C. og Hafdís Guðjónsdóttir. 2002. Framing professional discourse with teachers: Professional working theory. In J. Loughran & T. Russell (Eds). *Improving teacher education. Practices through self-study.* p.102–129. London, Routledge Falmer.

Fennema, E., Carpenter, T., Franke, M. L., Levi, L., Jacobs, V. R., & Empson, S. B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for research in mathematics education, 27*, 403–434.

Fennema, E., Carpenter, T. P., Franke, M. L., & Carey, D. A. (1993). Using children's mathematical knowledge in instruction. *American Educational Research Journal, 30*, 555–583.

Floden, R. and Meniketti M, (2005). Research on the effects of coursework in the arts and sciences and in the foundations of education. *Studying teacher education: The report of the AERA Panel on Research and Teacher Education*. M. Cochran-Smith and K. M. Zeichner. Mahwah: Lawrence Erlbaum Associates. 261–308.

Fullan, M. (1993) *Change forces: Probing the depths of educational reform.* Philadelphia: Falmer Press.

Fullan, M. (1999) *Change forces: The Sequel.* Philadelphia: Falmer Press.

Guðjónsdóttir, H., & Kristinsdóttir, J. V. (2007). Preparing teachers to teach all children mathematics. In P. A. Bartolo, A. M. Lous, & T. Hofsäss (Eds.), *Responding to student diversity: Teacher education and classroom practice. Proceedings of the International Conference on Teacher Education for Responding to Student Diversity* (pp. 44–61). Malta: Faculty of Education, University of Malta.

Guðjónsdóttir, H. (2000). *Responsive professional practice: Teachers analyze the theoretical and ethical dimensions of their work in diverse classrooms.* Unpublished doctoral theses. Eugene: University of Oregon.

Hiebert, J. et.al. (1997) *Making sense: Teaching and learning mathematics with understanding.* Portsmouth: Heineman.

Kristinsdóttir, J. (2004). Öll börn geta lært að reikna, *Glæður* 1.tbl. 2004

Kruger, T., & Cherednichenko, B. (2006). Social justice and teacher education: Re-defining the curriculum. *The International Journal of Learning, 12*, 1–8.

Lunde, O. (2002) *Rummelighed i matematik A-C.* København: Malling Beck.

Russel, T. & Loughran, J. (2007). *Enacting a pedagogy of teacher education: Values, relationships and practices.* London: Routledge.

Hafdís Guðjónsdóttir was a general and special teacher for 26 years before becoming a teacher educator. Teaching is her primary profession and she emphasizes partnership with teachers through teacher education, school projects, consultancy and research. Her focus is on inclusive practice, curriculum development,

authentic assessments, mathematics for all students, and collaboration with students' families. Her overall approach to research is qualitative and research priorities include teacher, action and self-study research.

hafdgud@hi.is

Dynamic assessment by overall evaluation in connection with difficulties in mathematics

Svein Aastrup
Trøndelag Resource Center

A dynamic assessment test has been developed in collaboration between schools, Educational Psychology services and practice-related further education courses. The test is a component in an evaluation model for pupils who are weak in mathematics. The assessment material is based on well known socio-cultural learning theory. The purpose of the test is not to present a diagnosis, but to provide a good basis for adapted teaching in mathematics. It provides knowledge about the pupil's mathematical thinking, both when the pupil is coping and when not. It assesses whether the pupil can apply knowledge within a context, task specific strategies, possible misconceptions and the pupil's learning potential: What can the pupil manage by himself and with a competent adult in the role of supportive scaffolding? There is a demand for good knowledge and awareness of professional practice and educational theory concerning one's own manner of communicating. Teachers consider the material demanding to become involved with, but that it nonetheless provides a great deal of relevant information for the planning of special needs teaching. The dynamic assessment material has been evaluated for several years by many teachers, advisors and students, as well as by the author. Using the material has had a positive effect on pupils and teachers, and when used for the right reasons it appears to be a good tool for planning and carrying out well adapted teaching in mathematics.

Introduction

Trøndelag Resource Centre, in collaboration with schools and the Educational Psychology Services, has developed a model regarding assessment and initiatives for pupils that experience difficulties in mathematics. The model will identify and assess pupils who struggle in mathematics. Central to the model is a dynamic assessment of the pupils' mathematical competence. The background for developing the dynamic assessment material is that we want a tool that can contribute to assessing what should be done to provide good, suitable teaching for pupils who are weak in the subject of mathematics. Dynamic assessment of a pupil refers to assessment through dialogue between an assessment leader and a pupil, where focus is placed on what shall be done to help the pupil reach a new functional level. The dynamic test has been developed in collaboration between Trøndelag Resource Centre and the North Trøndelag University College (HiNT) (2009)

Developing the assessment test

Background and goals

Olof Magne (1999) points to a range of factors which show that for a number of years, special teaching in mathematics has not been successful. Examples of methods used are:

- Teaching based on using textbooks from a lower level.
- Lessons often divided into two parts where the teacher first gives the pupils information collectively, and the students then sit at their desks reproducing a number of lined up tasks, with minimal communication.
- The teacher tryies to transfer knowledge to pupils through demonstrations.

Olav Lunde (2001) shows that the special needs help for the pupils who struggle with mathematics is rarely based on an analysis of what the pupils can and cannot cope with. Lunde (2006) also says that the criteria for identifying pupils' difficulties are unclear, the test procedures are varied and agreement on measures is slight. This reveals the need for good assessment methods that the teacher can utilise in order to provide the pupil with better adapted teaching. Lunde (1997) has developed a dynamic test which is specially directed towards pupils in primary education and has challenged us to develop a test for pupils from the 4th grade and later. Our ambition was to link the assessment tasks to mathematical competences detailed in the curriculum for the 10-year compulsory education in Norway.

Traditional assessment tests will, to a small degree, provide the information we want about the pupil. First and foremost they will illustrate the result the pupil achieved on the day he or she completed the assessment. However, this is information which is of little use for creating a suitable teaching plan for the pupil.

The purpose of the dynamic assessment is not to present a diagnosis, but to establish a good basis for providing the pupil with adapted teaching within mathematics. Therefore it is necessary to gather knowledge about how the pupil thinks during mathematical reasoning, when the pupil is unsuccessful but also when the pupil does in fact master the task. Furthermore, it measures whether the pupil can apply knowledge within a context, as well as which task specific strategies he/she has mastered and possible misconceptions they hold. With this kind of information, our goal is to reveal the pupils learning potential and explore what kind of support will help him or her to succeed.

Method

The dynamic assessment test is partly based on exercises from Norwegian National Tests for 4th Grade from the Learning Centre (2004, 2005). In a project in Klepp municipality, pupils in grade 8 that struggle with mathematics tried to solve problems given in national test for 4th grade. We have analysed 15 – 20 of these pupils' solutions, provided by Olav Lunde. The solutions showed that pupils in comprehensive schools struggle with basic knowledge they should have learned many years earlier. Our idea, then, was that one assessment test used dynamically could be useful for a wide age range of pupils. The National Tests was also closely linked to mathematical competences as given by Mogens Niss (2002) which might help assessment leaders analyse how to provide adapted teaching for the pupil.

The tasks are further supplemented by diagnostic exercises from material published by the Norwegian Directorate for Education and Training (1995). Diagnostic exercises are constructed to uncover misconceptions due the pupil's level of understanding mathematical concepts. An example of misconceptions is the understanding that a decimal number consists of a pair of numbers, the large ones on the left side of the decimal point and the small ones on the right side. Thus a pupil suffering from this misconception would say that 0.213 is a larger number than 0.5 because 213 is larger than 5. A task where the pupil is asked to compare 0.213 and 0.5 could highlight such a misconception and be characterized as a diagnostic exercise. On the other hand the task of comparing 0.2 and 0.57 would not be called diagnostic because the pupil would answer correctly, even when suffering from this misconception. Our intention is for diagnostic exercises in dynamic assessment to assist the assessment leader in obtaining relevant information about the pupil's mathematical competences.

The exercises based on National tests and diagnostic exercises have been redesigned to be used dynamically.

Our method for evaluating the material is described below; a lot of new ideas for exercises emerged through this evaluation process. Teachers and students trying out the assessment exercises suggested tasks related to new topics, new tasks related to existing topics and changing existing exercises due to context, to the given parameters and to aid facilities.

The assessment test has been developed over a period of 3- 4 years and during this period it has been tested in a variety of contexts:

- Teachers and Educational Psychology Service advisors from several networks in Trøndelag have used the material for a period of one whole school year. The three different networks each have 5 – 10 schools, each school with at least three teachers. Each school cooperated with an educational psychological service office in a network group. Each network group was linked to a chosen pupil in their own school who struggled with mathematics. Throughout the year the teachers and advisors attended nine or ten on-site sessions. Here they received theory lessons and examples of dynamic assessment as well as other, supplementary tests and mathematical didactics linked to central topics. The participants increasingly started their own assessment, discussing and explaining the results and planning adapted teaching for their own pupils.
- Students from North Trøndelag University College studying Difficulties in Mathematics, an in-service training programme for teachers, also tried out the material systematically. The course has been held twice using dynamic assessment exercises. The courses were based on the teacher students implementing theory on a chosen pupil at their own schools. Dynamic assessment was a central part of this, in addition to preventive teaching and planning adapted teaching for the

pupil. During on-site sessions, students presented their own assessment processes discussing how to use the results in their teaching. At one of these courses Master's student Heidi Kristin Holmen interviewed some of the teachers about the dynamic assessment and their teaching. The findings are described in her Master's thesis (2009); some examples are outlined under the heading "What we found – experiences".

- Students enrolled in the Special Education programme with advanced studies in mathematics difficulties at the South Trøndelag University College (HiST) have also tested the dynamic assessment material on pupils with special needs in the work experience periods of the study.
- I myself have tested the material in the cases of approximately 30 different pupils where Trøndelag Resource Centre has supported schools and the Educational Psychology Services with assessment and advice. The pupils were mainly from 4th to 10th grade, but also a few from upper secondary school.

Feedback on the content and design of exercises and guidelines has been discussed both in the actual network and courses as well as in the specialist environment of the resource centre and North Trøndelag University College. Some new suggestions have been evaluated, and the material has thus been developed to its final form.

The dynamic assessment is only one stage in an evaluation model. Before a pupil undergoes dynamic assessment, he/she has usually gone through some kind of screening process, observation in the classroom or that the parents have noticed that their child struggles a lot with mathematics. Such indications should lead to a more detailed investigation such as dynamic assessment. In parallel with a dynamic assessment, some supplying tests should also be carried out, for example including different cognitive tests, memory tests and a reading test. These will give additional information about other difficulties as well as the mathematical difficulty, but they will also indicate the pupil's strengths and weaknesses.

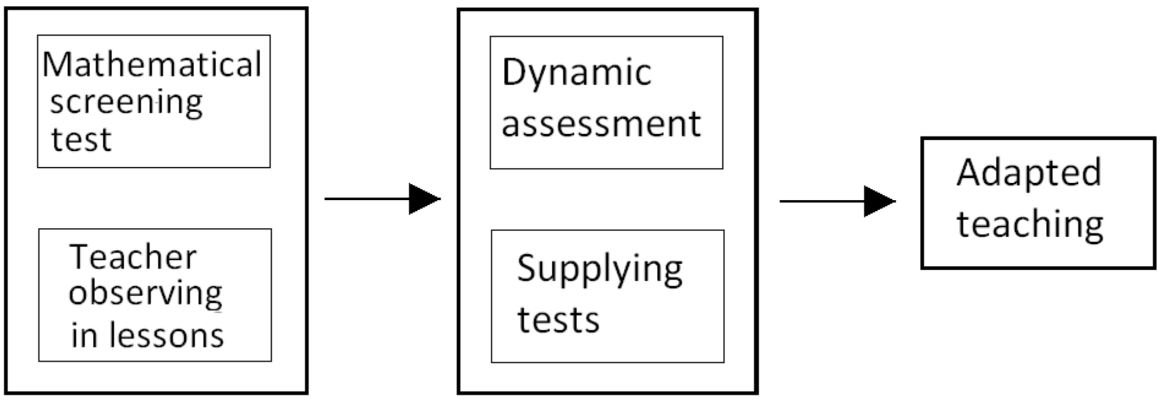


Fig. 1. The evaluation model

Dynamic assessment, however, is the focus of this paper and it may certainly be used as a stand-alone tool to assist schools in gathering information about pupils.

Theoretical background

Dynamic assessment provides information about what the pupil masters, both when working on his/her own and with the support of a competent adult. The theory is based on social constructivist thinking and especially Vygotsky's (1978) theory, where learning is described as the transition between two development zones, the actual to the potential zone with the help of supporting scaffolding. Since the aim of dynamic assessments is not to give any diagnoses, but to identify the pupil's competencies and qualifications in order to create suitable teaching, it is in its nature very close to a teaching and learning situation. An assessment leader will therefore take on the role of supporting scaffolding in the actual assessment situation. This is a demanding role where the assessment leader must possess good knowledge of professional practice and educational theory and be aware of his/her manner of communicating. Certainly the assessment leader should also be knowledgeable about mathematical learning and mathematical difficulties such as typical misconceptions. In this role it is very important that he/she is conscious about the interaction and about not trying to transfer knowledge but rather giving the pupil a chance to construct his/her own knowledge with appropriate support.

In dynamic assessment we want to identify some relationships which traditional assessment seldom detects (Lunde, 1997):

- *What previous knowledge does the pupil have and how is this used in the thinking process?*
- *How does the pupil think during mathematical reasoning?*
- *What is the quality of this knowledge; can the pupil apply the knowledge in a context?*
- *What task specific strategy has the pupil mastered?*
- *What are the pupil's possible misconceptions?*
- *Does the pupil have automating problems?*
- *What are the pupil's abilities, interests and needs?*
- *What will the pupil be able to learn in the future?*

In principle, the method of asking can be divided into two forms of questions:

- The *evaluating form of question*, where the teacher poses questions and the pupil's answers are basically right or wrong. Olga Dysthe (1995) calls this "question-answer-evaluating" such as is used in the primary school. The focus is product oriented, towards the answer, towards factual knowledge or towards "the correct procedure". The questions will often be leading and closed.

- The *assisting form of question* is a method where authentic, open questions (Dysthe, 1995) invite the pupil to reflect. The purpose is to help the pupil along, not by giving the answers, but by letting the pupil discover the possibilities him/herself. The pupil's reflections may go in different directions, but by posing new follow-up questions and clues the assessment leader leads the pupil toward the goal, and the pupil's mathematical thinking will be assessed.

It is important that the assisting form of questions should be employed. Through our learning networks and courses we found that many teachers are unused to ask assisting questions, are unaware of the form of question they use. Posing questions in this way might therefore be difficult, and for many teachers it will demand a good deal of preparation. However, suggestions for such dialogue are provided for each individual exercise in the guidelines for the assessment.

At the same time as the assessor offers support through dialogue, the pupil is challenged to present his/her method of thinking in a solution to the individual exercise. Through this communication and by simultaneously observing the pupil, the assessment leader forms a picture of what the pupil can or cannot do, how the pupil does it and why the pupil chooses to do it the way he or she does.

The assessor thereby provides a basis for the methodical teaching programme, learning content and educational strategies that should be chosen for the pupil.

My definition dynamic assessment is (Aastrup, 2009):

> *Dynamic assessment of a pupil refers to assessment where the communication between an assessment leader and pupil is based on dialogue and focuses on what shall be done to help the pupil reach a new functional level.*

Descriptions of the material

Content of the material

The dynamic assessment test consists of four main parts:

General guidance for dynamic assessment provides an introduction to dynamic assessment and a brief description of basic theory. It also illustrates the main principles of dynamic assessment and describes what kind of preparations the assessment leader should carry out before every test in general, and specifically before using the material for the first time. Some of the issues are described in the theory section.

The second part is the task specific guidance for dynamic assessment. This outlines some basic theory behind each task is outlined. This might be brief descriptions of different kinds of strategies pupils normally use, typical misconceptions, contents of some mathematical concepts and so on. In addition to this, there are suggestions for how to start a good communication with the pupil depending on what kind of answers the pupil gives to that specific task. These suggestions are not to be followed rigorously; they are just pieces of advice to support the assessment leader in posing meaningful, open, helpful questions to the pupil. There is also a link to the mathematical competences shown in the curriculum for the 10-year compulsory school in Norway.

One of the tasks relating to the position system is:

A specific number consists of 8 thousands, 4 tens and 3 units. Will you please write this number?

Oppgave 3 b Nullen som plassholder

Gjøres bare dersom eleven med eller uten støtte fra Kartleggingsleder lyktes i oppgave 3a).

K:	Et bestemt tall betyr 8 tusenere 4 tiere og 3 enere. Vil du skrive opp dette tallet?

Fig. 2. The zero's role in keeping a position

The guidance shows different, typical solutions, both correct and incorrect ones. One of the incorrect solutions in this example is the answer "800043". In this case the guidance manual provides some brief information about theoretical background as well as a proposal for starting a dialogue.

Alt. 3 **800043**	
Teoretisk bakgrunn	*Forslag til samtale*
Eleven kan ha lært seg hvordan 8000 skrives og skriver først det. Så kommer 43, som også huskes, etterpå. Dette <u>tyder</u> på en tenkemåte delvis basert på et additivt tall-system: 800043 betyr 8000+43.	*K: Vil du forklare hvorfor du mener dette er det riktige tallet?* *Mulige suppleringsspørsmål:* *K: Hvor mange enere har vi her? Og tiere? Hundrere? osv.* *En kan også spørre hva tallet 8043 betyr.*

Fig. 3. From the specific guidance. On the left is some brief information about the theoretical background, and on the right a proposal for starting a dialogue.

The text of the brief theory for this specific task is:

> *The pupil might have learned how 8000 is written, and writes that first. Then, afterwards, comes 43, which also is remembered.*
>
> *This <u>indicates</u> a way of thinking partly based on an additive number system:*
>
> *800043 means 8000 + 43*

The proposal to start a dialogue reads ("A" means the assessment leader):

> *A: Will you explain why you believe this is the correct number?*
>
> *Possible follow-up questions:*
>
> *A: How many units do we have here? And tens? Hundreds? Etc.*
>
> *One may also ask what the number 8043 means.*

The third part of the dynamic assessment manual are sheets for the assessment leader's notes. It is important to save all relevant information from the assessment situation such as a description of strategies and what kind of support would be useful for the pupil. This is described in “Carrying out an assessment” below.

Task sheets for the pupil are arranged so that there is only one task on each sheet. Thus, the pupil may concentrate on one separate task at the time. This also makes it easier for the assessment leader to drop some of the exercises if he/she finds that relevant. The pupil should be urged to put their solutions in writing on these sheets in order to collect as much information as possible.

Preparation for assessment

There are several reasons that support the choice of the pupil’s teacher as assessment leader:

- The assessment leader should know the pupil well, the pupil’s interests, weaknesses and strengths, as well as the linguistic expressions the pupil uses.
- It is important to create a confident assessment situation. This should be possible with normal good relations between pupil and teacher. However, sometimes the relationship between the pupil and the teacher is strained, in which case some other teacher should conduct the assessment (and probably also carry out the adapted teaching).
- The assessment is an important part of the basis for the adapted teaching which precisely the pupil’s teacher is supposed to plan.

The assessment leader should take time to familiarise him/herself with the assessment material in advance. This refers both to information about specific tasks as outlined in “Descriptions of material” above, and the more general principles and philosophy behind dynamic assessment (See “Descriptions of material” and “Theoretical background”). This is also mentioned in “What we have found - experiences” by teachers in the period of developing the assessment. It is of great importance to understand that the guidance is not a manual to be followed to the letter, but a tool to help create a good situation for carrying out a survey of the pupil’s competencies and learning potential. It is recommended that two teachers cooperate in carrying out the assessment when the material is used for the first time. This, in particular, would be useful in order to enter the role as supporting scaffolding (see “Theoretical background”), practising the assisting form of questions. Concretising facilities are not a part of the assessment material, but some simple kinds of concretes might sometimes be helpful to have in reserve.

Carrying out an assessment

The entire assessment could be carried out on the same day, but this is not necessary. For instance, the ability of pupils to concentrate varies greatly and some pupils may therefore need several breaks while others are motivated to complete the assessment continuously. Sometimes it is only relevant to conduct parts of the assessment, depending on the pupil’s competences. My own experience of carrying out the entire assessment suggests that it takes on average about two hours to complete.

Before setting the first task, the assessment leader should inform the pupil about the purpose of the assessment. . The leader should also tell the pupil that he/she will be asked to clarify mathematical thinking and strategies both when the solution is correct and when it is not. The assessment leader should also explain that in order to help the pupil in lessons in the future, knowledge about thinking and strategies is more important to the assessment leader than the correct answer. The pupil should also be informed that he/she may use the time he/she needs to solve the problems and that the assessment leader will help the pupil, not by giving solutions, but by asking new questions, giving hints and suggest the pupil to try in other ways.

During the whole assessment it is important to create a safe atmosphere for the pupil. The assessment leader should encourage the pupil, give positive feedback even to minor successful efforts and avoid focusing on correct and incorrect answers. He/she should also exert him/herself in order to support the pupil in mastering the exercise.

The assessment leader should be very careful to notice all relevant information, writing down necessary keywords to remember the pupil’s descriptions, strategies and other relevant information during the test. Although he/she needs time during the assessment to write this down, some important information may lack, and he/she should therefore complete their comments very soon after the assessment situation.

Here is an example from my own practice on using the assessment material. In this case, the resource centre was asked to carry out an overall assessment relating to specific mathematical difficulties and I carried out a dynamic assessment. The pupil, "Vegard", 14 years old, was in grade 9. The exercise was quite a difficult one, at least for pupils who struggle with mathematics. It is about modelling mathematics, "Vegard" was given a text, both verbally and in writing. The task is to find a correct arithmetic expression to solve the problem. The problem was:

> *Line is going to dress. She will put on a sweater and trousers.*
>
> *This can be done in several different ways.*
>
> *She has 4 different trousers and 5 different sweaters.*
>
> *In how many different ways can she dress?*

Oppgave 12 forts.

Tolking av tekstoppgaver – valg av regneart

Se på teksten i firkanten.
Hva slags regnestykke mener du skal til for å løse oppgaven?

---

e. Line skal kle på seg bukse og genser. Dette kan hun gjøre på flere forskjellige måter. Hun har 4 forskjellige bukser og 5 forskjellige gensere.

Hvor mange forskjellige måter kan hun kle på seg på da?

Fig. 4. The given task

This is a reconstruction of the dialogue situation:

> Vegard moves a little and ask: *How many different ways, what does that mean?*
>
> I answer: *She may for example put on one pair of trousers and then choose one of the sweaters. Then there will be several different ways.*
>
> Vegard: *Oh yeah, then it will be 4 + 5 = 9. He writes it.*

Skriv opp regnestykket som passer her. Så kan du regne det ut. Du kan også kladde, hvis du vil:

4+5
=9

Fig. 5. Vegard's first attempt

The challenge here is how to use language as a support for Vegard. We continued, and the reader may comment that first part of my answer may be more focused on the product than on the process!

> I: *That's not quite right. If she decides to use that specific sweater, in how many ways can she then dress with those trousers?*
>
> Vegard: *She's got 4pairs of trousers; I suppose it will be 4 ways…*
>
> I: *Yes, right. How many will it be if she may also choose between the other sweaters*?
>
> Vegard hesitates. After a while I try to help him.
>
> I: *For the first sweater there were 4 ways, how many if she chooses one of the other sweaters?*
>
> Vegard: *It's 4 ways there too. 4 and 4, that's 8 different ways!*

At this point I wonder; what does he mean? Is it 8 ways in total, or for the 2 first sweaters only?

> I: *How many different ways do you mean that she can dress in now?*
>
> Vegard: *It will be 8 different ways, not 9!*

Again I had to find a new way to support "Vegard". If dialogue alone does not help, the manual suggests that the pupil may draw a figure.

I: *Maybe you could draw all the sweaters and all the trousers?*

Vegard: *OK, but I'll just draw some simple rough drafts…*

I: *That's OK.*

Vegard draws quickly, with a steady hand, next to his writing.

I: *Take a look at the first sweater (Vegard marks it with a cross). With that one she can dress herself in…*

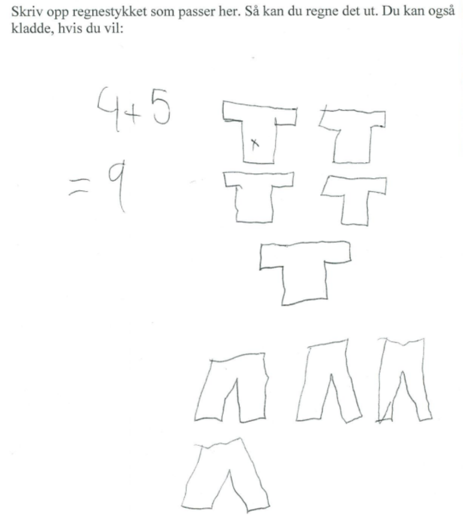


Fig. 6. Vegard's drawing

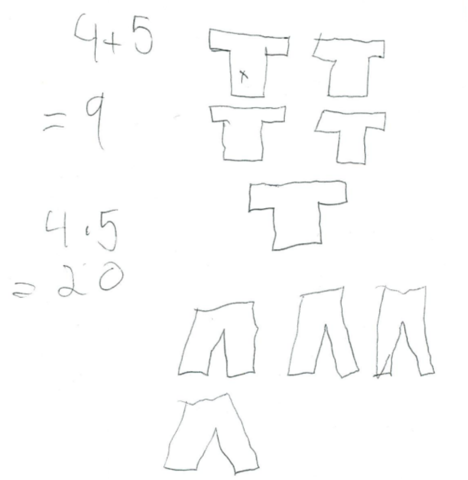


Fig. 7. The solution!

Vegard (interrupts): *In 4 ways. 4 + 4 + 4 + 4 + 4, that's 16. No… it's 20.*

I: *So which arithmetic expression can you use here?*

Vegard: *It will be 4 times 5 is 20.*

He writes it.

A little later Vegard exclaimed: *This was easy!*

What we have found - experiences

Reports from those who have used the material say it is quite demanding to become familiar with, but that it provides much relevant information about the pupil for the planning of special needs teaching. Evaluation carried out by teachers in learning networks and educational psychological services, trying out the material in real assessment cases in connection with their own pupils illustrates this. The responses to the evaluation question **"*How did the dynamic assessment test function for the teacher?*"**, for instance, included:

School A:

> *"The assessment test functioned well, but there was a lot of material to familiarise yourself with. We believe that when you have carried the test out a few times, you will become more secure about finding the follow-up questions."*

School B:

> *"We think it is comprehensive assessment test: It takes a long time to explain the matter to ourselves (the guidance), a long time to carry out, both for the assessment leader and for the pupil. Will this be too much for the class teacher to read up on?"*

School C:

> *"It was easy to get into both the method and the descriptions of the subjects, perhaps better than we had expected. We understand that it is a good idea to spend some time on preparations before the assessment in order to feel quit secure in what to carry out with the pupil."*

Many teachers see that their pupil has knowledge that they were unaware of. One evaluation question was: ***"Did you discover something new about the pupil, did you get confirmation of something you thought from before?"*** Typical answers were:

School A:

> *"We had thought the pupil was weaker than he appeared to be in the test situation. He had many thoughts about how he thinks and managed quite well to explain which strategies he uses in different tasks."*

School B:

> *"We are positively surprised about how much the pupil was able to do, but we also noticed some lack of understanding"*

School C:

> *We think the pupil mastered more of the exercises than we had originally expected.*

School D:

> *After the assessment, we have got a much more positive picture of the pupil's academic knowledge. This is both according to what she understands and masters, especially according to her interest and engagement. She appears to be better able to put into words /explain what and how she thinks when solving problems, more than we expected in advance.*

The teachers experienced that the assessment situation is regarded as positive for a large number of the pupils. The question *"**How did the assessment function for the pupil, was it a good/ bad experience?"*** gave answers like:

School A:

> *"It was a positive experience. (...) He expressed to us and at home that it was a positive experience and that he would like to continue."*

School B:

> *"The pupil and the teachers experienced the test situation as positive. We believe the pupil experienced success in relation to some of the tasks during the test."*

School D:

> *"She surprised herself (and the teacher) being more clever than she thought in advance. She was very pleased in retrospect, among other things she has told other pupils in her class what she have participated in and how enjoyable it was.*
>
> *As a positive additional result the pupil has been more enthusiastic and worked better in the lessons after finishing the assessment."*

School C:

> *"We experienced that the assessment functioned positively to the pupil. We are left with the impression that he experienced success and managed problems he originally said he would not master. "*

Some teachers explain that dynamic assessment leads to a change in their own teaching. Heidi Kristin Holmen (2009) has interviewed teachers participating in our study modules at North Trøndelag University College, about difficulties in mathematics after their dynamic assessment and adapted teaching in her master degree report about dynamic assessment. This indicates that carrying out dynamic assessment with their own pupils influences the way they teach pupils mathematics. One teacher says:

Another teacher says:

> *"I have noticed that when helping the pupils, I use the way of asking questions from dynamic testing in teaching situations with the kids. I spend more time listening to their thinking, and I try to guide them in a different way from before - I'm much more aware of this. In class situations I'm more conscious about focusing on their thoughts - they explain much better. We talk about how they think…. We establish a discussion in the class about strategies." (Holmen, 2009)*

A third teacher mentions the use of strategies in the classroom:

> *"I have listened to how they have been thinking, and they do think in different ways. So we have written it up, which made them listen to each other. Then they certainly get knowledge about many strategies. So, we use the word "strategies", yes, and I have become more conscious about that. Which strategy do you use? And, therefore they can explain to each other, so mathematics will not be the silent lesson."*

One teacher expresses that this process has also changed her view on what it is to be a teacher:

> *" I have not only changed my opinion about her, but I have also changed my view as a teacher. I am more conscious about many pupils' way of thinking (...). I still teach them a method, but I also emphasize that there are other ways of solving the problems. I do not think I would have said that before."*

The examples show that dynamic assessment might be a good tool to investigate a pupil's mathematical thinking in order to plan and carry out adapted teaching. It also seems to give pupils good experiences and influence teachers' ways of teaching mathematics and their views of what learning is.

Conclusions

The theoretical basis for dynamic assessment is well known but not always practised in assessments or teaching. So far experience indicates that dynamic assessment is a good tool to investigate a pupil's mathematical thinking in order to plan and carry out adapted teaching. However, using the dynamic assessment material does not guarantee that the assessment will be carried out dynamically. It is only meant to be a good tool in order to practise the dialogue in an appropriate way, discovering how to help the pupil reach a new functional level. Therefore, applying the philosophy of socio-cultural learning theory and training in order to establish good dialogues is important for the assessment leader.

Many pupils find the assessment situation positive, their knowledge is more appreciated than they are used to, they experience successes and a good atmosphere.

The dynamic way of meeting a pupil also influences the techniques teachers use in the classroom Teachers seem to be more open to alternative ways of solving problems. In many cases the focus of the lessons moves from product to process, inspiring pupils to talk about their own strategies and jointly reflect on different strategies.

References

Aastrup, S. (2009). *Dynamisk kartleggingsprøve i matematikk. For elever fra 4. – 10. trinn og elever i videregående skole.* Levanger: Trøndelag kompetansesenter.

Dysthe, O. (1995). *Det flerstemmige klasserommet.* Oslo: Ad Notam.

Holmen, H. (2009). *Dynamisk kartlegging. En kvalitativ intervjustudie om læreres erfaringer med dynamisk kartlegging av elever med særskilte behov i matematikk.* Masteroppgave i spesialpedagogikk, NTNU.

Lunde, O. (1997) *Kartlegging og undervisning ved lærevansker i matematikk*. Klepp: Info Vest Forlag.

Lunde, O. (2001). *Tilrettelagt opplæring for matematikkmestring*. Klepp: Info Vest Forlag.

Lunde, O. (2006). *Fra matematikkvansker til matematikkmestring*. Spesialpedagogikk nr. 4/2006

Magne, O. (1999). Kartlegging av matematikkforståelse. I *Pedagogisk-psykologiska problem.*

Niss, M. (2002) *Kompetencer og matematiklæring. Ideer og inspiration til udvikling af matematikundervisning i Danmark.* Nasjonalt læremiddelsenter/ Læringssenteret/ Utdanningsdirektoratet.

Diagnostiske prøver. (1995) Nasjonalt læremiddelsenter/ Læringssenteret/ Utdanningsdirektoratet.

Nasjonale prøver i Matematikk 4. trinn. (2004). Nasjonalt læremiddelsenter/ Læringssenteret/ Utdanningsdirektoratet.

Nasjonale prøver i Matematikk 4. trinn. (2005). Nasjonalt læremiddelsenter/ Læringssenteret/ Utdanningsdirektoratet.

Vygotsky, L.S. (1978). *Mind in Society. The development of higher Psychological Processes.* Cambridge, M. A., Harvard University Press.

As high school teacher I was responsible for special teaching in mathematics. I have been an assistant professor in mathematics for teacher and pre-school teacher students and had the main responsibility for developing an in-service training for teachers, ?Difficulties in Mathematics?. As senior advisor on Trøndelag resource centre I support schools and Educational Psychology Services on special needs in mathematics. I have developed a dynamic assessment material for students from 4th grade and up and will soon complete one for 1st to 5th grade. I am educated as a graduate engineer supplied with postgraduate certificate in education and special education.

svein.aastrup@statped.no

Arithmetica –Arithmetic art

Kristín Bjarnadóttir
University of Iceland –School of Education

Over the centuries, many Icelanders have striven to make knowledge of all kinds accessible to the general public through translations. They translated texts from Latin in medieval times, and added North European texts after the introduction of the Lutheran Evangelical religion in early modern times. Among texts preserved in manuscripts are arithmetic textbooks, which are clearly related to German textbooks. This paper will compare similar examples from four arithmetic textbooks and make conjectures on their connections as well as their motivation and interest to the general public.

Introduction

The concept of 'arithmetic for all' has different meanings, depending on time and location. This article concerns arithmetic in the early eighteenth century in Iceland, when arithmetic textbooks in the vernacular, Icelandic, were distributed from person to person in manuscript form.

Education in Europe was a privilege of the elite throughout the middle ages and far into modern times. Until the nineteenth century the elite knew Latin, the Lingua Franca of Europe. Iceland was, however, never densely populated, and the distance between the educated elite and the general public was not as great as in some more populated countries. Workers at farms were young people who could expect to become farmers in their own right, albeit as tenants.

A vast amount of many kinds of knowledge was translated into Icelandic from Latin from the twelfth century onwards. Therefore, access to knowledge was not restricted to the Latin-reading elite, but was available to all those who could read. Much of the translated material concerned the Christian religion, but all seven classical liberal arts were also presented: rhetoric, logic, grammar, arithmetic, geometry, astronomy and singing. However, geometry was only sparsely represented, and there is only sporadic mention of Euclid's Elements in Icelandic literature. Arithmetic, on the other hand, was introduced in a treatise in the thirteenth century, and over the centuries a number of treatises on chronology were written.

Arithmetic and the Reformation

In the sixteenth century the Reformation divided European culture between South and North. The Lutheran Reformation originated in Germany in 1517. One of its consequences was education of the public in the vernacular, primarily on religious matters and the Bible, but also other subjects. Iceland experienced greater German influence when the Icelandic Church adopted the Evangelical Lutheran faith in 1550.

German textbooks on arithmetic and other subjects were published in the Reformation countries: two German books exist under the title *Arithmetica Historica*, with the subtitle *Rechenkunst* / *Arithmetic Art*.

The first is *Arithmetica Historica*. *Die löbliche Rechenkunst* (Suevus, 1593). It was written by Sigismund Suevus/Schwob, and published in 1593 in Breslaw, now Wroclaw in Poland, a Lutheran Protestant town at the time. The longer subtitle explains that the examples are composed from the Holy Scripture and historical books. The book is comprehensive, 455 pages, in addition to an index and a foreword.

The other book in this category is *Arithmetica Historica. Das ist: Rechenkunst* (Meichsner, 1625a) by Georg Meichsner. It was published in 1625 in Rothenburg ob der Tauber, also a Lutheran Protestant town. The subtitle is long, and mainly a repetition of the title of Suevus' textbook. The content of the two books is very similar and many of the problems are the same. Meichsner's book is, however, much shorter, only 99 pages with the addition of the index. Neither of the books deals with fractions.

Meichsner also wrote two other books in the same series, *Arithmetica Poetica* (Meichsner, 1625b) and *Arithmetica Practica*, both published in 1625. All four books by Suevus and Meichsner are available digitally on the internet.

Icelandic Arithmetica Manuscripts

Printing originated in Germany in the fifteenth century and the technology spread rapidly. The first printing press in Iceland was established in the mid-sixteenth century. A number of religious books were produced in

Iceland around 1600 and onwards, but secular books were scarce. The printed material was important in order for the new religion to gain the public's trust.

In spite of the lack of printed books, Icelanders had ways to distribute knowledge that was of interest to them. At least three manuscripts from the first half of the eighteenth century exist, containing full arithmetic textbooks, i.e. with chapters on the number concept, numeration, the four arithmetic operations in whole numbers and fractions, and proportions in the form of the so-called *Regula de Tri*. The three manuscripts are preserved at the manuscript department of the National and University Library in Reykjavík. They are:

- Lbs. 1694, 8vo, *Arithmetica Islandica*, dated 1716, but examples point to 1733.
- ÍB 217, 4to, *Arithmetica – Þad er Reikningslist*, not dated, but examples point to 1721.
- Lbs. 1318, 8vo, *Limen Arithmeticum edur Eynfaldlegur Inngangur til Rettelegs Nams og Brukunar Þeirrar nafn frægu Reiknings Listar*, written by the Rev. Stefán Einarsson (1698-1754) in 1735, existing in a later manuscript.

All the manuscripts are of interest, but this article will mainly consider the first of them, ÍB 217, 4to, *Arithmetica – Það er reikningslist* (*That is Arithmetic Art*). Its subtitle is a direct translation of the subtitles from Suevus and Meichsner. Do they have anything in common?

The manuscript ÍB 217 cannot be the original of *Arithmetica*. For example it concludes abruptly in the chapter on proportions. In the catalogue of the library it is estimated to be written around the year 1750. On two instances in the text, the year 1721 is referred to as the 'present' year or the 'coming' year. It seems, therefore, that the existing manuscript is a copy of another text, closer to the original. According to its introduction it was comprised of four books, but the present manuscript comes to an abrupt halt in the second part of book III.

ÍB 217 Arithmetica – That is Arithmetic Art differs from the other two manuscripts in that it does not refer to local monetary units or units of measurement, and its introduction is rather theoretical. Its background is totally foreign, and no reference to Iceland or the Icelandic environment is to be found in it.

Euler's Einleitung zu Rechenkunst

Leonhard Euler was the most prolific mathematics writer of the eighteenth century. In 1738 he lived in St. Petersburg, Russia, when he published an elementary arithmetic textbook, *Einleitung zur Rechenkunst zum Gebrauch des Gymnasii bey der Kayserlichen Academie der Wissenschafften in St. Petersburg.* Gedruckt in der Academischen Buchdruckerey 1738; i.e. for use in the Grammar School of the Imperial Science Academy in St. Petersburg, printed at the Academic Book Press in 1738. The book is estimated to have been written in 1735 (Euler Archive). The extant version of Euler's *Rechenkunst* is a transcript of the entire text of the book, preserved on the Euler Archive website (Euler, 1738).

Its arithmetic procedures are more modern than in our ÍB 217 *Arithmetica*, and it is also dated at a later time. However there are a number of problems that ÍB 217 has in common with both Euler's *Rechenkunst* as well as the *Arithmetica-Historica-Rechenkunst* textbooks by Suevus and Meichsner. In the following we shall compare the four books to reveal their common examples, while the resemblance of the actual text of ÍB 217 to the texts of the other German books turns out to be minimal.

ÍB 217 Arithmetica Compared to Other Books

Numeration

What soon catches the reader's attention are the foreign and in many cases Biblical settings of the numerical examples in the Icelandic manuscript. In the chapter on numeration, the following examples illustrating number notation are found:

~ The number of years from the creation of the world until Christ was born: 3,970 years
~ The cost of the building of King Salomon's temple: 13,695,380,050 Coronatos Crowns
~ The annual cost of the government of the Emperor Augustinus: 1,200,000 Coronatos Crowns
~ The fortune of Sardanapalus, the King of Assyria: 145,000,000,000 Guilders
~ The number of grains of sand to fill the world: computed by Archimedes as 10^{63}, the unit with 63 zeros

The same examples are found in *Arithmetica Historica – Das is Rechenkunst* by Suevus, while the explanations are different. Suevus says about the example of the years from the creation of the world:

> *... Das ist die bestimpte zeit darin Gott seinen Son zu senden verheissen / auch seine zusage krefftig erfüllet hat /Galat. 3, daraus wir seine Treu und Wahrheit kennen lernen /... (Suevus, 1593, p. 4).*

This example is also used by Meichsner with reference to Psalm 33. In the Icelandic manuscript ÍB 217, this example is only used to demonstrate the base-ten placement system, while nothing is said about the content and there are no Biblical references. The number 3,970 is explained: 3 in the fourth place for three thousand, etc. Euler did not present this example in his *Rechenkunst*.

After some further explanation of the number notation, the example of the building cost of King Salomon's temple follows in ÍB 217. This example can also be found in the books by Suevus, Meichsner and Euler. All the books quote Bibliander, who states that the cost is 13,695,380,050 Crowns. The ÍB 217 says Coronatos Crowns. Suevus and Meichsner refer to the Bible, Exodus 34, while the ÍB 217 and Euler do not.

The books disagree on Emperor Augustinus's annual cost of running the state, i.e. for the army and defence: Suevus and Meichsner calculate the cost at 12,000,000 Crowns, while in the ÍB 217 the cost is one tenth of that, 1,200,000 Crowns. Euler's *Rechenkunst* agrees with ÍB 217.

The amount of the treasures and wealth of King Sardanapalus in Assyria is 145,000,000,000 Gillini (Guilders) in the ÍB 217. Euler's *Rechenkunst* agrees with ÍB 217, also on the currency, Guilders. Suevus and Meichsner have 154,000,000,000 Crowns.

Archimedes's number of grains of sand to fill the world, 10^{63}, is neither found in Suevus' book nor in Meichsner's, although Suevus calculates the number of grains of sand in the sea as $8 \cdot 10^{37}$ in an example, attributing the number to Archimedes. Euler states that the number of grains of sand in the world amounts to 10^{51}.

Problem	**Suevus**	**Meichsner**	**ÍB 217 manuscript**	**Euler**
The number of years until Christ was born	3,970 years (p. 4)	3,970 years (p. 3)	3,970 years (p. 4)	
Cost of the temple of King Salomon	13,695,380,050 Crowns (p. 4)	13,695,380,050 Crowns (p. 2)	13,695,380,050 Coronatos Crowns (p. 5)	13,695,380,050 Crowns (1, p. 11)
The state cost of Emperor Augustus	12,000,000 Crowns (p. 5)	12,000,000 Crowns (p. 2–3)	1,200,000 Crowns (p. 6)	1,200,000 Crowns (1, p. 11)
The fortune of Sardanapalus	154,000,000,000 Crowns (p. 6–7)	154,000,000,000 Crowns (p. 3–4)	145,000,000,000 Guilders (p. 6)	145,000,000,000 Guilders (1, p. 12)
Archimedes' Sand reckoning	$8 \cdot 10^{37}$ (p. 7)		10^{63} (p. 6)	10^{51} (1, p. 12)

Table 1: Comparison of examples on numeration

Suevus continues to discuss large numbers and also the number 7 in various religious contexts, Meichsner continues directly with addition, while ÍB 217 lays down common notions for the arithmetic operations.

Common Notions for Arithmetic Operations

The geometry of the 13 books of Euclid's *Elements* was little known in Iceland. However, Book 7 concerns arithmetic, and ÍB 217 *Arithmetica* quotes definition 2 of Book 7, for the number concept – that 1 is not a number but the origin of numbers – but reveals some doubts on this, thus agreeing with concurrent ideas by the Dutch mathematician Stevin, in his book *l'Arithmétique* 1585 (Struik, 1958, pp. 494-496). ÍB 217 goes on to list a number of rules, not common in arithmetic books, but originating in the common notions of Euclid's *Elements*. The references in parentheses are to Heath's edition (1956, I, pp.155, 223). Some of the other notions are found in some manuscripts of Euclid's *Elements* (Heath, 1956, p, 223).

1. The whole is more than its part (Common notion 5).
2. The whole equals all its parts.
3. Whose parts all are equal, are equal to one another.
4. If equals be added to equals, the wholes are equal (Common notion 2).
5. If equals be subtracted from equals, the remainders are equal (Common notion 3).
6. If equals be multiplied by equals, the wholes are equal.
7. The squares of equals are equal.
8. If square numbers and cubic numbers are respectively equal, then also their roots are equal.
9. Thus also their halves and doubles are equal.
10. No one can for himself measure all numbers.
11. All numbers have their measure of units.
12. The unit neither multiplies nor divides.

This list of common notions is not found in any of the above-mentioned books by Suevus, Meichsner or Euler, but it has some resemblance to a list of common notions found in the French mathematical writer Mersenne's edition of the *Elements* (Mersenne, 1644, pp. 3, 22–23).

Addition and Subtraction

The manuscript ÍB 217 continues with chapters on the four arithmetic operations. The following examples are found under addition:

- The age of Methusalem, who according to Holy Scripture was 187 years old when he begat Lamech, after which he lived for 782 years, to the age of 969.
- 'I want to know how many years have passed since the poet Homerus lived. Aulus Gellius writes that he lived 160 years before Rome was built, but the city of Rome was built 752 years before the birth of Christ, and the number of years since Christ was born until now is 1721 years.' This account provides important evidence about the origin of the text.

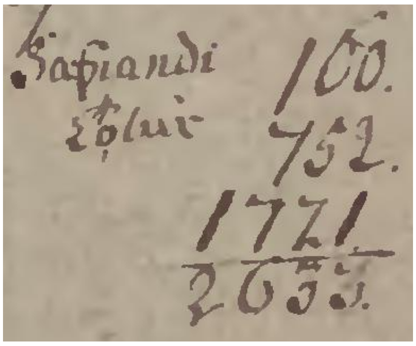


Figure 1: The age of Homerus, revealing the year when the text was composed.

- The total number of the Greeks and Trojans deceased in the Trojan War was 1,568,000.
- Four men owe 6,952, 8,346, 6,259 and 5,490 each, a total of 27,047 monetary units.

Below is a list of which of these examples are found in the German books by Suevus, Meichsner and Euler:

Problem	Suevus	Meichsner	ÍB 217 manuscript	Euler
The age of Methusalem	969 years (p. 22–23)	969 years (p. 5)	969 years (p. 10)	969 years (2, p. 6-7)
Example revealing the 'current' year	Number of years since the creation of Earth, 3,970 + **1,590** (current year) (p. 47)	Number of years since the creation of Earth, 3,970 + **1,625** (current year) (p. 4)	The number of years since Homerus lived, 852 plus the current year, **1721** (p. 10)	The number of years since Homerus lived, 852 plus the current year, **1737** (2, p. 7)
The number of men killed in Troy	1,566,000 men (p. 45–46)		1,566,000 men (p. 11)	1,566,000 men (2, p. 6)
Four men owing a total of 27,047 monetary units			27,047 [monetary units] (p. 11–12)	27,047 Rubles (2, p. 6)

Table 2: Comparison of examples involving addition

The example of four men owing 27,047 monetary units with the same four amounts, in that case rix-dollars, is also found in Lbs. 1694, 8vo, *Arithmetica Islandica*, dated on its front page from 1716, but an example counts the present year to be 1733, which could be a date of its extant copy.

ÍB 217 contains no historical examples on subtraction. Its examples involve people owing money or needing loans. Euler's examples are also of that nature, but not the same examples. The books by Suevus and Meichsner have many historical examples on subtraction.

Multiplication

In contrast, there are several historical problems on multiplication in ÍB 217. First, however, let us compare the multiplication tables, called *Abacus Pytagora* in ÍB 217 and *Tabula Pythagorica* by Meichsner.

Suevus presented a 9×9 table (p. 109), but Meichsner (p. 14) in *Arithmetica Historica* and ÍB 217 (p. 19) a 10×10 table. Euler, on the other hand, presented separate tables for each number, the tables becoming smaller and smaller, e.g. the 8-times table is 8·8 = 64, 8·9 = 72, 8·10 = 80 (Cap. 4, p. 9). This is also the case for Meichsner's *Arithmetica Poetica*, which indicates that both forms were commonly used.

The chapters on multiplication include a number of similar or identical problems, such as on the circumference of the earth, see Table 3 below. ÍB 217 gives an example on the average number of hours in a year, see Figure 2.

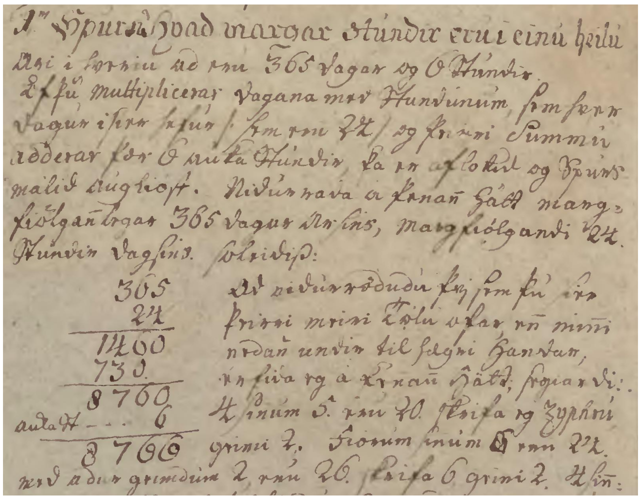


Figure 2: The average number of hours in a year is 8,766 in ÍB 217.

Problems	**Suevus**	**Meichsner**	**ÍB 217 Manuscript**	**Euler**
Circumference of the earth	360°·15 = 5400 miles (p. 128)	360°·15 = 5400 miles (p. 14–15)	360°·15 = 5400 miles (p. 23)	360°·105 = 37800 Werste (4, p. 13)
Number of hours in a year	(52·7+1)·360 +6 = 8766 hrs. (p. 127)	52·7·360 = 8,736 hrs. in 52 weeks in *Arithmetica Poetica* (p. 17)	365·24 + 6 = 8,766 hours (p. 22)	365·24 = 8760 hours (4, p. 14)
Size of a military group			264·100 = 26,400 soldiers (p. 27)	156·97 = 15,132 soldiers (4. p. 14)
Fortune in King David's grave	3000·600 = 180,000 Crowns (p. 171–172)		3000·600 = 180,000 Crowns (p. 28)	

Table 3: Comparison of similar problems involving multiplication

The manuscripts Lbs. 1694, 8vo, *Arithmetica Islandica* and ÍB. 35, fol., *Gandreið* (a five-page overview of basic arithmetic, composed in 1660, extant in a manuscript from 1770–1780) also contain the example on the circumference of the Earth as 5400 miles.

Division

Division is carried out in an old-fashioned way in ÍB 217 (see Fig. 3), and the same is true of Suevus. Meichsner did not illustrate the method, but Euler presented a more modern way, as far as can be seen from the on-line version of Euler's *Rechenkunst.*

ÍB 217: Dividing 17088 by 48

First 17 is divided by 4. Then subtract 3·4 = 12 from 17, makes 5, written above the 7 in 17, while the 4 is written below the 7 with 8 at its side. The 3 is written as the first digit in the quotient. Continued by 3·8 = 24, subtracted from 50, makes 26, 2 is written above the 5, 6 on the right side of 5, above 0, etc.

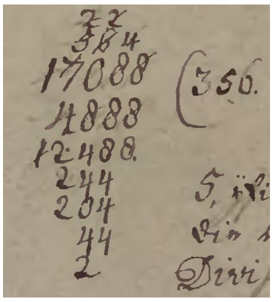


Figure 3: Dividing 17,088 by 48 in ÍB 217

Below is a table comparing similar examples involving division in the four books. The leap year example again indicates that 1720 or 1721 is the date of the Icelandic text, *Arithmetica – That is arithmetic Art*.

Example	**Suevus**	**Meichsner**	**ÍB 217**	**Euler**
Leap year	1591:4 = 397, remainder 3 (p. 175)		Coming year, **1721**:4 = 430, remainder 1 (p. 35)	
Golden number of year	(1591+2):19 = 83, remainder 15 (p. 176)		(1622+1):19 = 85, remainder 8 (p. 38)	

Table 4. Comparison of similar examples involving division.

Extracting roots
In ÍB 217 extracting roots follows on from division. Root extraction is not found in the three German books. The Icelandic manuscript presents a familiar example about the length of a ladder to climb a tower across a dike is presented: an entirely unfamiliar structure in Iceland. The height of the tower is 30 feet and the width of the dike is 28 feet. The tower is illustrated in Fig. 4.

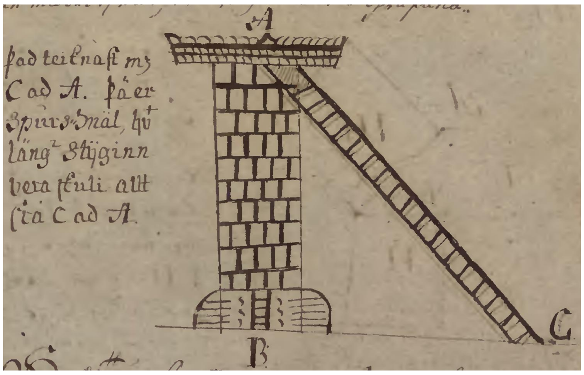


Figure 4: The tower, the dike and the ladder

The problem is solved by the Pythagorean Theorem. AB = 30, BC = 28.

$30^2 + 28^2 = 900 + 784 = 1684$

The square root of 1684 needs to be extracted in order to compute the length of the ladder.

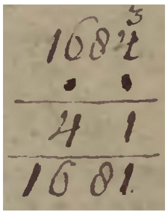


Figure 5. Extracting the square root of 1684

The remainder, when $41^2 = 1681$ is subtracted from 1684, is 3. The length of the ladder is approximated by 41 and $3/(2\cdot 41 + 1) = 41\ 3/83$ feet. There is no further discussion of the fact that the solution is an approximation only.

Discussion and Conclusions

The Icelandic ÍB 217 *Arithmetica* is clearly related to the books by Suevus from 1593 and by Meichsner from 1625, both published in Lutheran Protestant towns. There are, however, several examples in ÍB 217 which have a closer connection to Euler's *Rechenkunst*, written in 1735, than to the other two versions of *Arithmetica Historica*, while there are also examples which are common only to e.g. Suevus' book and ÍB 217, and not the other two. The dates of the three German books are fixed, while the Icelandic manuscript, ÍB 217 4to, which is estimated to date from 1750, contains 1721 as the 'present' or the 'coming' year. The original Icelandic text can therefore hardly be deemed to be modelled on Euler's *Rechenkunst*.

It seems logical to conjecture that there are links missing between the two *Arithmetica Historica* and Euler's *Rechenkunst*. One or more of the missing links could have served as a model for the Icelandic text *Arithmetica – That is Arithmetic Art*, which is clearly modelled after some foreign book, as there is not a single example that refers to the domestic environment. It is likely that more books of the same origin were in existence, circulating in the Protestant countries where Holy Scripture had been translated into the vernacular.

The arithmetic textbooks by Suevus and Meichsner may be considered as accompanying writings to the Bible, composed to educate the common people in arithmetic without leading their minds away from religion. The Icelandic *Arithmetica – That is Arithmetic Art*, and still more Euler's *Rechenkunst*, are less connected to religious topics, but contain many of the same historical references as the older books. Euler's book is in general a more modern textbook than the other three with respect to the number concept, computing procedures and other things.

It remains to be discovered how the examples reached Iceland. No Danish book has been identified containing similar examples to ÍB 217. Therefore it seems the logical conclusion to suggest that a German book was translated and slightly adapted to Icelandic. Indeed, nothing has been found in ÍB 217 to connect it to Iceland, except the language. After the mid-seventeenth century, Icelanders who went to university did so almost without exception in Denmark, due to certain privileges they had at the Regensen student residence in Copenhagen (Kristjánsson, 1999:12). German books were in circulation in Denmark around 1700, and Icelanders may well have found such a book there.

One possible candidate for translator is Bishop Jón Árnason, who studied in Copenhagen in 1690. Árnason wrote *Finger-Rhyme* (Árnason, 1739, 1938), a handbook to compute the calendar according to the Gregorian style, introduced in year 1700. The *Finger-Rhyme* is composed to educate the common people, so that every person, women included, could quickly and without hesitation compute the calendar.

Bishop Árnason wrote in 1739:

It is distressing to know that the art of finger-rhyme is mostly extinct in this country, which however was in my young days properly applied and used; many unlearned men and women could in a moment compute on their fingers both the dates of new moons and feasts … (Árnason, 1838, p. 11).

This reference constitutes evidence that education for all was valued in the early 1700s, for unlearned men and women.

My conclusion is that the ÍB 217 *Arithmetica – That is Arithmetic Art* is written in this spirit, to bring the arithmetic art to the common people, dressed in a familiar historical and to some degree religious form, in order to awaken motivation and interest. This was not only an Icelandic movement, but a part of the Humanist movement, to which Icelanders were strong adherents.

Acknowledgements
Many scholars have been kind enough to assist me in writing this paper. I wish to mention Albrecht Heeffer, professor at the University in Ghent in Belgium, who has studied Euler's sources of examples. He pointed out to me the websites with the books by Suevus, Meichsner and Mersenne. Professor Emeritus Harm Jan Smid in Leiden, and Professor Jan van Maanen, Director of the Freudenthal Institute in Utrecht, helped me to navigate in Dutch libraries and I enjoyed useful discussions with them.

References

Árnason, J. (1739, 1838). *Dactylismus Ecclesiasticus eður Fingra-Rím*. Copenhagen.

Euler Archive. Website: http://math.dartmouth.edu/~euler/. Accessed January 17, 2010.

Euler, L. (1738). *Einleitung zur Rechenkunst zum Gebrauch des Gymnasii bey der Kayserlichen Academie der Wissenschafften in St. Petersburg. Gedruckt in der Academischen Buchdruckerey 1738*. St. Petersburg. http://www.mathematik.uni-bielefeld.de/~sieben/euler/Rechenkunst.ocr.pdf. Accessed January 17, 2010.

Heath, Thomas L. (Ed.) (1956). *The Thirteen Books of Euclid's Elements*. Translated from the text of Heiberg with introduction and commentary. I–III. Second edition with additions. New York: Dover Publications.

Kristjánsson, A. (1999). *Nú heilsar þér á Hafnarslóð*. Reykjavík: Nýja bókafélagið.

Meichsner, G. (1625a). *Arithmetica Historica. Das ist: Rechenkunst durch alle Species vnnd fürnembsten Regeln, mit schönen Denckwürdigen Historien vnd Exemplis auß H. Göttlicher Schrifft, vnd guten Geschichtbüchern genommen, sampt deroselben bedeutung* ... Rothenburg an der Tauber. http://daten.digitale-sammlungen.de/~db/0002/bsb00024522/images/. Accessed January 17, 2010.

Meichsner, G (1625b). *Arithmetica Poetica. Das ist: Rechenkunst mit allerley Reymen, vnd Componirten Exempeln illustrirt*. Rotenburg. http://daten.digitale-sammlungen.de/~db/0002/bsb00024521/images/. Accessed January 17, 2010.

Mersenne, F.M. (1644). *Vniversæ Geometriæ, Mixtæqve Mathematicæ Synopsis, et Bini Refractionvm Demonstratarvm Tractatvs. Studio et Opera F.M Mersenni M*. Paris. http://gallica.bnf.fr/ark:/12148/bpt6k81532v.zoom.r=mersenne+universae.f57.langEN. Accessed 11 July 2009.

National and University Library of Iceland – manuscripts

ÍB 35, fol. *Gandreið* (composed in 1660/manuscript in 1770-1780)

ÍB 217, 4to. *Arithmetica. Það er reikningslist*. (1721 /c.1750).

Lbs. 1694, 8vo. *Arithmetica Islandica* (1716 / 1733).

Stevin (1585). *L'Arithmetique*. Leiden. In D.J. Struik (ed.) (1958). *The principal works of Simon Stevin*. Volume II B Mathematics. Amsterdam: C.V. Swets & Zeitlinger.

Struik, D.J. (ed.) (1958). *The principal works of Simon Stevin*. Volume II B Mathematics. Amsterdam: C.V. Swets & Zeitlinger.

Suevus, S. (1593). *Arithmetica Historica. Die löbliche Rechenkunst. Durch alle Species und fürnembste Regeln mit schönen gedenckwirdigen Historien und Exempeln; auch mit Hebraischer, Grichischer, und Römischer Münke, Gewicht und Mass, deren in heiliger Schrifft und gutten Geschichtbüchern gedache wird, der lieben Jugend zu gutte erkleret. Auch denen die nicht Rechnen können, wegen vieler schönen Historien und derselbigen bedeutungen lustig und lieblich zu lesen. Aus viel gutten Büchern und Schrifften mit fleis zusammen getragen*. Breslaw: Gbawman. http://www.archive.org/details/arithmeticahisto00suev. Accessed January 17, 2010.

Kristín Bjarnadóttir, associate professor at the University of Iceland, School of Education, completed a B.A.-degree in physics and mathematics at the University of Iceland, M.Sc. in mathematics at the University of Oregon and Ph.D. in mathematics education at Roskilde University Centre, Denmark. She taught mathematics and physics at secondary schools in Iceland and became employed at the School of Education in 2003. Her research field as well as field of interest is the history of mathematics education.

krisbj@hi.is

Learning Environment: Influences of teaching materials

Guðný Helga Gunnarsdóttir
University of Iceland, School of Education

This paper reports a study conducted in an eighth grade classroom in Iceland. The focus of the study is on how the learning environment in the class is influenced by the teaching materials used by the teacher. In the teaching materials an emphasis is put on investigative approaches. What happens when particular materials are used? Do the materials influence the learning environment? Eight lessons were observed during a 12 day period. The learning milieu during that time can be characterized as a landscape of investigation where the students engage in investigating numbers and number patterns and solving problems that have a reference to pure mathematics. This environment is influenced by the teaching materials even though the texts are not used directly in the observed lessons and the teacher's use of the teaching materials is more flexible than in most international studies on textbook use. It is also different from the environment observed in other Icelandic studies of mathematics classrooms

Introduction

For the last four years I have been involved in writing new teaching materials[8] (*Átta 10*[9]) for lower secondary schools in Iceland. My aim is to investigate the use of these materials in Icelandic mathematics classrooms. My primary interest is studying how students deal with the tasks outlined in the teaching materials and the development of the learning environment in these classes. I am also interested in finding out what purpose the students see in their mathematics learning.

This paper focuses primarily on the learning environment. When designing the materials we, the authors, emphasized an investigative approach that provides opportunities for students to reflect on their learning by discussing their solutions and strategies while working on problems and tasks. Investigative approaches, often referred to as "inquiries" in the literature, emphasize asking questions, seeking answers, recognizing problems and pursuing solutions. In contrasts with the almost exclusive reliance of traditional mathematics teaching on the presentation of mathematical information and the occupation of students in textbook exercises, the investigative approach provide multiple opportunities for teachers to meet the different needs of students, and for students to engage in wondering, exploring, investigating and looking critically at what they do and discover (Jaworski 1994).

This does not mean that informative and exercises will not not form a part of rich and responsive mathematics learning and teaching. However I have found it interesting to discover the range of learning environments in classes were the investigative teaching materials I have designed are used as a basis of instruction.

The emphasis on investigative approaches to teaching is coherent with National Curriculum Guidelines in Mathematics first published in 1999 and revised in 2007. These guidelines emphasize situated mathematical processes (mathematics as language, problem solving, logic and reasoning and connections to daily life) and the responsibility of teachers to meet different needs of students by emphasizing these processes and engaging students in varied classroom practices (2007, Aðalnámskrá grunnskóla, stærðfræði).

Theoretical background

In Iceland very little research has been carried out in mathematics classrooms. In the early nineties, research on mathematics classes in Icelandic middle schools showed that little whole class teaching

[8] I use the words teaching materials to emphasize that the materials consist of more than just textbooks for students. They consist of textbooks, teacher guides, assessment materials, interactive web programs, worksheets and thematic booklets. I find it important to look at all these materials as a whole.

[9] *Átta 10. Stærðfræði 1-6.* Teaching materials in mathematics for grades 8-10. Published by the National Centre for Educational Materials in 2005-2008. Authors: Guðbjörg Pálsdóttir and Guðný Helga Gunnarsdóttir

was taking place — students were mostly working individually with the textbooks (Sigurgeirsson, 1994). Recent research in grades 8 - 10 shows similar pattern where students work individually from the textbook and the teacher circling in the class helping individuals. The students work at their own pace according a teaching plan. Sometimes the teacher gives explanations on the board for individuals or small groups or sets up discussions related to items many students are struggling with. In most schools there are regular tests accompanied by a public review-process (Savola, 2008; Karlsson, 2009).

Internationally, many studies of mathematics classrooms conducted during the last 10-15 years have focused on different aspects of the teaching and learning relationship. (Stiegler & Hiebert, 1999; Clarke, Keitel, & Shimizu, 2006; Alrø & Skovsmose, 2002; David & Lopes, 2002; Goodchild, 2002; Kieran, 2001; Mercer & Sams, 2006). A number of classroom studies have taken into account the role of mathematical tasks and their influence on student learning and classroom discourse (Boaler, 2006; Henningsen & Stein, 1997; Stein, Grover, & Henningsen, 1996). Few have focused on the textbook and its use in the classroom. Johansson (2006) used data from the Swedish section of the Learner Perspective Study (LPS)[10] to explore how the textbook influenced what happened during lessons, and Pepin and Haggarty (2002) studied English, German and French textbooks and their use in classes. I have not come across any studies where textbook authors study how the materials they have designed impact on practice. It is of interest to study whether teaching materials that have a clear aim to influence classroom practices do so in reality. This study of a single classroom where particular teaching materials are in use provides an opportunity to investigate both the implementation of by the teacher and the milieu of learning created.

In this paper I focus on the learning environment in the class. Alrø and Skovsmose (2002) define two types of classroom practices (Figure 1). The Exercise Paradigm, often linked to traditional mathematics education, forms a milieu based on the traditional student textbook: prescriptive exercises dominate the work in the class, and classroom communication is in turn dominated by a quizzing pattern where the students try to find out what it is the teacher wants them to do. In contrast a Landscape of Investigation promotes a more open ended exploratory and problem-based milieu where communication takes the form of a dialogue between students and teacher. Within these contrasting learning milieus mathematical activity can also have different types of reference. It is thus possible to have reference to pure mathematics, to a semi-reality or have real-life references. Skovsmose and Alrø combine two different paradigms of classroom practice and three types of references into a matrix showing six different learning milieus

	Paradigm of exercise	Landscapes of investigation
References to pure mathematics		
References to a semi reality		
Real references		

(Alrø and Skovsmose, 2004 p.50)

Figure 1: Milieus of learning.

I will use the matrix of the learning milieus as an analytical tool to find out whether the learning milieu in the lessons observed can be characterized as a landscape of investigation or within the exercise paradigm. I use this framework to analyse the activities in the lessons I have observed with the following questions in mind.

- *Is there evidence of the students entering into an inquiry process?*
- *Does the setup of the tasks leave some room for the students to form their own questions?*
- *What kind of reality do the tasks refer to?*

---

[10] The Learner´s Perspective Study (http://extranet.edfac.unimelb.edu.au/DSME/lps/)

I will both look at the mathematical activities the students are working on, the setup of the tasks by the teacher and examples of the dialog between the students. I will also look at the classroom activity form a social perspective and analyze which classroom social norms and socio-mathematical norms come into play in the classroom (Cobb 2000, Yackel & Cobb 1996).

Data and methodology

Data was gathered in one 8th grade class in Iceland. Nineteen students formed a diverse group in the classroom. I chose a teacher who was using the teaching materials *Átta 10* in his class and who I also knew understood the ideas behind the materials. I observed all teaching in the class during a 12 day period or 8 lessons in total. The time of observation was chosen to my convenience and the teacher worked according to his usual plan. He knew which week I would be coming two weeks in advance. All lessons were video recorded. One camera was aimed at the teacher when he was speaking at the board in front of the class and 2-3 cameras were aimed at small groups two three of students working together. All students were then interviewed either alone or in small groups. The interviews with the students were stimulated recall interviews were the students were shown the worksheets and notes they had worked on during the period of observation and handed in to the teachers and short segments from the videos showing them working in the class (2003, Busse & Ferri). In the interviews I collected information on the students background, their views of themselves as mathematics students, their future plans and weather they thought mathematics would be important for them in that context. I also asked them what they thought of the assignments they had been working on in class during the period of observation, what they felt they had learnt from dealing with them and what purpose they thought the served. All interviews were transcribed and the students were given pseudonyms. The transcribed interviews and observations made in the classroom and by watching the videos from the classroom are the data used as a basis for analysis.

Analysis

The lessons took place in a small classroom that has been the teacher's mathematics classroom for several years. He has built up an environment that suits his needs as a mathematics teacher. The classroom has a computer and a smart board and the teacher also has a printer for his own use in the classroom. A portable laptop wagon with 6 laptops is kept in the room and it is frequently in use. Next door is a computer room that is often available and used during the lessons. There is a free seating arrangement and the students sit in rows of two and three but they are very flexible and move between groups and tables if asked to do so.

During the period of observation the students worked on numbers and number theory. An overview of the observed lessons is provided in figure 2.

	Numbers Overview of the teaching process
1.	Review of an end of unit test Students work on problem solving
2.	What different groups of numbers you know? Group work to gather information from the textbook and from the web.
3.-4.	Pascal's triangle – looking for patterns Numbers divisible by 3 and 5 Natural numbers, triangular numbers, square numbers in a row
5.	Pascal's triangle – looking for patterns Fibonacci numbers, powers of 10, Pascal petals http://mathforum.org/workshops/usi/pascal/petals_pascal.html
6.	Gauss – presentation and students work on solving problems in the textbook
7.-8.	Number square – summary and assessment.

Figure 2: Overview of the teaching process

Tasks were presented to students orally or with the help of a smart board and students were given open format worksheets to work with. Quite some time was spent on work with Pascal's triangle which the students first constructed and then used to look for patterns and different types of numbers.

In an interview about the teaching process the teacher described the main goal of the lesson as developing a basic understanding of different types of numbers. He considers this an important topic related to many other topics. One of his goals is also to do it in a fun and easy way where everyone is on the same level but with a possibility to move into different depths. He also mentioned that the students have just finished working through difficult algebra and functions so now it is important to do something different and have fun. He has been doing this in a similar way for the last 2-3 years and finds the process getting better every time. This time part of the teaching was carried out by two student teachers, but the overall process was designed by the teacher and the lessons planned with the student teachers.

The textbook was not in focus during the observed lessons. When asked about this the teachers pointed out that he used the ideology from the teaching materials and that the problems from the textbooks were used in many ways even though they were not literally taken from the book. The whole teaching process is influenced by the teaching materials and ideas taken from the textbook, worksheets and the teacher guide. The teacher explained that he found it good to take a rest from the textbook and not make it the main issue but rather a guide to the main content in his teaching. He also pointed out that if he had been working on another topic the textbook would probably have been more visible. The teacher described a flexible use of the *Átta-10* teaching materials in general where he chooses tasks from the teaching materials and other sources he finds important for the students to work on in each topic. The chapters in the textbook guide his choice of topics but he does not necessarily use the chapters in the same order. Now that he knows the grade levels 8-10 teaching materials well, this overview gives him flexibility and good possibilities to adjust differently to the needs of students.

From classroom observations and interviews it is evident that most students enjoy this work. While they do not find it too difficult, most are challenge, and most also continue to different levels in their investigations.

Katrín: *I found it very funny because it is not difficult to learn numbers and then it becomes amusing. It is amusing because you understand it.*

Anna: *A bit funny how this comes out like – you know – how this is according to the tables and can be in many different ways.*

Beta: *Always some pattern.*

Anna: *Yes.*

Beta: *Complicated a bit but still amusing .*

R: *Was there something funny about these tasks specially?*

Jónas: *Yes they were not difficult*

R: *Do you sometimes find it difficult in mathematics*

Jónas *No, just that if it is something difficult then it is – you know – boring.*

…

Two students who are well ahead of their classmates and usually work independently with grade 9 materials chose to take part in this teaching process. They became engaged in some challenging problems like finding the sum of the numbers from 1-150 before being introduced to Gauss´s method.

Einar: Yes I did that, I knew most of it but not all.

Magnús: This is more fun than something from the book, to do calculations on a paper or - you know - in a book.

When asked about the purpose of working with this topic the students find it difficult to find a practical use for this knowledge. They mention that this teaches them that there are all kinds of numbers that can be used in different ways and they think they will probably be able to use this knowledge sometime later in their studies. They are also fascinated by all the patterns they discover when working with the Pascal triangle.

R: What have you been learning in maths this week?

Anna : Only sit with this Pascal triangle, I had no idea that this, or you know I had some idea about this but I did not know how this worked.

Beta: There is so much of some patterns like this, which should not be possible, which is just so absurd - you know - should not really be possible, just it is.

Anna : Yes, exactly

Beta: Like with this triangle, there are always some patterns and something no one would have. thought about except some Pascal, Pascal something you know just some "skrilljón[11]*" patterns.*

Helga: To be good in it in high school or university or something to do with school.

Einar: There are probably many reasons for it, but I think we are being taught to find patterns in groups of numbers and something like that … and to teach us to look for it in an organized way.

Guðrún and Elsa mentioned that they think the purpose of these exercises is for them to get to know what mathematicians are thinking, so that they can also do and think like them. *Like the Greeks* they say *and when someone discovered the number 0*. Here they refer also to a film they have seen in school about Donald Duck in Mathmagic Land.

The number square

The most demanding task for the students was *Number square* (Fig. 3) which they worked on in two consecutive lessons. The task was used by the teacher as a kind of a summary and evaluation of what they had learnt during the teaching process.

Number square
Task for lessons 7- 8

Use the numbers 1-25

The sum in each row, column and both diagonasl is 65

5					
4					
3					
2					
1					
	a	b	c	d	4

- Square numbers in b5, d3, b1 and c1
- Prime numbers in a5, c5, e5, c4, a3, c3, e3, e2, and a1
- Triangular numbers in d5, e4, d3, a1, e1,c2
- Cube numbers in d3 and b2
- Numbers that are a power of 2 in b5, b2, e2, b1, an d3
- Two digit numbers that are the same if read backwards and forwards in a5 and d1
- Factors in 100 are in b5, d5, c4, b3, d3, a2 and e2
- The median of the numbers 1-25 is in c3
- Odd numbers are in row 3 and column c
- Numbers that are the same when they are turned upside down are in a5, d3,b2

Figure 3: Number square

[11] Icelandic children often count million, „skrilljón" - meaning some very big number.

The lesson started with a 10 minute introduction where the teacher explained the concept cube number which they had not heard before and reviewed the concept median which they had not worked with recently. He also gave them advice about how to start with the task and encouraged them to write down all numbers they thought could be used in an organized way. Then the students began the task in groups of two or three. All groups were engaged in the task but at different levels.

It took most of the students a full hour to complete the task. Árni, Páll and Gísli took the full time of two lessons to complete the task – they did not want to give up and continued even though almost all the other students had finished and had left the classroom.

R: *How did you find the problem you where working on yesterday*

Árni: *Difficult*

Páll: *I found it fun*

Árni: *Complex*

Gísli: *I found it kind of fun*

R: *When I observed you at first you were not really engaged – but when I came by later you were engaging in the problem - What got you started?*

Páll: *We thought we would be allowed to leave when we had finished it (laughing)*

R: *But you worked on it for the whole lesson?*

Páll: ***That*** *was not fun*

Gísli: *It was fun when you got started on it*

Páll: *Like a game –Sudoku*

Anna, Beta and Dóra worked on the task together, finishing the task in an hour. The audio tape of their conversations during the lesson showed that they struggled with it for the whole time — they hardly paused and they never moved off the task.

Anna: *No b5 – here is b5 og here is also b5 og here also, It is in all –factor in 100, power of 2 and a square number. Factors in 100 are 2 – 2 –hundred divided by 2 is fifty and fifty divided by 2 is and that - 25 by what – what do you get – wait a moment?*

Beta: *That is 25 divided by 2.*

Anna : *Yes here is just 1 – It is 2, 2, and 1 that are factors in 100. Is 25 a prime number? Is 25 a prime number? No – divided by 5, 5 times 5 is 25. Then it is 2, 2, 5,and 1 – then we have it. Ok, but (counts) 7 – it has to be 7 factors. You cannot divide 5 with anything. 5 is a prime number isn´t it?*

Beta: *Yes 5 divided by 5.*

Anna: *Yes 5 divided by 5 is just – I do not know how this is, we have not gotten this many (counts again 1-7)*

Here they encountered a problem because they understood <u>being a factor in</u> as <u>being a prime factor in</u>. This misunderstanding made the whole process more difficult for them, but at one point during the solution process, Anna: said *but let's continue to think and speculate*[12]. Eventually they set this problem to the side misunderstanding and continued with other parts of the problem. Later with some help from the teacher they realised their problem and then it all fell into place.

Guðrún and Elsa told me very proudly that they were the first in the class to finish the task. And it is also clear to them why.

Guðrún: *Because we wrote the numbers down.*

Elsa: *Because we wrote all the numbers down and that helped us.*

[12] *Icelandic: Höldum áfram að spá og spögulera.*

Several times during the lesson I observed the teacher asking students, who had problems and had requested his help, whether they had written down possible numbers for each clue they were given in the task (Fig 3). When the students asked the teacher for help he asked questions and gave them some clues to get them going. The students seem to value this kind of assistance.

Guðrún: He told us we could use a calculator and add some numbers here and here – but he never told us – he shoved us how we could figure it out.

Helgi could not finish the task. When I watched the videos I thought he had gotten very little out of this lesson but when asked about the task he could describe very well what he had understood and what he did not get. He could find and write down all the different types of numbers he was supposed to use but he did not understand how to use the clues to place them in the right box in the square.

Helgi: I found it a bit difficult – I was trying to ask Magnús what he was doing – I got what it was about – we had to find numbers form this triangle we had been doing (Pacals - triangle)and put them into a square where they fitted.

R: What was difficult about it? Should we have at look at it? (We observe a short video segment where the student is struggling with the problem and it does not look like he is doing much progress).

Helgi: Yes, look - I could find the numbers but I did not know which numbers were to go where, that was all mixed up I thought.

(Here we I show him the worksheet with the task)

R: Yes what was difficult in this task, what was it that you did not get?

Helgi: What I did not get – We had first to write down triangular numbers ant things like that. But what I did not get was that - here we had the triangular numbers written down but I did not know which triangular number to put in this box – do you get it. I never really got it – I tried to ask Magnús but he just continued.

The number square task is a demanding exercise for grade 8, but when used in this context it was a useful and a challenging task. All the students began the task and their work showed that they all had some knowledge about different types of numbers and their properties. I have used this activity with teachers and student teachers several times and it takes them at least 20 minutes to solve it. The task is published in a thematic booklet named *Pælingar í stærðfræði (Brain teasers in mathematics)* which is a part of the teaching materials *Átta -10.* The teacher is one of the two authors of this booklet but he used his own version of the task.

Discussion and conclusion

Most of the observed lessons can be characterized as a *Landscape of Investigation* (2001, Skovsmose; 2002, Alrø & Skovsmose). This does not mean that all students reached this stage, even though most did to some extent. During the whole teaching process students were active and engaged. Most of the tasks had an easy starting point and the last and greatest challenge (*Number Square*) had the setup of a game. The way the tasks were presented helped to preserve quality of the tasks. The openness of the tasks can be questioned – but the students found the fun and they discovered give at least some room for the students to speculate and ask questions as evident from the data. All the tasks have a reference to pure mathematics. The students find it difficult to find a practical purpose for dealing with the tasks but they engage in them, make some interesting discoveries and some see a mathematical purpose.

The learning environment is influenced by the teaching materials, *Átta-10,* that the teacher uses as his main reference when planning his teaching. The teacher used the materials (the textbook and the teacher guide) in a much more flexible way than reported in other studies of the use of textbooks (Pepin & Haggarty 2002; Johannsson 2006). The learning environment is different from the learning environment described in other studies of mathematics teaching in Iceland (Sigurgeirsson 1994, Savola 2008, Karlsson 2009). The students are all working on the same tasks at different levels. The tasks are presented orally by the teacher and the textbook is only used directly in one lesson. The students work in groups and there is a lot of dialogue both between the students and also between the teacher and the students.

Some social norms and socio-mathematical norms are also evident in the learning environment (Cobb 2000, Yackel & Cobb 1996). Most of the students seem to take responsibility for their own learning. They can always ask the teacher for help and he responds to them but he does not do the work for them and they appreciate that. It is a part of the classroom culture that the students can ask for a short break to stretch their legs if they have worked hard for some time and accomplished something important. This reflects the school policy to respect students' different learning styles and encourage them to find their strengths and use them to learn. The teacher also sends a clear message to the students that in mathematics it is important to organize your work well and answer the questions asked. And it is clear that the students are supposed to work together on tasks, discuss with each other and use and respect each other's ideas.

References

Aðalnámskrá grunnskóla, stærðfræði (2007). Reykjavík: Menntamálaráðuneyti.

Alrø, H., & Skovsmose, O. (2002). *Dialogue and Learning in Mathematics Education. Intention, Reflection, Critique.* (Vol. 29). Dordrecht: Kluwer Academic Publishers.

Boaler, J. (2006). Urban Success. A Multidimensional mathematics approach with equitable outcomes. *Phi Delta Kappan, 87*(5).

Busse, A., & Ferri, R. B. (2003). Methodological reflections on a three-step-design combining obsevation, stimulated recall and an interview. *ZDM, 35*(6), 257-264.

Clarke, D., Keitel, C., & Shimizu, Y. (Eds.). (2006). *Mathematics Classrooms in Twelve Countries: The Insider´s Perspective* (Vol. 1). Rotterdam/Taipei: Sense Publishers.

David, M. M., & Lopes, M. d. P. (2002). Student-Teacher Interactions and the Development of Students´ Mathematical Thinking. In S. Goodchild & L. English (Eds.), *Reseachin Mathematics Classroom*. Westport CT: Praeger Publishers.

Denzin, N. K., & Lincoln, Y. S. (2000). *Handbook of Qualitative Research*. Thousand Oaks: Sage Publications, Inc.

Goodchild, S. (2002). *Students´Goals; a case study of activity in mathematics classroom*. Bergen Caspar forlag.

Henningsen, M., & Stein, M. K. (1997). Mathematical Tasks and Student Cognition: Classroom-Based Factors That Support and Inhibit High-Level Mathematical Thinking and Reasoning. *Journal for Research in Mathematics Education, 28*(5), 524-549.

Jaworsky, B. (2005). I*nvestigating mathematics teaching, A Constructivist enquiry.* London: Falmers Press.

Johansson, M. (2006). *Teaching Mathematics with Textbooks. A classroom and Curricular Perspective.* Luleå University of Technology

Karlsson, Hafsteinn (2009). Kennsluaðferðir í íslenskum og finnskum grunnskólum. *Netla – Veftímarit um uppeldi og menntun.* Retrieved on June 20th 2009 from http://netla.khi.is/greinar/2009/001/prent/index.htm .

Kieran, C. (2001). The Mathematical Discourse of 13-year-olds Partnered Problem Solving and its Relation to the Mathematics that Emerges. *Educatoinal Studies in Mathematics, 46*(187-228).

Mercer, N., & Sams, C. (2006). Teaching Cildren How to Use Language to Solve Maths Problems. *Language and Education, 20*(6), 507-528.

Pepin, B., & Haggaty, L. (2002). Mathematics textbooks and their use in English, French and German Classrooms: a way to understand teaching and learning cultures. *ZDM, 33*(3), 158-175.

Pring, R. (2000). *Philosopy of Educational Research* London Continuum.

Savola, L. T. (2008). *Video based analysis of mathematics classroom practice: Examples from Finland and Iceland.* Ph. D. Thesis, Columbia University: New York. Retrieved on August 10th, 2008 from http://ru.is/publications/SoHE/LasseSavola2008.pdf.

Sigurgeirsson, I. (1994). *Notkun námsefnis og viðhorf til Námsgagnastofnunar í 200 grunnskólum.* Reykjavík: Rannsóknarstofnun Kennaraháskóla Íslands. Retrieved on June 25th 2008 from http://starfsfolk.khi.is/ingvar/namsefni/Notkun_namsefnis.doc.

Skovsmose, O. (2001). Landscapes of Investigations. *Zentralblatt für Didaktik der Msthematik* 33(4), 123-132.

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building Student Capacity for Mathematical Thinking and reasoning: An Analysis of Mathematical Tasks used in Reform Classrooms. *American Educational Research Journal 33*(2), 455-488.

Stiegler, J. V., & Hiebert, J. (1999). *The Teaching Gap: Best Ideas from the World Teachers for Improving of Education in the Classroom*. New York: The Free Press.

Guðný Helga Gunnarsdóttir, assistant professor in mathematics education at University of Iceland, School of Education. M.Ed in Mathematics Education in 2002 from Iceland University of Education. She has been a full time employee at the university since 2002. She has worked as a part time teacher in teacher education since 1980 and taught in compulsory school for 20 years. She has also worked as an educational consultant in the Ministry of Education and as a consultant and editor at the National Centre for Educational materials. She is an author of teaching materials in mathematics for compulsory schools.

gudnyh@hi.is

Teachers’ awareness of student learning

Ingemar Holgersson
Kristianstad University College, Kristianstad, Sweden

We report from a project aiming at stimulating primary and secondary school teachers to work with open-ended activities and using accounts-of to develop their awareness of the outcome of these activities. Results show that accounts-of seem to have a good potential for developing teachers’ awareness of student reasoning and thinking, providing teachers with a better position for following the learning of the students. Furthermore, open-ended tasks seem to have a great potential for enabling more inclusive mathematics teaching, in not setting special tasks or giving special instructions for particular students or smaller groups of students.

Background

This paper reports on a project which was carried out during the academic years 2006/2007 and 2007/2008 in collaboration with teachers in a small municipality in the Southern part of Sweden with the aim of stimulating the development of their mathematics teaching. The project was lead by the author and planned and executed in collaboration with Annika Palmgren for Svedala commune and Pesach Laksman, Birgitta Lansheim, Ulla Öberg, all in different ways affiliated to Malmö University College. The project was funded by a grant from Sparbanksstiftelsen Skåne.

Traditionally, mathematics education, in Sweden at least, is heavily focused on developing and practising skills. Teaching is often based on some prototypical task, where the teacher (and the book) demonstrates how to interpret and solve a problem. Thereafter, the students are allowed to practise on identical or very similar types of tasks, the aim being that this kind of problem becomes a kind of routine problem that the students immediately know how to solve when they encounter it later on. In this tradition there is naturally a strong focus on correct and incorrect. Particular care is taken to make the tasks manageable for students with special needs, often resulting in a reduced mathematical content and an even heavier emphasis on skills, in the way that those students separately practise different, more elementary and reduced skills.

The implicit learning through this tradition has been analysed, for example by Lampert (1990). She notes that working with mathematics in this tradition gives the student the impression that the important thing is to learn to follow the rules given by the teacher or the book. If you do not quite understand them, at least learn how to use them. To know mathematics becomes equivalent to remembering and using the appropriate rule, and correct or incorrect is determined by authority, either by the teacher or by the list of answers at the end of the book. As a consequence, the best way to learn mathematics is to pay careful attention to the teacher’s explanations and to practise often. Different aids, like a list of formulas, can be an important or necessary tool for solving different problems. Overall, the implicit learning and other kinds of experiences from this kind of traditional mathematics teaching often make students feel that mathematics is not for them, and similarly gifted students can find mathematics boring and uninteresting when faced with tasks that are not sufficiently demanding.

Many reform programmes all over the world have proposed different ways of trying to change this tradition (see e.g. Kilpatrick et al, 2001; Lampert, 1990; Sullivan et al, 2006). These programmes often place emphasis on increased work on genuine problem solving, leading to overt reasoning and discussions in the classroom, with a focus on developing the students’ mathematical thinking. Some researchers, as Sullivan et al (2004) and Sullivan et al (2006) have explored the potential of open-ended tasks for managing a more inclusive kind of mathematics, building on what students already know, and creating activities from which all students can benefit, despite their different levels of knowledge.

The need for developing practical and functional models for managing such an enterprise creates a demand for in-service training. However, traditional in-service training (at least in Sweden) often consists of some “expert” teacher showing other teachers how to teach. Their messages can be very normative with very fixed opinions on what works and equally fixed opinions on which types of activities are bad. The scientific foundations behind these opinions are often poor.

The aim of the project we will report from was to stimulate teachers to work with open-ended activities that give students opportunities to develop their problem-solving skills and thinking. We also wanted to stimulate the teachers to develop their awareness of their own actions and of how the activities they presented the students with affect student behaviour and ways of reasoning. Instead of telling teachers how to act, we wanted to empower them by stimulating their own reasoning and thinking, starting with their own beliefs and experiences.

Theoretical background

In this project, the overall framework for teacher development and for the research into the kinds of processes this entails has mainly been adopted from the Discipline of Noticing as formulated in Mason (2002). This means that teacher learning is seen as involving an increasing awareness of distinctions and relations between different phenomena and behaviour, and an increased ability to see additional opportunities in a situation. In such a research project, data therefore needs to be complex enough to genuinely mirror the situation, and the process of analysing it must be careful enough to maintain this authenticity.

In order to develop the teachers' awareness of what is happening in the classroom, we have therefore adopted a tool presented in Mason (2002) which is called making accounts-of, where the focus is on describing what has happened. In contrast, an account-for involves a description, and in addition interpretation and explanation of the reasons for the event developing as it did, as interpreted by the person making the account. The starting point for an account-of is something that happens in the classroom that you notice spontaneously. Often this entails elements of surprise or something that you do not quite understand. This can be about your own feelings or reactions or your students' behaviour and reactions. The task then is to give as descriptive an account of this event as possible, i.e. refraining from interpretation. In order to do that, you have to find a way to remember what it was that you noticed, and what your emotions and spontaneous reflections involved. Since these situations often arise when you are in the middle of teaching, this is not always easy. One way to manage this is simply to write down a few key words that will later help to trigger your memory of the event. When you can find the time, perhaps a few hours or days later, it will then be possible for you to write down more about what happened and what it was that caught your attention. You can even try to label the account-of, but the important thing is that your account is as descriptive as possible. The reason is twofold. Firstly, by focusing on the descriptive side of the event you can become more aware of what is happening in contrast to your own interpretation of what is happening. Secondly, when you share a description with colleagues, they will spontaneously interpret what is happening from the background of their own experience, in a way that they will not do when listening to an account-for. In this way accounts-of can give a base of identifiable events, which, when they are shared among a group of mathematics teachers, generate different interpretations based on the participants' different experiences, and thus can be used as a basis for exchange of these experiences.

Our work with teachers is also inspired by the Cognitively-Guided-Instruction-project (CGI) in the USA (Fennema et al, 1996; Franke et al, 2001). This project centred on stimulating teachers to use problem solving in mathematics and to focus on the development of student thinking as one of the most important goals for mathematics education. The project research team acted as experts on problem solving and the strategies students use to solve different kinds of problems. However, the teachers and not the researchers were regarded as the professional capacity to develop activities in the classroom (Fennema, 2002). As a further outcome of the project a scheme for analysing teachers' engagement with student thinking was developed (Franke et al, 2001); this scheme will be applied in some modified form to analyse the professional development of the teachers in our project. This opportunitiy for teachers to engage with student thinking, using materials from their own classes, with the possibility of reporting to a group of other teachers, where experiences can be discussed and reflected upon in a professional community, can be seen as a hallmark of successful in-service teacher programmes (Kazemi & Franke, 2004).

This project attempted to stimulate the participants to use open-ended tasks and investigative activities of different kinds. One source of inspiration for this work has come from a work with open-ended questions carried out by Peter Sullivan and colleagues in Australia (Sullivan et al, 2004; Sullivan et al, 2006) working to find productive ways of managing open-ended tasks in an inclusive classroom. Here, the term open-ended task refers to starting within a context that has the capacity to engage all students, and that has more than one possible solution. This openness gives the students an opportunity to spot and investigate patterns in the different solutions, which can also encourage them to work in a more systematic manner to find all the solutions. This type of investigative work very often also leads to trying to find new questions worth investigating in their own right.

We have also been inspired by the work carried out by John Mason in the UK aiming at exploring the potentials of different kinds of investigative tasks (Mason & Johnston-Wilder, 2006).

Research Questions

- In what ways can work with accounts-of stimulate teacher's awareness of and engagement with student thinking and learning?
- What potential have Open-ended tasks for facilitating a genuinely inclusive mathematics education?

Method

At the outset we approached the educational department of a municipality in the southern part of Sweden to inquire whether they were interested in supporting a project where mathematics teachers at different levels of the school system would get the opportunity to develop their mathematics teaching skills. We asked for participants from three groups of teachers that have different planning and teaching conditions: A group of pre-school teachers (Pre), working with children from 0 to 6 years old; they have very limited time for planning and curricular goals are not necessarily specified in detail. A group of primary school teachers (Prim), working with students from 7 to 12 years old; they have responsibility for almost all of the subjects taught in a class. And finally a group of secondary school teachers (Sec), working with students that were 13 to 16 years old; they have responsibility for about two subjects, including mathematics. We also asked for participants that were interested in, and willing to develop their ways of working with mathematics. The selection process resulted in 10 participants in the Pre-group, 10 in the Prim-group and 6 in the Sec-group, and the project was carried out during the academic-years 2006/2007 and 2007/2008.

The primary activity offered were workshop style meetings in these basic groups. Meetings were held regularly every third week, and involved two different kinds of activities. At each of the meetings one, or more often two, members of the research group were present. Usually, we started by giving each participant the opportunity to report on one or two of the accounts-of that they had gathered since the last meeting, followed by discussions and exchanges of experiences. Thereafter we introduced different kinds of activities that aimed at stimulating the participants to try out some open-ended tasks in their classes before the next meeting. Depending on the themes that the discussions had opened up this could be letting them try out different examples of open-ended tasks or presenting reports of research findings that were considered relevant. However, it is important to remember that the teachers themselves were always responsible for planning and selection of the kind of activities and tasks to use in their teaching, taking the topics covered in the class and the age of the students into consideration.

A second type of activity offered involved workshops held twice a semester, where all the participants from the different groups met and worked in mixed groups. The motivation for these workshops was to stimulate discussions that crossed the borders of the usual groups and working conditions, and to stimulate a more longitudinal perspective of the mathematical development of different students.

Since our primary goal was to study the impact of the meetings for the development of the teachers' experience, the meetings in the basic groups were audio-taped, and written versions of the accounts-of were collected continually. The large workshops involving all participants were not recorded. Due to limitations of time and resources, we did not observe any lessons. At the outset and the end of the project the participants filled in a questionnaire about their view of themselves as mathematics teachers. At the end of the first academic year they participated in an evaluative task and interview; in the second academic year they were given a free evaluation task that was formulated as:

> *Tell us about your experiences of your own developmental process during the project. Write at length and with as many distinctions as possible.*

This paper will report almost exclusively from this last evaluative task; due to the theme of the conference only the Prim and Sec groups will be included.

Results

As yet only the evaluations have been analysed in detail, these will therefore form the basis of the main results reported. The results will follow themes, which the analyses so far have identified as recurring in the different teachers' evaluations.

Accounts-of

Most of the participants considered writing accounts-of a fruitful tool for documenting student learning. Some have even adopted them for other subjects as well. The participants also witnessed an increased awareness of the students' different ways of reasoning and observed that they had a tendency to use more prolonged discussions about a fewer number tasks than before. However, the accounts-of have also been a somewhat demanding tool that teachers have needed time to get accustomed to. Not everyone finds it easy to write down what they observe: It takes time, time you must find worthwhile.

Most of the teachers consider the accounts-of a good tool for increasing the awareness of nuances in student's learning and for making new things visible in different situations. In this way the accounts-of function as a means to broaden the perspectives of the teachers. They also work as a tool for documenting the learning of individual students, and thus becomes a valuable means for use in contacts with colleagues and parents. The participants also mention that the accounts-of have helped them better to understand the difficulties students can have in understanding mathematics. This is witnessed in the following citation (responding teacher identified in parenthesis by group and number):

> *Gradually ... more easy and natural to notice what happened among the students (Prim 4)*

Although the accounts-of are beneficial, many participants also describe the difficulties they have had in adopting them as a regular habit. Some say that accounts-of have been hard to become accustomed to and to carry through, especially at the beginning of the project.

> *They must be done at once, something I have found it difficult to manage. (Sec 4)*
>
> *The part that has been hardest during the project ... but ... I was forced to look at and listen more to the students ... saw and heard my students developing (Prim 7)*

The number of accounts that were produced in the different groups varied. In the Pre- and the Prim-groups almost all of the participants wrote accounts-of regularly, and some of these were quite elaborate. In the Sec-group, however, some of the participants found it very hard to overcome the threshold of starting to write down an account-of. These teachers found it much easier to give an account-of orally instead of writing it down, remarking that this preserves more nuances, nuances that are easily lost in the process of writing. This resistance to writing was also explained as a preference of interest or cognitive style: as a mathematics teacher in a secondary school it is more likely that you are not good at writing or especially interested in it.

Open-ended tasks

The majority of the participants report that the open-ended questions have challenged all the students, more so than traditional tasks. Students have also enjoyed these tasks more and been looking forward to the mathematics lessons. They have also listened more actively to each other during the discussions and developed their confidence in working with mathematics.

Open-ended tasks were tried out in particular among the Prim-group: at the beginning of the project one teacher even contrasted a traditional closed task with a reformulated open variant in her year 4 class in order to find out what the students thought about the different types of tasks. The closed task she had chosen was task 1:

> *Emma makes 80 SEK per hour. How much has she made in all, if she works for 5 hours?*

The open variant was formulated as task 2:

> *Emma made 400 SEK. How many hours did she work, and how much did she make per hour?*

Table 1A. Student notes in response to task 2: Group A.

80 kr	5 timmar
100 kr	4 timmar
25 kr	16 timmar
12,50	32 timmar
6,25	64 timmar
≈3,20	128 timmar

Table 1B. Student notes in response to task 2: Group B

15 min	400*4=1600
7½ min	8=3200
3¾ min	6=6400
1 7/8 min	32=12800
0≈95/100	64=25600
0≈42½/100	128=51200
0≈21¼/100	256=102400

Then she asked her students:

> *Which task did you enjoy the most?*

To her surprise one of her students answered:

> *I liked the open task better, because in the first one you don't learn anything, you only show if you already know how to solve it.*

The majority of her students agreed that this was the case. As an example of what the group work resulted in, she gave the fragments of results displayed in tables 1A and 1B. They have been typed from the original scrap notes as faithfully as possible, preserving the original Swedish language.

In table 1A the students start with the data of task 1, but then give the most obvious solution, i.e. 4 hours and 100 SEK per hour. After that they start to repeatedly halve the money earned per hour, resulting in very long work hours. As illustrated in table 1B, Group B on the other hand, starts by halving the time spent on the work resulting in "fast cash". We can also note the abbreviated notation for the multiplications in the right column resulting in a free use of the equal sign. Interestingly, the students in this class had not previously encountered other fractions than halves and quarters, but Table 1B gives an indication about their ability to handle halving fractions in a more general way. The notation 95/100 can be understood if you convert 1 min into hundredths. Then one and seven eighths will become approximately 188 hundredths which will be about 95 hundredths when halved. So retaining the unit of one minute the fractions given are surprisingly correct. The only odd thing is the use of zero to denote a small number rather than giving a more correct approximation of the fractions given.

To most of the students, the open-ended tasks made mathematics more enjoyable and engaging, although this was not the case for all, at least in the first year. For those students whose self-image as high-achieving rested on their speed of completing tasks in the book, open tasks could be seen as a threat. The tasks were too time-consuming and it was very frustrating not being able to know whether the answer was correct or not. It was a relief for other students that the focus on correct and incorrect had diminished, and many struggling students apparently improved as can be seen in the following quotations:

> *The existence of more than one solution makes it even possible for a "slow starter" to come up with ideas, and it is not as easy to remain passive as when there is only one answer. (Prim 2)*

The "weaker" students … suddenly nearly all of these students began to develop. They are coming closer to the others in giant leaps.   (Prim 7)

This brings us to the topic of inclusion.

Inclusion

Some of the teachers in the Prim-group testified how the open-ended tasks helped them to achieve more inclusive teaching as in the following example:

I have worked a lot with open-ended questions and suddenly the problem with age-integration and integration of special school students disappeared. There was no need to give special instructions only to a small part of the class. … Now the children are more active, and can take part in investigations and make discoveries on their own level. It now feels as if the children seek knowledge instead of expecting me to deliver it by instruction. … They have no hurry and do not compete about leading the race, since there is no race. (Prim 5, year 2-3)

A teacher in the Sec-group also commented on the topic of inclusion:

To have students that are very capable and students that don't know the tables together within the same class is not easy. But it is a tremendous challenge that I personally, after our group meetings, tackle in a completely different way. (Sec 3)

Managing instruction

The teachers in the Prim-group were more concerned with assessment topics and the role that accounts-of play in giving information about the learning of individual students in connection with the open-ended tasks,:

Short notes on every student provide a good starting point for the evaluative conversations with parents. In addition I feel certain about what every student understands and can do. (Prim 2)

Previously I always started a new topic with an exposition. Now I look for open tasks that start with what the students know, and that lead them into the new topic. (Prim 2)

In the Sec-group the teachers focus more on the dominating tradition of skills and the practice of these. Through their participation in the project this heavy tradition is beginning to loosen up:

The endless practising of skills has given way to… . My students now work more practically with maths and above all more problem solving. (Sec 4)

Practising of skills (which of course still must be there) has given way to other types of tasks, i.e. activities, thought provokers, and open tasks. (Sec 5)

Reported development

In their written evaluations, the teachers are generally very positive about what they have learned during the project. The themes of these reports concern both their self-images as teachers and the role that student thinking plays during the lessons:

I now feel more secure as a maths teacher, … and I think that my students enjoy their maths lessons much more. What has changed the most is the content of the lessons. (Prim 6)

My way of looking at teaching maths has changed. I listen more to my students' questions and ideas. And I myself pose different questions now. (Prim 4)

I listen much more to what the students are thinking, how they tackle the problems. More time is spent on students expressing their thoughts, both to their peers and to me. (Sec 5)

As time went on I realised that the active lessons gave me and the students more than the textbook did. (Prim 7)

I have stopped looking for what the students have done … and still I have a clearer picture of what the students know. (Prim 5)

What the teachers express in their evaluations after the second and final year of the project is more positive overall than what they did in the interviews after the first year. This can be interpreted as an indication that the kind of learning processes that this project aims for take time and perseverance.

Group meetings

Those who mention the group meetings in their evaluations mostly refer to the meetings as stimulating and inspiring for the development of individual participants. Some also mention the importance of an open and permissive atmosphere:

> *A possibility to discuss mathematics on a different level than it is usually done (Sec 4)*
>
> *I have got ideas from the other members of the group that I have tested with my own students (Prim 3)*
>
> *We tested different tasks, and we did it in different ways, experiences differed and gave the breadth and the depth. There was plenty of room during our discussions, we felt free to risk trying out ideas. (Prim 7)*

The members of the project team were also reported as important for securing the depth in the discussions and for providing a connection to research.

Conclusions and discussion

Firstly the witnessed effects of working with accounts-of will be considered. Although this tool is sometimes seen as rather cumbersome, especially for participants that have resistance to writing, it has worked productively as a way of gathering the kind of information that Kazemi & Franke (2004) have identified as essential for good in-service training. Many of the participating teachers testified directly or indirectly and in different forms about an increased awareness of student thinking, and its importance for instruction. Effects of this kind are reported through most of the different themes identified in the results.

Being a part of the “social community” of a professional group that share the same kind of professional experiences stands forth as being very important. An essential part of this seems due to the way that the accounts-of are treated. The supervisors, i.e. the project leaders, are important in more than one way. They are responsible for assuring a climate of openness, where different views and experiences are received and treated respectfully. They are also important for bringing in new perspectives from research and other areas of reflected experiences that ordinary teachers very rarely have the opportunity to encounter. Similar results are reported from the CGI-project (Fennema et al, 1996).

We now turn to the effects of working with open-ended tasks. Some of the participants report positive effects as regards inclusion, both in primary and in secondary classrooms. These findings are similar to the ones reported by Sullivan et al (2004) and Sullivan et al (2006), albeit less systematically and ambitiously investigated. At the same time as we report the positive and encouraging effects of the use of accounts-of to increase teachers’ awareness of student thinking and variability, and the effects of using more open-ended and investigative tasks, we must also conclude that not all teachers learn as much as others. Since this kind of developmental project really is about teachers’ learning how to deal with a new way of looking at the enterprise of mathematics teaching, it would be strange if there were not an individual variation in this learning process as well. We wished to recruit participants that were willing to experiment and to try things out in their mathematics teaching. The vast majority of the participants met these criteria, but not all. Nevertheless, change (learning) takes time. Our overall impression is that the participants needed the first year to come to terms with a new way of working and thinking about mathematics teaching and learning, and it was not until the second year that their grasp of the aims of the project and their reported teaching started to change significantly. Thus it seems that two years were a necessity to achieve the positive experiences that the participating teachers had. To change your way of working, and to start building your teaching more on an increased awareness of student thinking and learning takes time and perseverance.

In conclusion, accounts-of seem to have a good potential for developing teachers’ awareness of student reasoning and thinking, i.e. enabling teachers better to follow their learning, and open-ended tasks seem to have a potential for achieving more inclusive mathematics teaching.

References

Fennema, E. (2002). Personal communication.

Fennema, E., Carpenter, T. P., Franke, M. L., Levi, L., Jacobs, V. R., & Empson, S. B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education, 27*(4), 403-434.

Franke, M. L., Carpenter, T. P., Levi, L., & Fennema, E. (2001). Capturing teacher's generative change: A follow-up study of professional development in mathematics. *American Educational Research Journal, 38*(3), 653-689.

Kazemi, E., & Franke, M. L. (2004). Teacher learning in mathematics: Using student work to promote collective inquiry. *Journal of Mathematics Teacher Education, 7*, 203-235.

Kilpatrick, J., Swafford, J. & Findell, B. (2001) *Adding it up: helping children learn mathematics.* Washington, D.C

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal 27, 29-63*

Mason, J. (2002). *Researching your own practice. The discipline of noticing.* London: RoutledgeFalmer.

Mason, J. & Johnston-Wilder, S. (2006). *Designing and using mathematical tasks*. St. Albans: Tarquin.

Steinberg, R. M., Empson, S. B., & Carpenter, T. P. (2004). Inquiry into children's mathematical thinking as a means to teacher change. *Journal of Mathematics Teacher Education, 7*, 237-267

Sullivan, P., Mousley, J., & Zevenbergen, R. (2004). *Describing elements of mathematics lessons that accomodate diversity in student background.* Paper presented at the PME 28, Bergen, Norway.

Sullivan, P., Mousley, J., & Zevenbergen, R. (2006). Teacher actions to maximize mathematics learning opportunities in heterogenous classrooms. *International Journal of Science and Mathematics Education, 4*, 117-143.

Ingemar Holgersson is a senior lecturer in mathematics education at Kristianstad University, Sweden. His main research interests are in how children develop competences in arithmetic and why this is so hard for some children, and in processes that can stimulate teachers to develop their ways of teaching mathematics that build on children’s reasoning skills and encourage children to develop their problem solving abilities.

Ingemar.Holgersson@hkr.se

Empowering mathematical thought

Halla Jónsdóttir
University of Iceland – School of Education

This paper discusses Gudrun Malmer's, (2002) *Analys av läsförståelse i problemlösning* (Analysis of reading comprehension in problem-solving) and the conclusions of American professor Matthew Lipman (1998, 1991). Lipman's research group investigated how teaching philosophy to children supported their success in mathematics and became a motivation for seeking new methods of teaching mathematics for disengaged children with poor results in the 9th grade of elementary school. I will also describe my use of the tools developed Gudrun Malmer, the definitions and ideas of Lipman and others regarding philosophical thinking and the recount the way that the students were led. Finally, some issues regarding which parts of mathematics are mainly being influenced, and whether these ideas can be utilized in general mathematics tutoring, will be discussed.

The theoretical framework

American professor, Matthew Lipman pioneered the introduction philosophy to children. For decades he taught, developed written material and researched with co-workers in the field. The conclusions of these studies are significant in our context, particularly the consistent ability of children to wonder and think reflectively, in addition to logical and critical thought. These studies, many of which were performed in liaison with, and by, The Advancement of Philosophy for Children, have also shown progress in mathematics with children who work with philosophy. Lipman emphasizes such things as conceptual explanation and logical inference, which are central to philosophical thinking. (Lipman 1988, 1991)

Lipman's definitions of critical thinking include: guided by criteria, self-corrective and sensitive to context. (Lipman, 1991, p. 193)

> *Philosophy is a discipline that takes into consideration alternative ways of acting, thinking or creating. To discover these alternatives, philosophers examine presupposition, question what is taken for granted and speculate imaginatively, taking into account broader frames of reference. That is why philosophy brings to education critical and creative thinking. (Daniel, De la Garza & Slade, 2001, p. 89)*

Daniel, De la Garza & Slade pointed out the importance of the philosophical method both as a methodology in teaching mathematics and also in support of critical and creative thought (Daniel, De la Garza & Slade, 2001, pp. 92-103)

Both Dewey and Lipman emphasize the notion of community in learning as well as understand thinking as a process of inquiry (Dewey, 1916; Jónsdóttir, 2010)

As a Community of Inquiry, the classroom is a pluralistic group of learners, centered on dialogue and collaborative activity (Cam, 2006, p. 8) The learning and teaching of philosophy takes place within such a space as teachers and students organize and navigate the process. The main goal of teaching philosophy is to evoke curiosity among children and support them to participate in critical and logical conversations, that at the same time benefit their learning and their participation in a democratic society (Lipman, 1988, Malmhester & Ohlson, 1999). Constant focus on the communication components such as listening and expression ensures that participants learn and get to know themselves as they are active in studying, and correcting where needed, their own ideas and the ideas of others. Thinking about thinking is an important aspect of every lesson.

Working in the Community of Inquiry nourishes and develops six dimensions of thinking:

Critical thinking

Logical thinking

Creative thinking

Concern and empathy

Communicational skills

Meta-cognition (Thinking about thinking) (Lipman, 1988, pp. 38-41)

My practice has a theoretical basis in the work of a number of educators. Gudrun Malmer with her decades of experience, plus her post-retirement research, is very experienced and wise in teaching mathematics, not least regarding students experience difficulties. (Malmer,1990, Malmer & Adler, 1996). Lev Vygotsky's (1981) work was also important, primarily for his understanding of a person´s intellectual development as a process through which the inter personal communicative functions of language are transmitted into verbal thought, his recognition of pupils' need for scaffolding learning, and the "Zone of Proximal Development". J. Bruner's was significant for theory on instrumentalism, and his understanding of how schools need to supply pupils with a toolkit that can be useful to them for simplifying the learning process and becoming qualified to study (Bruner, 1996).

Gudrun Malmer´s tools

In 2003, I travelled to Gothenburg to participate in Gudrun Malmer's program on mathematics teaching methods. As experienced teachers and scientists discussed various ways and attitudes, I began thinking about my professional experience and re-evaluating my mathematics teaching methods.

Malmer introduced Requirement Analyses Material (Analys av Läsförståelse i Problemlösning) with the intention of analyzing the impact of reading difficulties in mathematics. This effective tool can be used to examine students' reading difficulties with regard to decoding and interpreting text. It is critically important because of poor knowledge of concepts (in both general and abstract mathematics) and incompetence in logical and constructive thinking. (Malmer, 2002b) Gudrun Malmer gave me kind permission to translate and use her material for my teaching.

This simple analysis tool assisted me to differentiate problems and respond in new ways. I immediately began using the material, primarily with my students who had significant problems. For example exercise from the material will provide further clarification:

A medium portion of soup is 2.5 dl.

How much is a medium portion of soup?

How many medium portions of soup are in 1L?

How much soup is needed so that 20 people get the medium portion? (Malmer, 2002b, p.13)

Information from student responses included:

Examples 1 & 2 examine student's ability to decode and interpret text. Students with reading difficulties had problems answering these questions.

In part 3 we see things that relates to the usage of words and concepts

Part 4 deals with logical thinking.

While interviewing students who participated in the test, we recognized that the instrument examined both simple and complicated reasoning among the participants. Some students could read and understand the contents of part 2, but had difficulties with the reasoning, even if it was simple. Others had problems expressing themselves about what they did and why.

Unpredicted issues also arose during follow-up discussions with students. Thinking about their own thinking was challenging and in turn made it difficult for students to describe how they did and how they thought. A number lost faith in thinking, and so could neither see the overall or the connections — they lost hope and had to be really pressed to try.

The classroom

A range of elements came together to set the scene for a three year experiment that connected philosophy and mathematics learning.

Students in the 9th grade, were given the opportunity to choose supplementary classes in mathematics (2 extra hours in addition to obligatory hours), and I was chosen to be responsible for these extra classes.

For many years, I taught an optional philosophy class which was popular with students.

I also taught mathematics for several years at the same school, and primarily worked with students who had difficulties with the subject.

Approximately 24% of 9th grade students chose the optional mathematics classes, and of those, there were more than 80% in the philosophy class.

Existing issues for students of mathematics and/or philosophy:

Eight per cent of students found reading difficulties an obstacle. Problem areas included difficulties with reading, processing and understanding concepts, plus issues related to delivery of content.

Twelve per cent of the students showed significant concerns with simple reasoning and over 30% were challenged by more complex reasoning.

Many of the students also had difficulties using concepts and communicating about mathematics. That is one of the most fundamental issues that need to be considered when teaching mathematics (Malmer, 1996; Dalvang & Lunde 2006, pp.37-65).

I thus decided, in a conscious way, to connect the mathematics lessons to the philosophical lessons and use Lipman's methods.

The following objectives were set up from the start, and discussed with students throughout the semester.

To build trust and reliance for own thinking.

Training the students in Analytic thinking.

Training the students in Logical thinking.

Training the students in Thinking about thinking.

Working with philosophy and mathematics depends on communications, and also on respect.

We noticed that student understanding and application of these objectives grew during the semester

A typical lesson in philosophy is likely to include the following elements:

Motivation: The teacher begins by presenting a motivation to the students. A motivation is an experience that gives students a common starting point to consider, e.g., a story, picture, music, a puzzle. The students are taught to analyze and to pose questions about the motivation. As the emphasis is on thinking before speaking, students are expected to reflect and reason privately before responding. They are also encouraged to point out examples to support their opinions.

Choosing Shared Questions: Together the students choose one significant question to discuss. They make the choice based on the potential for the question selected question to lead to a significant debate topic Groups and individuals set themselves goals, both as for the discussion.

Debate: Once the topic is negotiated the debate itself can take a variable amount of time as is decided by the group. The main role of the teacher is to make sure that the rules, that have been agreed, are followed, that students stick to the subject at hand and that criticism and logic is used.

Evaluation: Reviewing the conversations, examining how group and individual personal goals were met/not met, and evaluating the discussions are important elements of evaluation, e.g., looking at what ideas were put forth, questioning how logic, explanations and imagination were used, and reflecting on interpersonal action, respect when communicating with each other (Malmhester, Ohlsson & Halldén, 1991).

Did this method in philosophy-lessons become successful in supporting children with difficulties in mathematic? Did it increase the interest of those who had lost interest? Were there any visible changes in the logical thinking and communicational skills regarding mathematics?

Discussion

Interesting changes occurred for individual participants in this study. Firstly, philosophy students soon forgot about mistrusting their own thinking as they dealt with subjects connected to their lives and formed opinions that made them individually stronger. Communications also improved, and a distinction was soon evident between those who participated in the philosophy classes and those who did not. Throughout this section I have included examples of mathematics problems that students have worked on as a group, with individual explanations, discussions and self-evaluations.

Many of the pupils commented that when they were working with mathematics problems, it was easier than before to "find out what they should take with them and what they shouldn´t". Simple problems from Gudrun Malmer assisted them to develop these skills: (For example Pia is 10 years old and 141 cm high. She has grown 90 cm since she was born. How long was she then?) In this case, the analytic thinking and training helps. A big change was evident in the group's thinking and "trying to describe it". They quickly grasped and connected with the mathematics problems in various ways. Some interesting changes could be seen there, with regards to both the individual and the group. Thinking about thinking and skilled communication are the prerequisites to the mathematics discussions that enable solutions to be found.

When seeking solutions for problems in Gudrun Malmer's questionnaire, it was obvious that many of the students, who studied philosophy, chose another path. Most of them, aware of what they were doing, applied both analytic and logical thinking in new ways, and were more confident in their communication and explanations. For example, they thought to take a thinking break before they spoke, took advantage of their creativity and imagination, and above all they were more confident and more able utilize the tools that they had acquired during the philosophical work. When students received the mathematical problems, some of them dealt with them in new ways. Sometimes they viewed them in a wider perspective, or maybe created a story around the problem. Some used other techniques, for example drawing.

However, there was no connection to mathematics problem with numbers such as 2x(2-4) — it neither easier to deal with, clearer to see how they could process the problem. Even though students applied (or tried to apply) the new abilities and confidence they developed to mathematical problems, they were unable to transfer these strategies to numeric calculations. They needed support to make the connections.

This experiment was very exciting. Every year, there were a handful of students that were unaffected by this work, and the largest obstacle was to persuade them to see the connection with mathematics. During the last of the three year period, I tried to do this by using mathematical problems in the philosophy class, from time to time.

There were no complications that I can point to, but the transfer does only occur for specific factors. It is an exciting undertaking to examine the impact on the practice of mathematics among younger students, and it is clear that this is a path that offers new possibilities that need to be investigated, and there are without a doubt several more angles to this connection that I have not identified.

Conclusions

Why do so many students experience difficulties in succeeding in mathematics?

There are many and different reasons (Dalvang & Lunde, 2006). Ways out of the problem are therefore presumably many and only one of those ways has been discussed here.

As has been discussed previously, difficulties in mathematics often have an extensive influence on the students' thinking and attitude towards themselves. It is therefore important to look for ways to work with these important aspects of confidence, trust in their own thinking, logical thinking and thinking about their own thinking.

De la Garza & Slade (2001) and Lipman (1998) raised the issue that school curricula are not sufficiently meaningful for children.

> *It is based on different way of thinking about mathematics, as well of doing philosophy, …in philosophic-mathematical community of inquiry that will help them tame*

mathematics, understand better and have more pleasure in doing it. (Daniel, De la Garza & Slade, 2001, p. 98)

Lafortune (1999) observed that the school system does not favour creativity, nor invite dialogue about mathematical concepts and problems. She presents a Philosophy for children approach adapted to mathematics as a possible and successful path.

Can mathematics use philosophical methods to strengthen certain areas of mathematics? And is that possible with general mathematical teaching? How can these results be utilized in general mathematics teaching?

Let us imagine different ways. What about working with subjects that have a single objective to strengthen logical thinking, ability to think about thinking and work with concepts, for instance twice a week. The subjects are not connected to mathematics in any other way. That has the advantage that those students who don´t trust their own thinking in the mathematical context, have given up or maybe haven´t even tried, may try. It is important that the subject is exciting and appeals to the students. The rules have to be clear in the group and everyone has to agree, that they should be followed.

The other way is without a doubt being used more in teaching mathematics today. There, mathematical puzzles, or mathematical concepts are used. The same thing applies regarding the rules, that they are clear; everyone should have their turn, to rationalize their opinion and explain what happened during the discussion.

Critical thinking as skillful, responsible thinking that relies upon criteria, is self-correcting, and is sensitive to context, and if schools are to be succeeded in teaching it the educator must have a clear idea of what it is, and be willing to walk that road, not always easy but a new one, a new solution to old problem (Lipman, 1991, p. 190).

References

Bruner, J. (1996). *The culture of education.* Cambridge, MA: Harvard University Press.

Börresen, B. & Malmhester, B. (2003). *La barna filosofera. Den filosofiska samtale i skolan.* Kristiansand: HöyskolaForlagt.

Cam, P. (2006). *20 Thinking tools.* London: Camperwell

Dalvang, T. & Lunde, O. (2006). Med kompass mot mestring. Et didaktisk perspektiv på matematittvansker. *Nordic Studies In Mathematics Education*, 11(4), 37–63.

Daniel, M-F., De la Garza, M. T. & Slade, C. (2000/2001). Philosophy of mathematics in the classroom. Aspects of a tri-national study. *Analytical Teaching,* 20(2), 88–104.

Dewey, J. (1916). Democracy and Education – An Introduction to the Philosophy of Education. New York: A Division of the Macmillan Company.

Jónsdóttir, H. (2010). In J. Einarsdóttir & Ó. P. Jónsson (Eds.). *John Dewey í hugsun og verki: Menntun, reynsla, lýðræði,* pp. 131–143. Reykjavík: Háskólaútgáfan

Lafortune,L,Daniel,M.-F.,Pallascio,R and Schleifer,M. (1999) Evolution of pupils' attitudes to mathmatics when using a philosophical approach in *Analytic Teaching* 20(1), p. 33–44.

Lipman,M. (1988). "Critical Thinking – What Can It Be?" *Educational Leadership,* 16(1), 38–42

Lipman, M., Sharp, A. M. & Oscanyan, F. S. (1980). *Philosophy in the classroom.* Philadelphia, PA: Temple University Press.

Lipman.M. (1991). *Thinking in education.* New York: Cambridge University Press.

Malmer, G. (1990). *Kreativ matematik.* Solna: Ekelunds förlag AB.

Malmer, G. & Adler, B. (1996) *Matematiksvårigheter och dyslexi.* Lund: Studentlitteratur.

Malmer, G. (2002a). *Bra matematik för alla.* Stockholm: Studeltlitteratur AB.

Malmer, G. (2002b). *Analys av läsförståelse i problemlösning.* Lund: Gudrun Malmer och Firma Bok & Bild.

Malmhester, B. & Ohlsson, R. (1999). Filosofi med barn. Reflektioner över ett försök på lågstadiet. Stockholm: Carlsson.

Malmhester,B., Ohlsson,R., & Halldén O. (1991). *Barn som filosoferar.* Centrum för barnkulturforskning, Stockholm: Stockholms universitet.

Vygotsky, L. S. (1981). *Thought and Language*. Cambridge, The M. I. T. Press.

Halla Jónsdóttir is an assistant professor of history of ideas, science and ethics at the University of Iceland, School of Education. She worked for 20 years as a general classroom teacher in a secondary school. Teaching is her primary profession and she has been active in curriculum planning. She teaches undergraduate and graduate courses on didactic, ethics and history of education. Her research interests are in learning and teaching in diverse classrooms. Current research and writing include democracy in the classroom, research on teachers students, and multicultural education.

hallaj@hi.is

Teacher development in a community of inquiry: primary school teachers rethink their mathematics teaching in diverse classrooms

Jónína Vala Kristinsdóttir
University of Iceland – School of Education

This paper reports on an on-going collaborative inquiry into mathematics teaching and learning where primary school teachers and a teacher educator research their practice together. The teachers work in schools with diverse groups of students and have found it difficult to include all children in meaningful mathematics learning. Preliminary results indicate that participation in the project has influenced the teachers' beliefs on children's learning and their way of teaching mathematics in diverse classrooms.

Introduction

Iceland has changed in the last decade from being a homogeneous society to developing as a multicultural community and children with additional learning needs are now included in local schools. In consequence teachers are currently faced with new challenges to differentiate teaching. The spectrum of students and learning needs requires them to adopt flexible approaches to teaching and learning.

Throughout my work as a teacher educator[13], I have recognised that many teachers find it difficult to teach mathematics in diverse classrooms in ways that are coherent with the goals of the curriculum guidelines (Menntamálaráðuneytið, 2007). Observations in classrooms and discussions with pre-service teachers and primary school teachers have revealed that interactions between students in mathematics classrooms are limited and the teachers often have a passive role as educators. Their own experience as mathematics learners was typically as passive receivers who practiced rules and procedures, introduced by teachers and textbooks. Teachers lack experience in investigating, communicating, reasoning and making connections. Additionally they feel incompetent in using these approaches in inclusive schools (Guðjónsdóttir & Kristinsdóttir, 2007).

The aim of this paper is to discuss an on-going study where primary school teachers and a teacher educator research their practice together. The main goal of the study is to understand how teachers meet new cultural and mathematical challenges and how participation in a learning community with their colleagues and a teacher educator can lead to changes that are valuable for their work. Teachers in two neighbouring schools in Reykjavík initiated a plan to develop their teaching and improve their competency as mathematics teachers' with diverse groups of students. The study involves seven 5th and 6th grade classroom teachers researching their own practice with my support. I also studied my development as a researcher and the collaborative process as a whole. As such I am working with the following research questions.

What characterizes the processes that emerge through collaborative inquiry between classroom teachers and a teacher educator?

- How does the collaborative research process emerge?
- How do the teachers' beliefs and practices change?
- How do the teachers perceive that participation in the project is reflected in their teaching?
- How do I perceive that I have changed as a researcher?
- How have my attitudes to the teachers practice changed and my beliefs on mathematics teacher development?

[13] In-service, pre-service and graduate courses

The research questions were derived from my interest in mathematics teaching and learning and the professional development of teachers. The focus of this paper will be two-fold: 1) teaching children with diverse backgrounds and learning styles, and 2) understanding how the teachers involved in this research developed their capability to teach these children mathematics.

Including all children in mathematics classrooms

Classroom teachers are required to teach diverse groups of students and are held accountable for covering the curriculum in a manner that enables all students in the class learn. Students do not learn in the same way or use the same amount of time to learn. For instance, if the teacher decides to continue without giving the students the time they need, some of them will experience failure as they are moved through the curriculum without understanding. Moreover, students with learning difficulties in mathematics are highly likely to experience narrowly defined learning opportunities and one of their most common instructional activities is traditional algorithms for performing the four basic operations (Woodward & Montague, 2002).

Fosnot and Dalk (2001) have also addressed this tendency to emphasize practicing traditional algorithms. Many schools today are still teaching algorithms as proficiency in them is the main goal of elementary mathematics instruction. They are still teaching the goal of seventeenth-century mathematics education when it was necessary to be able to carry out computations directly with paper and pencil. Today we have access to more powerful tools and it is important for children to learn to use computers as a tool for calculations. They need effective mental strategies to be able to assess whether the answer on the calculator is reasonable sense, and well-developed understanding of the basic operations to be able to calculate mentally. Traditional algorithms lead us to think of digits rather than the composite numbers that the digits make up — they are therefore not helpful in developing meaning within our number system. Invented algorithms in contrast are number orientated and flexible. When children are trusted to look at the numbers before they start to calculate and get a sense of what strategy is useful to solve the problem their understanding of mathematics will improve. Children learn flexible strategies by discussing with their classmates rather than using invented algorithms that they do not understand themselves.

Learning mathematics with understanding should not only be for students considered to be bright, but also accessible to all students, although all students will not learn the same mathematics to the same degree. According to Marlowe and Page (2005) students with learning difficulties in mathematics are less likely to receive constructivist approaches and more likely to spend time on tasks requiring little more cognitive energy than rote memorization — an approach historically supported by many mathematics teachers and special educators. Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray, Oliver & Human. (1997) confirm that many classrooms give students with additional needs more instruction on basic skills and less opportunity to develop conceptual understanding. In addition a high percentage of children from poor families, minority groups, and/or girls receive more instruction on the mastery of basic skills and less instruction on developing conceptual understanding and learning how to apply that conceptual understanding to solve authentic problems. The most important role for the teacher is creating a classroom in which all students can reflect on mathematics and communicate their thoughts and actions. Building a community of mathematical practice requires teachers to take the lead in establishing appropriate expectations and norms (Hiebert etl al, 1997).

Teacher educators must be aware of the challenges experienced by teacher learners as they seek to transform their knowledge of mathematics learning into their practices. In 1996 Zeichner advised teacher educators to support the belief that all students can succeed and help prospective teachers to learn scaffolding techniques to bridge students' home culture with the school culture. He encourages course work and field experiences in which prospective teachers learn about languages, cultures and socioeconomic circumstances of the particular students in their classrooms. They should also learn instructional strategies that value sense making and knowledge construction and acknowledge the involvement of parents and other community members in authentic ways in education.

Schools that make progress towards inclusive ways of working develop the capacity for teachers to learn from one another, so that they share ideas and practices, and spend time discussing how teaching can be improved. Teacher educators who embrace diversity and inclusion also need to learn how to observe carefully, so that they continue to understand practice as it is carried out in their own classrooms and countries. Such processes become starting points to continue the journey of new learning (Ainscow, 2007).

In mathematics education transformative pedagogy is equally important. Moore (2005) discussed the importance of transformation from theory to practice and concludes that if teachers are expected to teach for diversity and understanding, they need opportunities to develop and enhance their mathematical pedagogical knowledge. It is important for them to experience their own mathematics learning in an environment that reflects the one they are expected to create in their classroom. Teachers are empowered to practice a culturally responsive and socially relevant pedagogy as they begin to look critically at their classroom environment. The practitioner becomes the action researcher, transforming theory into practice and research on that practice.

The teachers' conceptions of mathematics, teaching and learning, learned from their educational experiences, make it difficult for them to act in new ways to foster a different quality of mathematics learning for students (Crawford & Adler, 1996). Teachers need more knowledge and experience of how to explore and experiment with mathematics and engage in discussions about their own mathematical thinking. They also need knowledge of the development of children's mathematical learning (Bredcamp, 2004). Problem-solving activities may help teachers to experience and discuss difficulties similar to those met by students in class and understand the importance of evaluating the process in mathematical activities (Boero, Dapueto, & Parenti, 1996; Crespo & Sinclair, 2008).

The terms *integration* and *inclusion* have been used to describe different levels of including children in schools. Integration typically makes a limited number of additional arrangements for individual pupils with special educational needs in schools, but the schools do not change much (Ainscow 1995). Inclusion implies more radical changes through which schools restructure themselves to be able to embrace all children. Armstrong, Armstrong & Spandagou (2010) present inclusive education as a process or an on-going project where difference is the central. Inclusion refers to all students and thus goes beyond special education.

Inclusive education thus implies that each student is accepted in school on her/his own premises and diversity in classrooms is seen as a source of the richness, interest and enhanced learning possibilities that bring to life the notion that children learn in various ways, make different contributions to the learning community, use different amounts of time, and need different instructions and different tasks to learn (Fullan, 1999).

Lately the focus in the Nordic countries has been on finding ways to respond to students' difficulties by building on their strengths. According to Dalvang and Lunde (2006) the main obstacle in special education is the focus on students' disabilities instead of their abilities. They present a model that they call the compass-model as a guide for analysis of students' learning conditions and their mathematical competencies. The model assists teachers in creating learning communities for students based on analysis of the current situation with a distinctive focus on each student's strengths. Students' prerequisites are discussed and the content and design of the current teaching is reconsidered with support from the Danish "KOM" project (Niss & Højgaard Jensen, 2002). The "KOM" project defines important mathematical competencies or proficiencies for pupils to acquire. The compass model is built on Niss's and Jensen's definition of mathematical competencies. According to them it is important for all mathematics students to learn to question and answer with mathematics as well as to use its tools and language.

According to Pugach (2005) perspectives and attitudes of teachers involved in inclusive education are more positive to diverse group of students than those who have not had the opportunity. The most important role for the teacher is creating a classroom in which all students can reflect on mathematics and communicate their thoughts and actions. Building a community of mathematical practice requires teachers to take the lead in establishing appropriate expectations and norms. In classroom cultures that promote mathematical learning all students have a voice and are supported to develop their understanding of mathematics through exploring, investigating, discussing, reflecting and drawing conclusions.

Results from a study of four teachers' collaboration and reflective discussions indicate that teacher discussions about their own students' way of learning mathematics and reflections on their teaching can influence teaching in diverse classrooms. Through constant discussions on the children's solution strategies and reflection on their teaching the teachers developed their understanding of their students' learning (Kristinsdóttir, 2010). According to Mason (2002) such systematic reflection on mathematical interactions that focus on student's learning and understanding of processes, as well as on one's own interaction behaviour, represents an essential professional competence of teachers.

Methodology and methods

This collaborative inquiry into mathematics teaching and learning had a twofold origin (Goos, 2004; Jaworski, 2006). The study brought together a group of teachers with a strong interest in improving their mathematics teaching, and a teacher educator and researcher enthusiastic for meaningful mathematics learning opportunities for all students. The teachers are all working as primary-school teachers in Icelandic schools, and I, the researcher and teacher educator, will be working with them on the journey. Improvement of mathematics learning in classrooms is fundamentally related to development in teaching and teaching develops through learning processes in communities of inquiry (Jaworski 2006). Jaworski also proposes inquiry as a fundamental theoretical principle and position in order to avoid perpetuation of undesirable practices. Communities allow participants to ask questions about improving students' opportunity to learn mathematics and in doing so aim to learn about their own learning. In a community of inquiry the inquiry is seen both as a tool for developing practice and as a way of being in practice.

Self-reflection is central to participatory and collaborative inquiry. Action Research methods (Kemmis, 1999; Kemmis & Wilkinson, 1998; McNiff, 2002) are therefore useful tools in framing the research process. Collaboration implies that teachers and an external researcher have different roles. The teachers are the insiders because the research is focused on their practices. Individual research can take place in a collaborative environment involving teachers either within a school or across a number of schools (Jaworski, 2003). Outsiders may take various roles— they can assist with the formation of a community of teachers and educators where the teachers can share their research practices and discuss their ideas. They may themselves conduct research into classroom learning or teaching and be engaged in research into the collaborative programme. They might also be researching their own practices as educators in supporting teacher research in which case they become insiders in researching their own practice. In co-learning, the learning of one is dependent on the participation and learning of others.

Starting the project I interviewed all the teachers and school principals and visited all their classrooms. The interviews and classroom observations provided me with insight into the teachers' world and their visions for the project, as well as guiding me in leading the research. On a monthly basis we meet in workshops where we explored mathematics and reflected on our investigations, discussed teachers' stories from their classrooms and reflect on their students' learning. We also reviewed new research on the topic and teachers' professional development. The workshops are audio and video taped and I write my reflections after the workshops.

Case writing base on classroom experiences was an important part of the teachers work. Cases stimulated their inquiry and analysis of the real challenges and dilemmas of their practices. Describing their practice they adopt discourses for interpreting the action and constructing their personal theory of the practice described. Theorized practice presents practitioners with opportunities to propose and trial new practices, make decisions or conclusions, and develop and improve their practice (Kruger & Cherednichenko, 2006). Through refinement of their teaching I hope that spirals of experience will emerge and we can learn from former cycles while building new.

The workshops have mathematical focus and all participants explore together with mathematics. The 'spiral of building confidence' manipulating; getting sense of; capture in pictures, words and symbols; fodder for further manipulation etc (Mason, 1999) is used as a guiding tool in our explorations with the mathematics. The problems we explore with have the potential to promote mathematical activity and thinking and stimulate collaboration where discussion and sharing thinking is meaningful (Jaworski, 2007).

Research requires evidence of reframed thinking and transformed practice, which according to LaBoskey (2004) are derived from an evaluation of the impact of developmental efforts. It is interactive and demonstrates discourse with colleagues, students, the educational literature and our own previous work confirming or challenging our developing understandings. It employs multiple, primarily, qualitative methods many of which are used in general educational research and some that are innovative.

Austin and Senese (2004) encourage teachers to include self-study in their list of professional expectations and responsibilities. It is about who teachers are and there is no better way to strengthen teaching practices than to recognize personal values and beliefs, and to enrich students' learning. Mathematics educators emphasize the importance of teacher research in their own teaching. (Stigler & Hiebert, 2004) claim that teachers have a central role to play in building a useful knowledge base for the profession. They need to analyze what happens when they try something new in their own teaching and record what they are learning and share that knowledge with their colleagues. Artigue (2009) stresses the importance for research within schools to take into account factors internal to the development of the field itself. The progression of research has made more and more evident that research methodologies have to organize a relationship with the situational, institutional and cultural dimensions of learning and teaching processes. Researches within schools where teachers are active participants in the research process meet these requirements.

Preliminary results

In the initial phase of our collaboration the teachers expressed their need for help in teaching mathematics for understanding in inclusive schools. They are accustomed to tracking students into ability groups in mathematics classes and believed that this is important for their possibilities to develop in their mathematics learning. The students that need much support according to the homeroom teachers' analysis of their mathematical ability work in small groups with help from special-education teachers and those labelled as more able work in larger groups with their homeroom teachers.

Visiting their classrooms, I found that most lessons started with a review of homework, followed by the teachers' description of the subject of the lesson and a brief exchange with students about procedures for solving the problem types on the agenda. The students then worked individually or in small groups with problems in their textbooks and the teachers explained procedures for students who asked for help. There were no common discussions during lessons or round up of the topic at the end. In the special-education classrooms teachers worked hard to motivate students to do their work, but the children did not seem interested in discussing their work. The focus was on solving problems with traditional algorithms and rote memorization. In the homeroom teachers classes I saw more initiative from students and different ways of solving problems accepted by some of the teachers that even urged the children to use flexible strategies.

The first workshop was based on my reflections on the observations and interviews with the teachers. As I planned the workshop I referred to my former experience of working with teachers, as well as research on mathematics teacher education and children's learning in inclusive schools (Ainscow, 2007; Boero, Dapueto, & Parenti, 1996; Crespo & Sinclair, 2008; Zeichner, 1996). The first activity required the teachers to work on a problem that inspired them to explore their own understanding of mathematics and discuss how their students might have solved the problem. To begin with the teachers seemed frustrated and insecure about how to respond to these challenges, but after I urged them to share their ideas they started to draw pictures and write numbers to explain their thinking. Our discussions on their different ways of solving the problem and our shared recognition of how difficult it is to explain our thinking paved the way for new ways of thinking. We began to talk about children's way of learning, how they might have solved the problem and how they explain their thinking. A number of teachers were sure that their own students would not be able to solve the problem unless the teacher explained the procedures carefully to them. Others believed that their students could solve it if they were inspired to approach the problem from their own way of understanding and discuss their thinking with their classmates and their teacher.

We also discussed possible goals for posing this problem to young children, teenagers etc. What will children learn from solving the problem? Will all children learn the same and how can the teacher understand what each child is learning? I encouraged the teachers to think about individual students and how they might approach the problem. These questions lead to shared reflecting on individual children's needs and support and extended our understanding of how all children can contribute to the learning in the classroom.

At the end of the first workshop, when I asked the teachers what they would like to do at the second workshop, they all asked to learn more about cooperative learning. Some of them had taught their children to work cooperatively but not in mathematics classes. We began the next workshop by forming groups to discuss different ways of computing number sentences. We also assigned roles for each participant to play during the discussion as we might have done with children (supporter, writer, time keeper etc.) Afterwards we discussed the goals for teaching children cooperation, the different roles and what they felt about being assigned roles to play in their groups. We also discussed traditional algorithms and different ways of computing numbers. The teachers expressed their reluctance to trust that children can use flexible ways of calculating and even invent their own algorithms. They admitted that they themselves are not used exploring flexible ways of computing numbers but were willing to try. When we discussed different ways of solving problems one of the teachers often indicated that her students would be confused to hear so many different explanations. She believes that it is important that the teacher gives clear instructions to avoid confusion in the class. The other teachers were more open to discuss children's strategies and they often tried strategies that they had seen their children use, thus trying to relate their own thinking to their students'.

In the following workshops the teachers solved many different mathematical problems and explored them together. We have discussed both their way of solving them and their students' way of thinking about mathematics. The teachers experienced that exploring with mathematics opens up diverse way of thinking about mathematics and thus helps them to understand their students' diverse ways of learning. I have introduced the teachers to different ways of working with mathematics in inclusive classrooms; given them articles to read about other teachers' way of working and showed video clips from both my own teaching and from other research. (Boaler & Humphreys , 2005; Carpenter, Franke & Levi, 2003; Fosnot & Dalk, 2001). We have also explored different ways of working with mathematical concepts like using concept cartoons (Dabell, Keogh & Naylor, 2000).

Inspired by their experiences at the workshops the teachers have investigated problem solving and cooperative learning in their classes. At our workshops they tell stories from their classes and discuss what they have learned from their work. We compare their work with our own experience at the workshops and they support each other in their reflections on their teaching. The teachers' concerns for children who don't speak the language used at school and those who have difficulties with mathematics have been discussed frequently: for example we have questioned if they would gain from working with children who are quicker to find solutions to problems. In one of the schools it was decided to try to work with heterogeneous groups. Two of the teachers in that school have reported that they find it more meaningful and the children support each other. Their comments have affected the teachers at the other school who want to try to mix children in their groups though only for a few lessons to begin with.

Writing about their experiences and using the guidelines for case and commentary writing (Kruger & Cherednichenko, 2006) has been demanding for the teachers. They are not used to write about their work and find it difficult to analyse their practice. One of the teachers has been reluctant to bring her writing to our workshops and she asked where her worked would be stored. She feels that she works as a professional and responds to students needs in her daily work and does not need to write about it. I suggested that she, and the others in the group, keep track of their own data to be able to go back to them and use as resources in their analysis of their work and their own development. I also encouraged them to write what they themselves think is important to write, not what they think I would like them to write.

All our workshops are videotaped and the teacher who has concerns about where her writings will be stored was concerned about the filming to start with. She is also concerned about all information about her students and is reluctant to permit any recording in her classes. My proposal for audio- or videotaping in their classrooms to discuss their work at the workshops also seemed to be overwhelming to all the teachers at the beginning of our collaboration. I therefore decided to propose that the teachers observed each other classrooms to be able to learn from each other. This teacher immediately responded by telling that she does not believe that they gain anything from visiting each other classes, "we often do that and we of course see how the other teacher conducts the lesson, but we know pretty much how our colleagues work". The other teachers wanted to observe each others classroom and decided to organize mutual visits to discuss at our workshops.

The mutual visits have been fruitful and enabled the teachers to relate their work to their colleague's way of teaching. They are willing to discuss individual student's way of thinking about mathematics. One of the special-education teachers told about a boy who preferred to use mental computation and was reluctant to write down his thinking using traditional algorithms. "If I had not been to these workshops I would not have understood what he was doing and forced him to calculate in the way I have tried to teach him. I now understand that his thinking is much more developed than I had realized before".

In one of the schools the teachers have now decided to try to work with heterogeneous groups for several months and explore how this is affecting their teaching. The special-education teacher in this school has expressed that she finds this fruitful and she has noticed that her students from last year classes gain from working with students that have more developed mathematical thinking. She also found that children with other mother tongue than Icelandic gain much from working with those who are willing to discuss their work.

Three of the teachers have now audio taped some of their lessons. They found it difficult to discuss their experience at the workshops and wanted to have private discussion with me to start with. When they listened to their recording they all experienced that they had said things they had not planned to say and that they messed up with mathematical concepts. We discussed their experience and how useful this tool is to stimulate reflection on one's teaching. They have all planned to record more from their work and discuss their learning at our workshops.

Conclusions

The preliminary results of this study are promising. The teachers are slowly adapting to the processes of reflective practice and studying their own practice. Their belief that tracking students in ability groups in mathematics classrooms secures children good learning opportunities has been questioned and I see their schools slowly moving from integration to inclusion in their work. Our different backgrounds, both mine and the teachers, affect the process of our work. I see professional mathematics teachers as teachers who are capable of using their knowledge of mathematics, and the teaching and learning of mathematics in their classrooms; they are aware of the learning that takes place in the classroom, both students learning and their own learning developed through critical awareness and reflection. The teachers have expressed similar visions for their teaching but are not used to study their own teaching. Telling them what to do and what to avoid is attacking their core beliefs and identity as a teacher and a person and I therefore try to listen to their expressed desires and help them find effective ways to reflect on their work and unlock practices that limit their freedom to help their students learn mathematics.

Referrences

Ainscow, M. (1995). Education for all: making it happen. *Support for learning,* 10(4), 147–154.

Ainscow, M. (2007). Teacher development in responding to student diversity: The way ahead. In P. A. Bartolo, A. M. Lous & T. Hofsäss (Eds.), *Responding to Student Diversity. Teacher Education and Classroom Practice. Proceedings of the International Conference on Teacher Educatiuon for Responding to Student Diversity* (pp. 1– 22). Malta: Fauculty of Education, University of Malta.

Armstrong, A.C., Armstrong, D. & Spandagou, I. (2010) *Inclusive education: International policy & practice.* London: SAGE.

Artigue, M. (2009). Didactical design in mathematics education. In C. Winsløw (Ed.) *Nordic Research in Mathematics Education. Proceedings of NORMA08* (pp. 7–16). Rotterdam: Sense Publishers.

Austin, T. & Senese, J. C. (2004). Self-study in school teaching. In J. J. Loughran, M. L. Hamilton, V. K. LaBoskey, & T. Russell (Eds.), *International handbook of self-study of teaching and teacher education practices* (pp.1231–1258). Dordrecht: Kluwer.

Boaler, J. & Humphreys, C. (2005) Connecting mathematical ideas : middle school video cases to support teaching and learning. Portsmouth, NH: Heinemann.

Boero, P., Dapueto, C., & Parenti, L. (1996). Didactics of mathematics and the professional knowledge of teachers. In A. J. Bishop (Ed.), *International handbook of mathematics education* (pp. 1097–1121). Dordrecht; London: Kluwer Academic Publishers.

Bredcamp, S. (2004). Standards for preschool and kindergarten mathematics education. In D. H. Clements & J. Samara (Eds.), *Engaging young children in mathematics* (pp. 77–82). Mahwah, NJ: Lawrence Erlbaum Associates.

Carpenter, T. P., Franke M. L & Levi, L. (2003). *Thinking Mathematically. Integrating Arithmetic and Algebra in Elementary School.* Portsmouth NH: Heinemann.

Crawford, K., & Adler, J. (1996). Teachers as researchers in mathematics education. In A. J. Bishop (Ed.), *International handbook of mathematics education* (pp. 1187–1205). Dordrecht; London: Kluwer Academic Publishers.

Crespo, S., & Sinclair, N. (2008). What makes a problem mathematically interesting? Inviting prospective teachers to pose better problems. *Journal of Mathematics Teacher Education, 11*(5), 395–415.

Dabell, J., Keogh B., & Naylor, S. (2000). Concept Cartoons in Mathematics Education. Sandbach, Cheshire: Millgate House.

Dalvang, T. & Lunde, O. (2006). Med kompass mot mestring et didaktisk perspektiv på matematikvansker. *Nordic Studies in Mathematics Education,* 11, 37–64.

Fostnot, C. T. & Dolk, M. 2001. *Young mathematicians at work: Constructing multiplication and division.* Portsmouth, NH: Heinemann.

Fullan, M. (1999). *Change forces: The sequel*. Philadelphia: Falmer Press.

Goos, M. (2004). Learning Mathematics in a classroom community of inquiry. *Journal for Research in Mathematics Education,* 35(4), 258–291.

Guðjónsdóttir, H., & Kristinsdóttir, J. V. (2007). Preparing teachers to teach all children mathematics. In P. A. Bartolo, A. M. Lous, & T. Hofsäss (Eds.), *Responding to student diversity: Teacher education and classroom practice. Proceedings of the International Conference on Teacher Education for Responding to Student Diversity* (pp. 44–61). Malta: Faculty of Education, University of Malta.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K. C., Wearne, D., Murray, H., Oliver, A. & Human, P. (1997). *Making Sense. Teaching and learning mathematics with understanding.* Portsmouth NH: Heinemann.

Jaworski, B. (2003). Research practice into/influencing mathematics teaching and learning development: Towards a theoretical framework based on co-learning partnerships. *Educational Studies in Mathematics, 54*(2/3), 249–282.

Jaworski, B. (2006). Theory and practice in mathematics teaching development: Critical inquiry as a mode of learning in teaching. *Journal of mathematics teacher education, 9*, 187–211.

Jaworski, B. (2007). Itroducing LCM - learning communities in mathematics. In B. Jaworski, A. B. Fuglestad, R. Bjuland, T. Breiteig, S. Goodchild & B. Grevholm (Eds.), *Learning communities in mathematics*. Norway: Caspar Forlag.

Kemmis, S. (1999). Action research. In J. P. Keeves & G. Lakomski (Eds.), *Issues in Educational Research* (pp. 150-160). Amsterdam: Pergamon.

Kemmis, S., & Wilkinson, M. (1998). Participatory action research and the study of practice. In B. Atweth, S. Kemmis & P. Weeks (Eds.), *Action Research in Practice. Partnership for Social Justice in Education* (pp. 21-36). London/New York: Routledge.

Kristinsdóttir, J.V. (2010). Teachers' development in mathematics teaching through reflective discussions. Í B. Sriraman, C. Bergsten, S. Goodchild, G. Palsdottir, B. D. Søndergaard & L. Haapasalo (ritstj.). The Sourcebook on Nordic Research in Mathematics Education (bls. 487–494). Charlotte, NC: Information Age Publishing.

Kruger, T., & Cherednichenko, B. (2006). Social justice and teacher education: Re-defining the curriculum. *The International Journal of Learning, 12*, 1–8.

LaBoskey, V. K. (2004). The methodology of self-study and its theoretical underpinnings. In J. J. Loughran, M. L. Hamilton, V. K. LaBoskey & T. Russell (Eds.), *International handbook of self-study of teacher education practices* (pp. 817–870). Dordrecht: Kluwer.

Marlowe, B. A. & Page, M.L. (2005) *Creating and sustaining the constructivist classroom.* Heatherton, Victoria Australia: Hawker Brownlow Education.

Mason, J. (1999). *Learning and doing mathematics.* 2nd rev. ed. York: QED.

Mason, J. (2002). *Researching your own practice. The dicipline of noticing.* London Routledge Falmer.

Menntamálaráðuneytið. (2007). *Aðalnámskrá grunnskóla, stærðfræði.* Reykjavík: Menntamálaráðuneytið.

McNiff, J. (2002). Action research for professional development: concise advice for new action researchers Retrieved 20.02., 2008.

More, J. (2005). Transformative mathematics pedagogy: From theory to practice, research, and beyond. In A. J. Rodriguez & R. S. Kitchen (Eds.) *Preparing mathematics and science teachers for diverse classrooms.* Mahaw, New Jersey: Lawrence Erlbaum.

Niss, M. & Højgaard Jensen, T. (Red.) (2002). *Kompetenser og matamatiklæring – ideer og inspirasjon til udvikling af matematikundervisning i Danmark.* København: Undervisningsministeriets forlag.

Pugach, M. C. (2005) Research on preparing general education teachers to work with students with disabilities. In M. Cochran-Smith & K. M. Zeichner (Eds.) *Studying teacher education: The report of AERA panel on research and teacher education.* Mahwah, NJ: Lawrence Erlbaum associates.

Stigler, J. W., & Hiebert, J. (2004). Improving Mathematics Teaching *Educational leadership, 61*(5).

Zeichner, K. (1996). Educating teachers for cultural diversity. In K. Zeichner, S. Melnick & M. L. Gomes(Eds), *Currents of reforms in preservice teacher education* (pp. 133–151). New York: Teachers College Press.

Woodward, J., & Montague, M. (2002). Meeting the challenge of mathematics reform for students with LD. *The Journal of Special Education, 36*(2), 89–101.

Jónína Vala Kristinsdóttir is an assistant Professor of Mathematics Education at the University of Iceland, School of Education. She worked for 20 years as a general classroom teacher in an elementary school. Teaching is her primary profession and she has been active in curriculum planning and writing in mathematics for elementary schools. She teaches undergraduate and graduate courses on mathematics education and education in general. Her research interests are in mathematics learning and teaching in diverse classrooms. Current research and writing include teachers' development in mathematics teaching in diverse classrooms and a self-study of mathematics for all learners.

joninav@hi.is

Mathematical communication in “Träningsskolan”

Ann-Louise Ljungblad

The National Agency for Special Needs Education and Schools (SPSM)

This work reports on a three year development project, which focused on the highly complex learning inherent in mathematics education. The teaching of mathematics is particularly challenging because of the subject’s complexity, but the project provided groundbreaking results. The pupils learnt mathematical terms and concepts in a way that we did not believe possible at the outset. Thus the expectations of the pedagogues, parents and the tutor involved have been raised constantly throughout the project. Träningsskolan in Sweden is a special needs school for children with very severe difficulties at an early developmental stage, with intellectual disabilities/mental retardation. Most of these children also have physical disabilities. A project was launched at Regnbågens Träningsskola in Falköping, Sweden in collaboration with Ann-Louise Ljungblad, an adviser at The National Agency for Special Needs Education and Schools (SPSM). The aim was to discover how to overcome obstacles in the mutual mathematical communication between pupils and pedagogues and to explore new learning possibilities (Ljungblad, 2003) by taking digits as artefacts (Säljö, 2000) into the classroom, focusing on mathematics didactic structures: *Sense of space, Sense of time, Sense of numbers, Pattern and categorisation.*

The pedagogues analysed the initial extent of every pupil’s mathematical vocabulary: What have they learnt so far? New goals were set out from a lifelong learning perspective in collaboration with parents: What mathematical terms and concepts do we want our child to master in her/his life? These goals were written down in an individual development plan and were recurrently analysed and discussed with the pupil’s parents every term.

Filming was a frequently used tool and the pedagogues filmed different sequences in the daily work and education. After the school day the personnel met to reflect on the film sequences. The intention was to capture the communication that took place between the pupils and the pedagogues, which was hard to grasp in the classroom. These analyses led to the development of new ideas and innovative approaches in mathematics education.

During the whole process values like participation and potential were also discussed. What can we do in the learning environment to encourage each pupil to participate in the mathematical work? How can we make the mathematical language accessible to each child? The basic idea of the project was to improve the didactic environment so that mathematics education is characterised by *Equality, Participation, Accessibility* and *Companionship.* The school and education system need to work towards a realisation of the right to *an inclusive mathematics education for all children* (Ljungblad, 2006).

The children involved, who had previously been excluded from the mathematics social communicative culture, have inspired *a fundamental value and belief that all children can learn mathematics, in their own personal way, thus to live a good life.*

A unique mathematics project

For a number of years an exciting development project on mathematics education has been carried out at Regnbågens Träningsskola in Falköping, Sweden. The teaching of mathematics is particularly challenging because of the subject’s complexity.

The project provided groundbreaking results. The pupils learnt mathematical terms and concepts in a way that the personnel involved did not believe possible at the outset. These children do not use verbal communication, some communicate with traditional sign language and other pupils use and develop more personal signs. The expectations of the pedagogues, parents and the tutor involved have been raised constantly throughout the project. Even though we at the outset hoped for, and expected, some progress in children’s mathematics learning, this occurred much faster and became a reality in a way we had not expected even in our wildest dreams. There was a natural variation in the extent of the children’s development during the project and the range is wide. A couple of children learnt some maths terms. Other children experienced a tremendous development: they began to count and work in an abstract way that was beyond our imagination. Even though it is impossible to get a clear view of how the children experience these new maths terms that were applied in the teaching, it was actually possible to see pupils on a very early level respond to maths terms in a new and active way. It was not of major interest to quantify each child’s progress; instead we searched for the quality of how a child can use a new maths term or concept, and the impact of this in his or her life. That was the most important outcome of this development project.

When a child learns a new maths term and can communicate and use it in different situations - at home, during schoolwork or with companions at leisure time - this can give a new qualitative dimension to the child's life. This was perhaps the most important result we saw: the pupils mastering new math terms and being able to use them in an active, communicative way. It astonished us pedagogues to see how these new maths terms changed the pupils' lives in many important ways. For example, by learning a few time words, a girl named Klara suddenly had control over changing activities without any frustration. She was also able to start negotiating about her schoolwork, what to do first, and what she was going to do later, and be able to influence it. This is not only a question about *Equality* in education, or a matter of how the mathematical language can become *Accessible* in some ways, but it also gives the child an essential right to *Participate* in the maths world, in a new and active way, where she would earlier have been excluded on account of tradition and culture. *The school and education system need to work towards a realisation of the right to an inclusive mathematical education for all children* (Ljungblad 2003; 2006).

Träningsskolan Regnbågen

The Swedish school system consists of three types of schools. The local authorities operate both regular schools and "särskola" for those with intellectual disabilities/mental retardation as identified by doctors and psychologists. Särskolan again has two branches: regular school for those who have the capability nearly to follow the regular plan and curriculum for all children and "träningsskolan" for children at an early developmental stage that have severe difficulties. Most of these children also have physical disabilities. Träningsskolan sets down targets for the pupils to aim for, but these documents do not contain a single reference to mathematics as a subject. The children we have worked with in this project belong to this last group.

Regnbågens Träningsskola has 14 pupils aged six to16 years, who all participated in this project along with their 20 pedagogues. The term pedagogue here refers to both teachers and assistants, i.e. all the adults at the school who work with the pupils during the entire day. These pedagogues do not only work with the children during school hours, but also before and after school during leisure activities. Regnbågens Träningsskola is a school with a tradition of good progress in involving all colleagues when working on projects. An earlier major project in collaboration with The National Agency for Special Needs Education and Schools (Specialpedagogiska Skolmyndigheten, SPSM), involved the development of the child's Swedish vocabulary, which for some children meant learning completely new personal signs. This project was very successful and the interesting results were documented in the report "From me, to you, to everyone" (Från mig till dig till alla, 2005).

A new maths project arises

This pilot project led to new ideas and challenges at Regnbågen. In collaboration with Ann-Louise Ljungblad, an adviser at The National Agency for Special Needs Education and Schools, new ideas about a project with focus on *"Mathematical Communication in Träningsskolan"* emerged. We asked ourselves whether we could continue the interesting work in developing pupils' communication, only this time from a mathematical perspective. After some discussion a new school development project arose. The general idea was based on teaching and learning mathematical terms and concepts and to use digits as artefacts (Säljö, 2000). We also focused on how to overcome obstacles in the mutual mathematical dialogue between pupils and pedagogues and trying to discover new learning possibilities (Ljungblad, 2001; 2003).

The main goals for the project were the following:

- The starting point was to identify each pupil's current mathematics ability.
- Another important goal was *to develop tools for mapping a child's development in learning maths.* The background for this was that we did not find pre-existing mapping materials of this kind, concerning children at a very early level and their mathematical learning.

- We also realised that we would probably have to develop laboratory materials and artefacts to support mathematical learning for children at an early level.

- To summarise we can say the main goal for this school development project was to develop the didactic learning environment in träningsskolan, encouraging pupils' communication and learning in mathematics.

In the project we took a purely mathematical perspective, and simply lifted maths terms and digits into the classroom and worked with four didactic structures:

- Sense of space
- Sense of time
- Sense of numbers
- Pattern and categorisation

Initially, thinking in mathematics didactic structures in this way was difficult for the pedagogues, but after a while they described their own development by saying: "Now that we have put on our mathematical glasses we suddenly see learning and teaching with our pupils in a totally new way." This was a tipping-point in the project, and it became obvious to me as a tutor that new things started happening and exciting progress was made.

Theoretical framework

The analyses in this project are founded on two theoretical perspectives. Firstly, socio-cultural theory (Säljö, 2000) with Vygotsky's (1999) thinking about *learning through communication between people* is an important foundation for this project because of the focus in education on mutual communication in education between children and pedagogues. Socio-cultural theory helps us understand interplay in learning situations and why learning takes place in each generation. This way of seeing the *human being* ***in*** *the world* provided a new perspective when the *learning context* was analysed. We saw clearly how *learning is situated*; something could work very well in one situation but might not work at all in a different situation, however similar. This showed us the importance of awareness about *mediation*, which is the voice of the teacher when we teach and the interpretation of the mathematical terms and concepts we want a child to learn. A part of this included a constant search for, and attempts to overcome, obstacles in the mutual communication with the children. Socio-cultural theory also emphasises the importance of *artefacts* for learning. Artefacts are tools that express knowledge and can be of a tangible or intellectual nature. Säljö (2000) illustrates individual learning as a question of how to access a tool and learn to handle it. In this educational development project one focus has been on how children on a very early level can learn with artefacts. Thus the pedagogues also developed new artefacts, for example counting boards in different designs, in order to support children's learning.

The other theoretical component of the framework is *the Pedagogy of Differences* (Säfström, 2005), which is based on Hanna Arendt's and Emanuel Levinas' ethical ideas about peoples' relationships. In Sweden, Professor Carl Anders Säfström has developed this into new and interesting ideas in education. Viewed from this ethical approach, differences between people are not seen as something that disturbs the education. On the contrary, differences in children's ability are seen as the potential for learning and education as a process where differences can be inquired. Therefore education must take differences very seriously, and raise ethical and moral questions to discuss how to create good meetings in schools with unique children. This is a choice to live ***for*** other people (Säfström, 2005).

Mapping

To begin with, the pedagogues analysed the initial extent of each pupil's mathematical vocabulary. What had they learnt so far? When we investigated the children's maths ability it turned out to be very limited. This result was not unexpected as teaching had not focused on mathematics at all. The curriculum for pupils in träningsskolan does not contain a single reference to mathematics as a subject. We discovered that two pupils could count to five or six and some other pupils knew a couple of maths terms. Thus it was clear that for all the pupils we would have begin at an early level of learning and teaching maths.

After this initial investigation we met with the parents and discussed new goals from a lifelong learning perspective. What mathematical terms and concepts do you as parents want your child to master in her/his life? What goals do we want Emelie to accomplish, to enable her to live a good life? These goals were written down in an IUP ("individuell utvecklingsplan", *e.* individual development plan), which was analysed and discussed with each pupil's parents each term. The targets in this plan were considered from two time-perspectives: targets to achieve in a short time and targets that might take longer to reach. Through the IUP, mathematical learning became a natural part of each child´s education.

Values like participation and accessibility were also discussed amongst the pedagogues throughout the course of the project. How can mathematical concepts become available to the pupils? How can Klara develop her sense of time, and participate in planning her daily work together with the teachers? We saw tremendous development occur when she mastered two time words – *first* and *then.* These words were signed as two personal signs to Klara and thereafter structured deliberately and used in communicative learning situations. Klara learnt these new time tools and a lot of frustration related to changing activities was eliminated. She was now participating and could influence activities and tasks, when there was going to be a change and what was going to happen afterwards. These two small mathematical terms had a strong impact on Klara's life and her ability to participate in the daily school work, and also created personal freedom.

It is important to point out that this teaching project did not focus on quantity, in the sense of how many new maths terms children learnt. Instead we looked for qualitative effects in each child's life. As a by-product, however, some pupils actually learnt many new terms, thus exceeding our expectations. In this analysis of the results, however, we investigate qualitative effects that give new dimensions in the child's life.

Filming as a tool

The teachers had learnt from earlier experiences to use a video camera in the classroom to capture the children's signs of communication, which can be hard to grasp in everyday situations. The team of pedagogues found it successful to analyse film sequences after each school day, this time with focus on four mathematics didactic structures. This led to the discovery of new possibilities for learning. Together the team chose a film clip that they viewed again and again, in different ways:

- *The first time – watch and experience without interpretation.*
- *The second time – write down what the pupil is doing.*
- *Thirdly – write down what the teacher is doing.*
- *Then, focus on what is happening in the learning environment.*
- *Next time, close your eyes and listen without watching.*
- *Then – turn off the sound and watch without listening.*
- *Finally – analyse the entire complexity!*

This way of working was very educational and we learnt a lot through the use of this reflective tool. One of the pedagogues described a sequence the staff had watched perhaps 50 times, and new things were still emerging. This encourages a profound understanding of how necessary patience is in educational work like this.

Filming teaching situations in the classroom was also an important tool to use in discussions with the parents. One example is a boy working on developing his sense of time: The teacher showed a film clip where everyone could see that he was very much in the present – *here* and *now*. Taking his sense of time into account, it would be a natural goal to work towards for him to get a feeling for a whole *day*. The work could be extended by discussing what happened *yesterday* when he comes to school in the morning – and giving instructions about what is going to happen *tomorrow* when he leaves in the afternoon. The parents did not actually think it would be possible for their son to reach this goal, but agreed to give it a go. When they saw new film clips during the following term showing his development and that he already had reached this goal they started to cry from happiness.

Tutor

One of the tutor's main tasks was to support the teachers in crossing pedagogical boundaries, with focus on mathematics didactic. These pedagogues work with children with huge disabilities in an educational system where neither tradition nor culture have previously dictated any ideas or expectations regarding teaching maths to this group. Another goal was to inspire and explore ways for mutual work in developing a learning environment where these pupils can be able to learn mathematical terms and concepts, a didactic environment where mathematics education is characterised by *Equality, Participation, Accessibility* and *Companionship.*

One of my first actions as a tutor was to observe how the pedagogues handle diversity and differences between the children, as this was one of the fundaments of the project. It was fascinating to see a culture of constant awareness of differences and diversities develop. Differences were not seen as something that disturbed the education, instead they were the starting point and these differences were deeply explored. The pedagogues invited pupils to good learning meetings; they took an active responsibility for doing this and regarded each child as a unique person. This was an important result in the outcome of the project. For example, even though a pedagogue was responsible for one or two children, they were also mindful of the other pupils' learning. This was an ongoing positive and professional search for possibilities and potential and the pedagogues explored mutual ideas about finding ways for all the children at the school.

As a tutor I challenged the pedagogues to think in new ways. At the outset this was a difficult task because of the tradition and culture in träningsskolan, where the curriculum is written without focus on maths. Another important outcome was how these children, who had previously been excluded from the mathematical social communicational culture, throughout the process showed us and inspired a profound awareness that *mathematical learning actually is possible for all children*. To achieve this, the adults around a child must cooperate and have a clear focus on developing the child's communication and simultaneously be mindful of mediation. This is also an opportunity to look at the possibilities of education in träningsskolan in a new way: from a purely didactic viewpoint. The complexity of working with these children lies in the differences in the children's personal conditions as they have intellectual disabilities, usually in combination with physical disabilities. This has certainly constituted a challenge in this educational project.

From a mathematics didactic point of view, we simply took the digits as artefacts and brought them into the classroom – an active choice that took some courage for all the adults involved, including pedagogues and parents. In the beginning some parents asked whether this really was possible and, naturally, we did not know at first what the outcome would be. It is also important to note that this has been extremely challenging educational work. We had to rethink, be constantly innovative and reflect upon new ideas during the process. Last but not least, we also had to raise our expectations and give the children new challenges, careful of not holding them back from fear that they might fail.

Mediation

This development project about complex learning in mathematics has to a great extent been a work about *mediation.*

In education, mediation (Säljö, 2000) is for example the voice of the teacher when she interprets a new maths term we want a child to master. It is a question of how to pre-think the communication in mathematics. What words can I use, what models can I choose, and what kind of tools can we apply? And how do I, with a high awareness, combine them in my communication with the child, with a deep understanding about how learning is situated. Mediation exists in two time zones. The first one is prior to the teaching situation, when a teacher pre-thinks and pre-plans, a preparation based on contextual experiences and understanding. The next time phase is the actual teaching situation with the pupil. In that situation in the classroom, the teacher must encourage, adjust, develop the capability to listen, and last but not least grasp the situation that occurs.

Another important resource we developed was the ability to reflect on the present context. A frequently asked question was: in what context do we find ourselves right now? We tried to maintain a high awareness about our mediation, with the aim of facilitating the conversation in the present teaching situation. We studied the context and tried to understand how the child experienced the current situation. Even though the situation was similar to another in our view as teachers, the child would not necessarily experience similarities. Our awareness of mediation, artefacts and context in the discussion between the pedagogues and tutor, gave a very high-quality result.

Another example of a tool for mediation was, the computer program “Musse”. In this program it is possible to plan different working processes for each child. Part of the program is a feature called “Count and there is music” where the child can count up to five, for example, and there are five different samples of music behind the digits 1 to 5. The child, then, learns what music appears after counting the five different numbers. Some children loved this task and could sit for an hour and count, deciding for themselves what kind of music they wanted to listen to. One day, however, we suddenly became aware of our instructions in the program from a different angle. We had recorded our voices saying: “Count and there is music, 1, 2, 3, 4, 5”. When we came to the end of a recorded sequence we automatically raised the intonation, which is a normal feature of the Swedish language. At this point we started to discuss whether the children perhaps do not know how to count, but rather only follow our intonation. This was an important issue to explore. Therefore, we recorded our instructions again, only this time without a change in our intonation – and the children could actually count. The intonation did not interfere in this context. However, with a particular child, June, we always had to raise the intonation at the end of an instruction, because that is what she needs. To her counting is more rhythm in the communication, in conjunction with the intonation it helps her to understand that something is going to happen.

In another situation when using the computer program, we discussed *when* the music sample should appear after the child has counted. Should there be a pause of three or four seconds before the music sample comes as stimulation? If the pause is too short the child might just push the touch-control linked to the computer and it might not actually be counting. Conversely, if the pause is too long, the children might lose interest. The balance in the mutual communication between a child and a teacher in this mediated situation is extremely difficult, it comes down to questions of intonation and pauses of three or four seconds before the next word follows in the communication. As a tutor in this project I would like to emphasise how this illustrates the depth of mathematics didactics and mediation: it does not get more intricate than this, with a profound contextual understanding, analysing new patterns and creating possibilities in order for new learning actually to occur.

Learning mathematics

We observed exceptional changes beginning to take place in the children’s development. They learnt mathematics in their own personal way and the potential was astonishing. An example is Rikard, who was 11 years old when the project started. He did not understand letters and could not recognise colours. In this type of pedagogical situation it is unusual to introduce digits and counting. But this is what happened to Rikard through the project and he turned out to love counting! After three years the pedagogues suddenly realised that Rikard had explored a computer program about mathematics all by himself, and without any instructions. This illustrates how important it is not to have preconceived ideas and instead focus on working in the “Zone of Proximal Development” (Vygotsky, 1999). Thus it is important to have focus and awareness about mediation, structured pedagogical and didactical mutual work – based on the belief that the “impossible is possible” – and to have real faith in children’s potential.

Another example is Klara who is now 16 years old. She is blind and in a wheelchair. There is a huge difference between being pushed in a wheelchair from one room to another, compared to having control over the sense of space. Klara can now communicate with her pedagogues, and give directions: she tells the way with signs like *right, left, forward*, when they move from the classroom to the sports hall.

With Klara’s parents we discussed that it might be of importance for her to reach an understanding of two time words *first* and *then.* By mastering this she might become able to handle situations such as changing activities, in a better way than before. It did not take long before Klara had learnt new signs for these two maths terms and could use them in an active way. It had a great impact on Klara’s life. She became an active participant that could handle and change situations and negotiate about her work during the day.

Another aim Klara's mother considered important for her daughter was being able to count, for example to buy a double chocolate bar and know that it consisted of two pieces. She would then have control over a situation where she could eat one piece and there would still be one piece left. Then it would not be frustrating for her when that last piece was eaten and there would be no chocolate left. This goal involved counting and handling the first numbers, so we started by making her a personal counting board showing 1, 2 and 3. It did not work out at all, however, and we could not figure out why. Then one day in discussion with the pedagogues I mentioned that it might be of importance to understand the concept of 0 – *the empty quantity* – before you can learn and understand the quantity of 1. This turned out to be the key: Klara understood 0 in a short period of time and her progress could then continue. We developed several different counting boards for Klara, which she used frequently and in many different situations every day, for example when she and the pedagogue discuss whether she wants to take part in the next activity or not. The counting board has different cups; one is empty and the other contains many beads. She makes her choice by putting her hand into the cups; we consider her mathematical choice to be the last cup she has chosen before she pulls the counting board away. Thus, she decides if she wants to take part in the activity many times, or not at all.

Another pupil is David, who has made great progress in his mathematical learning. The personnel now believe that David could eventually work in a small café and handle money, naturally with the aid of a calculator and a supporting tutor. This is an achievement we did not imagine possible at the beginning of the project, not even in our dreams. When we see how these pupils on a very early level are learning mathematics, it gives us a new dimension to special pedagogic in relation to mathematics education. As adults we now have a responsibility to preserve a result like this for the future.

The cases of June, Klara, Rikard and David illustrate the wide range of learning that has taken place through this project. Some children have learnt to handle some new maths terms and other children have shown such extraordinary progress that it could eventually be possible for them to get a job! The important outcome is that *mathematics learning can occur in every child's life* and *the positive impact these maths terms can have in their life*, although we do not fully know how and in what way. One of the teachers said at the end of the project: "It is now the journey starts! It is now we can start to discuss what teaching mathematics to children at a very early level can encompass."

Two years into this successful project, I as a tutor had discussions with the pedagogues and head teachers about the possibilities of sharing and spreading the result of this pioneering work to other schools. The huge documentation in the form of film sequences was a treasure chest available to use. We produced a DVD film from the mathematical structures: *Sense of space, Sense of time, Sense of numbers*, *Pattern and categorisation,* with several subheadings. Alongside the film a teacher's guide was produced. This material will be edited by SPSM (www.spsm.se) during the spring of 2010, and will enable other schools to be inspired and learn about really complex and challenging mathematical teaching situations from an inclusive perspective.

Conclusion

As an adviser at SPSM and a tutor on the project, thus having had the opportunity to be a part of this pioneering work, one must ask oneself: What is the essence of such a positive successful, special pedagogic project, about extremely complex mathematical learning? How is it possible that the children learnt mathematics even quicker than we managed to capture on film? How can they learn maths even though their personal circumstances are extremely complex? Mathematics education does not get more complex than this. Perhaps we also have to ask ourselves, why there previously have not been any expectations that these children should learn mathematics? Träningsskolan in Sweden has been criticised for placing too much emphasis on care and support, and too little on the pupils' actual learning. At Regnbågen there is a lot of love, support and caring – but what this project did was to add something new; a pure mathematics didactic focus. To develop the culture in träningsskolan in general will require education for the personnel, where concepts like fear and courage must be discussed. By doing this a professional environment can grow, where pedagogues together create trust, use creativity and share new findings in a mutual professional learning in practice.

It is very hard to describe the process and progress of children on a very early level and how they have achieved mathematical knowledge. The results clearly illustrated, however, that these children, who previously were excluded from the mathematical social communicative culture, inspired a profound understanding of the fundamental value and belief that all children can learn mathematics in their own personal way and thus become able to live a good life.

Some factors of success have been discovered;

- Mathematics must be seen as a language, which children can experience in many different personal ways.
- These differences in learning mathematics must be the starting point and can be investigated.
- Focus on work with mathematical artefacts such as words, concepts, digits and physical laboratory tools.
- Awareness in mediation and focus about context in the actual educational situation.
- Systematic structure in the pedagogical and didactical mutual communication.
- Filming as a tool, and time for analysing film sequences within the teaching team.
- Good communication with the pupil´s parents, based on lifelong learning perspectives to live a good life.

Finally, all children should have the right to learn mathematics based on their own personal needs and life situations (Ljungblad, 2006) with an education that can handle diversity and explore human differences. We can learn from these differences, with an open mind regarding to pluralism and diversity in the educational system (Säfström, 2005). The work regarding inclusion and mathematics, and the visions we need to put on the agenda for discussion include questions about *Accessibility, Participation, Companionship* and *Equality*.

A journey has just begun, where schools and the education system need to work towards a realisation of *the right to an inclusive mathematics education for all children* (Ljungblad, 2006).

References

Ljungblad, A-L. (2001). *Matematisk Medvetenhet.* Varberg: Argument.

Ljungblad, A-L. (2003). *En studie av hur barn använder siffror, tal och antal i en matematisk diskurs.* Magisteruppsats i specialpedagogik. Institutionen för pedagogik och didaktik. Göteborg: Göteborg universitet.

Ljungblad, A-L. (2006). *Matematik en mänsklig rättighet.* Varberg: Argument.

Säfström, C. A. (2005). *Skillnadens pedagogik.* Lund: Studentlitteratur.

Säljö, R. (2000). *Lärande i praktiken. Ett sociokulturellt perspektiv.* Stockholm: Prisma.

Vygotsky, L. S. (1999). *Tänkande och språk.* Göteborg: Daidalos.

Ann-Louise Ljungblad is a special pedagogue, and works as an adviser at the National Agency for Special Needs Education and Schools in Sweden. She is now responsible for a four year long national project - Mathematical difficulties in an inclusive environment. During many years she has worked with action research in the classroom with children in mathematical difficulties, with focus on developing the teaching dialogue to create new possibilities for learning.

ann-louise.ljungblad@spsm.se

Further education for in-service teachers with focus on special needs in mathematics

Håkan Sollervall
Linnaeus University, Sweden

This work discusses the content knowledge and pedagogical content knowledge in mathematics that needs to be prioritized in further education programmes for Swedish in-service teachers wishing to focus on teaching students with special needs in mathematics. The study is based on interviews with teacher educators as well as recent curriculum development in Sweden. We conclude that in order to meet the needs of all students, and specifically those who experience difficulties with prescribed methods and strategies, emphasis on teachers begin able to develop a deep understanding of elementary school mathematics, including one's own competencies and how translate between different mathematical representations as well as understanding the educational value of such competencies, is of utmost importance.

Background

Several universities in Sweden, including ours, have recently started to offer further education at an advanced level for in-service teachers who wish to become special needs teachers in mathematics. Course elements based on content from the regular teacher education programme have in our new programme been placed in a pedagogical rather than a mathematical setting. In the first version of the programme it was taken for granted that the teachers, who were qualified mathematics teachers, had sufficient command of mathematics to be able to discuss advanced pedagogical issues in relation to school mathematics. However, some teachers experienced difficulties in analyzing students' mathematical performance. They were sometimes unable to identify certain qualities in students' mathematical reasoning and were not always convinced of the relevance of the course content offered within the programme.

The author of this paper was given the task to identify aspects of mathematics, specifically taking into account both content knowledge and pedagogical content knowledge (Ball, 2000), that should be included to improve the new version of the programme.

Rationale for the project

The study presented in this paper is primarily based on discussions with colleagues in mathematics education at our university, with the aim of making use of their accumulated competencies and experiences for identifying appropriate content for the programme. The discussions have been followed up with individual interviews with colleagues that have taught on the programme.

Our colleagues in mathematics education have varying backgrounds and professional duties and include several active researchers and staff with extensive teaching experience from the Swedish school system. We have an ongoing professional discussion about desirable learning outcomes and content in the courses in mathematics and mathematics education within the teacher education programme, including issues regarding special needs. The staff members have a good command of trends and initiatives from both research and political perspectives and have good insight into current teaching practice through in-service training of students and examination projects.

This study relies on the accumulated experiences of the participating colleagues, and in particular the author, who is well acquainted with the governing documents of the Swedish school system and has served in the national reference group for development of new curriculum in mathematics for primary schools in Sweden. The author has written several text books (for engineering, teacher education, and upper secondary school), as well as conducting research in mathematics education with a particular interest in design based research (Nilsson, Sollervall & Milrad, 2009), mathematical competencies (Niss & Jensen, 2002) and registers of mathematical representations (Duval, 2006).

Drawing on a parallel with applied and pure mathematics, this study may be understood as "applied mathematics education" where the focal point is its implications for practice, as opposed to mathematics education as a "pure" research discipline where the primary focus is theoretical contributions to the research field. The approach in this study relies heavily on the actual competencies of the participating colleagues and their judgment of the in-service teachers' understanding of mathematics and what they need to know as prospective special needs teachers in mathematics. By account of the accumulated competencies and experiences from both regular teacher education and special needs teaching we find this approach sufficient regarding the educational aim of the project, especially considering the limited time scale (a few months) allowed for the project.

Results of interviews with colleagues

According to the teacher educators, the students in the special needs programme have been teaching for several years and are not used to research based approaches from their teacher education. The teacher educators express that some students have very specific expectations concerning the courses and primarily expect handbooks and instruction on how to act in specific situations. They do not expect to learn mathematics within the programme but are otherwise described as highly motivated.

The teacher educators express that all teacher students need to develop a view of mathematics as a subject that differs from the view most of them are used to from school. This "school view" has an algorithmic character, where the focus is on mastering specific techniques and solving problems using prescribed methods. A common view among the teacher students is that the "best method" should be taught in school, that pupils should not be confused by being offered several methods for solving one problem. Hence they often question the need to learn more mathematics.

Furthermore, teacher students often discriminate against "simple" solutions and show preference for algebraic methods. The teacher educators express that it is unfortunate that this view is confirmed in some of the course literature used in teacher education and also in grading criteria for Swedish national tests in mathematics. They claim that this view may be one of the reasons why some of the students in the special needs programme question some of the courses in the programme that are intended to broaden their mathematical understanding by discussing different solution methods and how to pose mathematical problems that may stimulate use of different strategies.

The teacher educators also express that the students are eager to discuss pedagogical issues but are reluctant to discuss issues regarding mathematical content, where the latter is more or less taken for granted. "I have to look it up in the book to make sure" is a common answer regarding questions about mathematical concepts. For some students, there is nothing to discuss.

Reflections on the interviews

The "school view" as identified by the teacher educators is challenged by the national curriculum for elementary school mathematics in Sweden that emphasizes *a balance between creative, problem solving activities and knowledge about mathematical concepts, methods and forms of expression* with emphasis on communication (verbal, written, actions) and own investigations and using different forms of mathematical representations (numerical, symbols, tables, pictures) and solution methods (Skolverket, 2009). Even if the current teacher education and initiatives that focus on teaching development support these standpoints, the teaching practice in schools is still dominated by the instrumental, tradition mainly because of the rigidity of the school system (Skolverket, 2003).

Based on the interviews, we conclude that some features which are nowadays consider central aspects of mathematics were previously not prioritized in teachers' training and have only recently been prioritized in school mathematics. Consequently, we conclude that further education of in-service teachers has to address features such as multiple representations (Ainsworth, 2006; Duval, 2006) and problem-solving based on students' own investigations. Effort has to be made to motivate the students with respect to the value of such features, obviously with reference to the national curriculum but also by stimulating own reflections on specific instances and examples relevant for school teaching, as well as by raising awareness about the fact that different forms of representations may put different strain on working memory and hence be more or less suitable for students with limited capacity in working memory (Baddeley, 2000).

It may be noted that none of the teacher educators expressed that the in-service teachers needed to study mathematics at higher level. Our interpretation is that they believe that teachers benefit more from strengthening their pedagogical content knowledge regarding elementary mathematics.

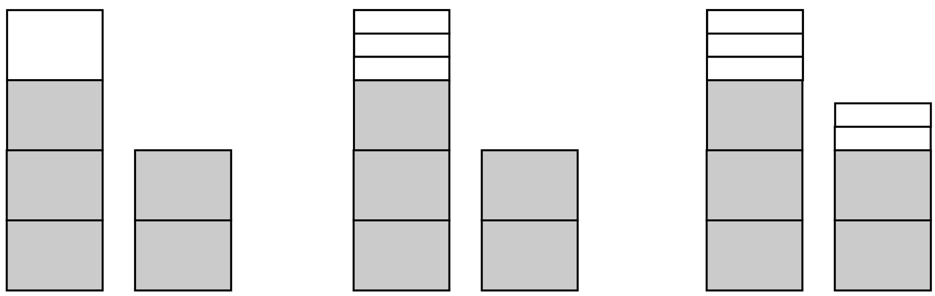
Regarding the preference for algebraic methods, we find support in Arcavi (2003, p. 235) to argue against devaluation of visualization and other "simple" methods. Instead, as Arcavi also argues, such methods may both support a solution based on algebraic methods and provide a solution in their own right (ibid., p. 223).

Implementation

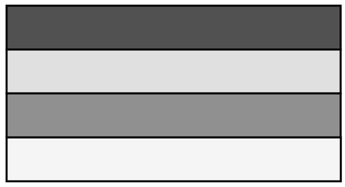
Teachers and teacher students have to be motivated and convinced about the value of using and being able to translate between different representations. They have to experience mathematical investigations based on numerical examples or pictures and not only using advanced symbolic methods which only the brightest students will be able to master. Particularly, they ought to work with mathematical tasks that highlight features such as: 1) sometimes a "simple" representation, for example reasoning based on a picture, may be the most efficient way to understand and solve a problem; 2) how conversions between representations support problem solving processes and understanding of other representations; 3) how multiple representations support and strengthen concept formation and understanding of mathematical content; 4) how different representations may put more or less strain on working memory. Below, we provide examples of these four important features by treating tasks from a text book used in teacher education (Sollervall, 2007).

1) **A simple representation may be the most efficient**
   Lisa has discovered that she obtains a nice wall colour if she mixes 3 parts yellow paint with 2 parts purple paint. She has 4 litres of yellow paint that she mixes with 2 litres of purple paint. Now she has 6 litres of paint but not 3 parts yellow and 2 parts purple. Help her work out what colour and how much of that colour she should add to the 6 litres to obtain the correct proportions.

   This problem may be solved algebraically, but the resulting rational equation is so complicated that most students get stuck in the solution process. Even if they manage to solve the equation, they realize that a solution based on a sequence of pictures is both efficient and may be considered fully acceptable from a mathematical point of view.



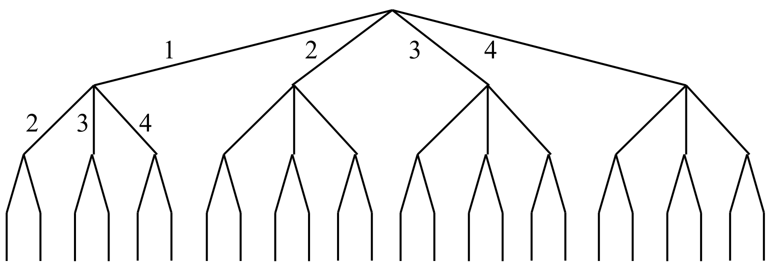
2) **Conversions between representations may support mathematical understanding**
   A flag consists of four rectangular sections that are placed on top of each other. Each section will be coloured in a distinct shade of grey. There are four different shades to choose between. In how many ways may the flag be coloured? One possible flag is shown below.



A natural start to solve the problem is to draw all possible flags. If the (24 possible) drawings are structured convincingly, the drawings may be used as a basis for a solution. An alternative solution might be based on coding the colours/shades, for example as 1, 2, 3, and 4. One might then be represented as 1234, another as 4132, and so on. The flags may be listed as

1234	1243	1324	1342	1423	1432
2134	2143	2314	2341	2413	2431
3124	3142	3214	3241	3412	3421
4123	4132	4213	4231	4312	4321

Again, this listing could provide the basis for a solution. The flags may also be represented in a tree diagram, where each branch represents a choice of colour/shade:



The branching in the tree diagram in turn supports thinking in terms of multiplication, and hence we may conclude that the problem can be solved algebraically by calculating $4 \cdot 3 \cdot 2 \cdot 1 = 24$.

This problem may stimulate generalization with respect to number of sections and number of shades and also stimulates problem posing in other settings.

3) Multiple representations support and strengthen conceptual understanding
As Duval (2006) states, *a representation is something that stands for something else*. For example, the abstract concept of "five" may be represented in many different ways; as a symbol 5, five objects, five dots, position five on a number line and so on. A reasonable standpoint, in line with the constructivist tradition in the Swedish school system, is that none of these representations constitutes the concept "five". Rather they all contribute – in different ways – to our understanding of what "five" may be. This view on what it means to understand a concept supports a developmental approach to education, which may be more favourable regarding students with special needs than a positivist attitude according to which these students may simply be classified as those who "do not understand" (at all).

4) Different representations put more or less strain on working memory
The computational scheme to the left is the most common in Swedish schools. This scheme is efficient for those who already know how to use it but some of its characteristics may cause unnecessary problems in the learning process. The scheme to the right requires more writing but is more transparent (both visually and mathematically) and puts less strain on the individual learner's working memory (Baddeley, 2000). Especially for students with special needs, the scheme may be supported by use of an addition table as compensatory aid.

$$\begin{array}{r} {\scriptstyle 1\,1} \\ 237 \\ +\ 594 \\ \hline 831 \end{array} \qquad\qquad \begin{array}{r} 237 \\ +\ 594 \\ \hline 700 \\ 120 \\ +\ \ 11 \\ \hline 831 \end{array}$$

Conclusions

Sometimes numerical and/or geometrical representations of mathematical tasks may be sufficient to stimulate students' creative mathematical thinking. Teachers in mathematics need to develop understanding of the educational value of using and translating between different mathematical representations and not only using advanced representations. The teachers have to develop their own competencies regarding using and translating between representations during problem solving activities where solution strategies are chosen based on comparison and own judgment. Furthermore, reflecting on their own learning processes within familiar mathematical domains stimulates teachers' reflections regarding teaching strategies as well.

References

Ainsworth, S. (2006). DeFT: A conceptual framework for considering learning with multiple representations. *Learning and Instruction,* 16 (3), 183-198.

Baddeley, A. (2007). Working memory, Thought, and Action. Oxford University Press, Oxford.

Ball, D. L. (2000). Bridging practices: Intertwining content and pedagogy in teaching and learning how to teach. *Journal of Teacher Education,* 51 (3), 241-247.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, 61, 103-131.

Nilsson, P., Sollervall, H., Milrad, M. (2009). Collaborative design of mathematical activities for learning in an outdoor setting. *CERME 6 - Sixth Conference of European Research in Mathematics Education.* Lyon, France, Jan. 28th - Feb. 1st, 2009. Accepted for publication.

Niss, M., Jensen, T.H. (eds.): *Kompetencer og matematiklæring*. Uddannelsesstyrelsens temahæfteserie, nr. 18, 1-334. Undervisningsministeriet, 2002.

Skolverket (2003). *Lusten att lära – med fokus på matematik.* Stockholm: Fritzes.

Skolverket (2009). *Kursplan med kommentarer*. Stockholm: Fritzes.

Sollervall, H. (2007). *Tal och de fyra räknesätten.* Studentlitteratur, Lund.

Håkan Sollervall is a senior lecturer in Mathematics Education at Linnaeus University in Växjö, Sweden. Current research efforts put focus on scaffolding students' learning of mathematics by designing innovative learning activities involving digital media and outdoor settings that support cooperative and collaborative activities. These activities involve a variety of external representations, with the aim to provide unique and efficient learning opportunities for each individual student.

hakan.sollervall@lnu.se

A theoretical framework for understanding students with learning difficulties in mathematics

Erika Stadler
Linneaus University, Sweden

I will present a theoretical model for mathematical learning in a specific setting, taking the students' previous experiences into account. Learning difficulties and special needs in mathematics constitute a complex issue and to reduce the complexity, researchers and educators tend to pre-define the character of the issue. I suggest another approach, which places students' perspectives as a starting point. By focusing on students as active learners, we can observe their mathematical learning objects and mathematical resources in order to identify learning difficulties and needs. This can serve as a point of departure for pedagogical development and for further research.

My contribution addresses the issue of how research output can be used in practice and for further research. I will present a theoretical model to describe students learning mathematics within a specific setting, taking their previous experiences into account (Stadler, 2009) and argue that this model can be used to describe and analyse the situation of students with special needs. Students with special needs constitute a complex issue in mathematics education (Miller & Mercer, 1997). Difficulties in learning mathematics can be ascribed both to societal and organisational factors. Another basis for explanation is the actual learning environment within classrooms, including teachers, peers and textbooks. Finally learning disabilities can be regarded as a psychological and medical issue, an explanation commonly referred to when individual students do not fit into the general educational system (Persson, 1998). The existence of learning disabilities should not be regarded as an epistemological question but the reasons for them should rather be considered for specific students who experience difficulties and problems with learning mathematics in a particular situation. Students have access to educational resources that aim to improve the situation, make mathematics more meaningful and help them pass exams. Instead of an observer's pre-definition of what learning disabilities involve, an alternative approach is to place the students at the centre. By examining their work in mathematics, we can adopt a student's perspective that accounts for the individual, situated and socio-cultural aspects of the issue.

The theoretical model for mathematics learning in a specific setting where the student's previous experience is taken into account has emerged from an empirically based research project about students' transition between upper secondary and tertiary level (Stadler, 2009). Many students experience considerable difficulties when they begin studying mathematics at university. To examine this, novice students were observed and interviewed during their first course in mathematics at university. Data was analysed with methods inspired by grounded theory (Charmaz, 2006), resulting in a theoretical model for describing the learning of mathematics from a student's perspective (Stadler, 2009). The research areas of learning disabilities and transitions are complex areas that both encompass individual, situated and socio-cultural aspects of learning mathematics. This theoretical model may therefore also be used for research concerning students' learning disabilities and special needs as regards learning mathematics.

The three categories in the theoretical model

The model consists of three main categories. The category *mathematical learning objects* applies to the main target of mathematics studies in a wider sense. This concept captures the very essence of what mathematics is and what should be learnt from a student's point of view. Specific mathematical content as well as meta-mathematical phenomena can provide mathematical learning objects for individual students in different learning situations. Fractions, percentages and equations are some examples of different possible mathematical learning objects, but the term can also refer to mathematical processes and the understanding of mathematics in a wider sense. Some examples of meta-mathematical learning objects are the use of physical manipulatives, interpretation of pictures, communication with the mathematics teacher and interpretation of an exercise answer and comparison to the answer in the book.

Mathematical resources are anything that students use to acquire a mathematical learning object. Textbooks, teachers, peers, mathematical pre-knowledge and logical thinking are some examples of entities that can constitute mathematical resources. The definition of mathematical resources is made on a relational basis and the main point is that these items constitute mathematical resources when students use them as such. Thus, the teacher is not a mathematical resource in itself, but when a student asks the teacher for help and explanation or attends a lecture, the teacher becomes an external interactive mathematical resource for that student. Another consequence of this definition is that what constitutes a mathematical resource for one student may not be a resource for another. Also, the same potential mathematical resource can be used in various ways by different students. Mathematical resources can be categorised according to their character. They can be internal or external, material or immaterial, interactively dynamic or static, personal or public. These attributes have an impact on how they are used as mathematical resources.

The actions and statements that students make in different situations for learning mathematics, and the intentions behind them, identify *Students' actions as learners*. This category frames mathematical learning objects and mathematical resources into one model because students' actions and statements are closely related with their use of mathematical resources and mathematical learning objects

There seems to be a dialectic relationship between mathematical learning objects and mathematical resources. Students use mathematical resources that they perceive as useful with respect to a specific mathematical learning object, but on the other hand, the availability of different potential mathematical resources determines which mathematical learning objects students focus on. In relation to different mathematical learning objects, students' actions as learners can be dependent or independent. When, for example, a student is unable to achieve a specific mathematical learning object because of lack of access to or inability to use necessary mathematical resources, the student becomes dependent as an active learner.

The model and students with learning disabilities in mathematics

The three category model originally emerged as an attempt to describe mathematics learning from a student's perspective in relation to transitions between different educational levels. However, because this issue is closely connected to students' experiences of learning difficulties, I suggest that these categories can also be used to describe and understand students with special needs and learning disabilities in mathematics. From the perspective of the individual student, the model can be used to account for the situation. Observations and descriptions of the students' actions as learners in terms of present mathematical learning objects and mathematical resources can constitute the base for suitable help and support to the student. By analysing desirable changes in terms of the student's mathematical learning objects and his or her use of mathematical resources, it supports the possibility of designing learning situations the student considers favourable. A more pragmatic approach is also possible by shaping a hypothetical desirable learning situation with focus on students as active learners. Desirable mathematical learning objects and meta-mathematical learning objects should be identified. Accordingly, potential mathematical resources should be mapped out.

The model can also be used as a methodological approach as regards students who are assumed to have learning disabilities. In accordance with the three categories, a student's situation can initially be analysed by observing his or her actions. This can be followed by interviewing the student about his or her view of mathematics and learning mathematics. Results that are presented in terms of students' actions as learners, mathematical learning objects and mathematical resources, provide a solid foundation for characterising students with learning disabilities in mathematics. For students who are regarded as having learning disabilities or special educational needs, this *model can be used to find out how these difficulties can be understood from the students' views of themselves and in their work with mathematics.* Mathematical learning objects and mathematical resources can be discerned in students' statements and actions, by observing students when they are working with mathematics. From this analysis, conclusions about students' actions as learners can be drawn and can be used as a basis for both developmental work and action programmes for individual students and for further research about students' learning difficulties and obstacles in mathematics.

From an individual perspective, it is important to reveal misconceptions as well as anxiety regarding mathematics as a subject. For example, do students focus relevant mathematical learning objects or do misconceptions lead them in a wrong direction? Are there students, whose main goal is to conceal their shortcomings in mathematics? These, among other things, can be discovered in an analysis of students' actions as learners with regard to mathematical learning objects and mathematical resources. A specific learning situation can also cause learning difficulties for individual students. This can become explicit for example in students' use of mathematical resources. Are students' uses of mathematical resources relevant to the mathematical learning object? Do students identify their learning objects according to their mathematical content or in relation to the learning situation? Answers to these questions can be used as clues to understanding learning disabilities and special needs, which can be associated with the learning situation and pedagogical issues pertaining to it.

A sample episode

In the following section I will present a tentative analysis of a sample episode. Data consists of an observation of Robin and his teacher. Robin is 18 years old and is undertaking his final semester on the individual program at upper secondary school in Sweden. This national program is specifically developed for pupils with particular educational needs. Robin's study-group in mathematics involves four male students and a female teacher. The teacher works individually with each student, giving "mini-lectures" about the mathematical content that each student is working with at a given time. For Robin, the goal is to pass the course Mathematics A. During the lesson observed, Robin is practicing fractions and percentages and the transformation between them. He uses computer software. Ten tasks are shown on the screen and Robin is meant to write the answer in the empty boxes after each task.

Robin: *What the hell should I do?*

Teacher: *Well, let's have a look...* [Looks at what Robin has written on the screen.]

Robin: *Is this the right answer?*

Teacher: *Yes... okay. Here you have written the number in decimal form, right?*

Robin: *Yes.*

Teacher: *So, how can you transform decimal form to fraction form?*

Robin: *You turn it the other way around?*

Teacher: *And what is… if we look at the last exercise here.*

Robin: *It is… If you move the number of zeros…*

Teacher: *Yes, and you have written the number in decimal form here. How can you re-write it in percentages? Zero points eighteen, how many percentages is that?*

Robin: *Zero point eighteen… it is there!* [Robin points at the screen.]

Teacher: *What is, if we think about… what does percentages mean?*

Robin: *Hundred!*

Teacher: *Parts of hundred, yes, exactly. So, if we have decimal form and want to transform it into percentages, then we multiply by hundred, right?*

Robin: *Yes.*

Teacher: *And if we take this one hundred times, what answer do we get?*

Robin: *I need a calculator!*

Teacher: *But did you understand how we reasoned?*

Robin: *Yes.*

Teacher: *Did you?*

Robin: *Fuck!*

Teacher: *Is it hard for you, Robin?*

Robin: *This gets on my nerves!*

The teacher wants to use pictures to explain the idea about percentages, fractions and decimal numbers, so she gets a sheet of paper and starts drawing.

Teacher: *Let's see what we got here…*

Robin: *What the hell are you going to do? Fuck!*

Teacher: *Calm down, I am just going to make things a little easier. Here we have ten. I*
[The teacher marks ten squares on the paper.]

Robin: *I hate these disgusting percentages!*

The teacher goes on, working with a few more exercises, focusing the general relation between decimal numbers, fractions and percentages. She encourages him to try some exercises on his own and then she leaves and moves on to help another student. But almost immediately Robin cries for help from the teacher and walks over to her.

Robin: *Fuck, I need help!*

Teacher: *I'll be right there, but go back to your seat while you're waiting*.

Robin moves back to his seat but starts talking again.

Robin: *I must leave soon*.

Teacher: *Why do you have to leave?*

Robin: *I must talk to my lawyer*.

Tom: *Don't lie! You suck!*

Robin: *I've been physically abused!*

Tom: *You were hit by a police, not abused!*

Teacher: *Are you kidding with me?*

Tom: *It's on YouTube. There you can watch him being hit.*

Teacher: *Oh, so you have video recorded it?!*

Tom: *No, but it was football at Ullevi*.

Teacher: *Nevertheless Robin, you know how it works! Have you applied for permission to leave earlier?*

Robin: *I know, but I can be here until the lesson is finished. I just have to hurry*.

Teacher: *All right. Good*.

Robin: *But do you want to see the video? I can show you the video!*

Teacher: *No thanks, I don't want to see it.*

Robin: *Come on, as a memento!*

Analysis of the sample episode

The dialogue between Robin and the teacher is in itself an illustrative example of some concrete problems that can occur in an actual learning situation. However, the main aim with my tentative analysis is not primarily to discover interesting things about Robin and his teacher per se, but rather to put the three categories into work and examine how they can be used for analysing the behaviour of students with learning difficulties in mathematics.

Students' mathematical learning object refers to the main target with their mathematics studies in a wider sense. Robin's opening line in the dialogue implies that he wants to use the teacher as a mathematical resource to find and define a mathematical learning object that is comprehensible for him. He knows the rules for the computer software, but he needs help to find a strategy to find the right answers. Instead of helping Robin to find an answer to each exercise, the teacher tries to explain the overall idea with fractions, decimal numbers and percentages, probably with the aim of making him an independent active learner. At the same time, it seems to be an insuperable mathematical learning object from Robin's perspective to understand the general relation between fractions, decimal numbers and percentages. His recurrent swearing and expressions of frustration can be interpreted as an indication for that. This implies a possible deadlock. From a student perspective, one common learning strategy is to focus on mathematical learning objects in terms of what to do to gain a functional understanding. This can later be transformed into a more comprehensive understanding of the mathematical content at hand. But Robin does not seem to recognise either learning what to do, nor understanding how it is, as possible mathematical learning objects. This implies that the mathematical learning objects that students with learning disabilities in mathematics focus on might not be an accessible way to begin to research and explore these students' situations and their learning difficulties.

As far as mathematical resources are concerned, Robin wants to use a calculator to work out zero point eighteen times one hundred. The calculator is a well-known mathematical resource and feels comfortable and secure to use. Instead the teacher offers him another potential mathematical resource, namely a drawn picture to illustrate the relation between hundredths and percentages. One possible explanation of Robin's vehement rejection of this aid as a mathematical resource is that from his perspective, the picture and the explanation are not a potential mathematical resource, but rather become a mathematical learning object in itself. This puts obvious demands on Robin to learn something new in a situation that already seems uncontrollable from his perspective.

For students with an explicit intention to learn mathematics, the teacher is an essential mathematical resource (Stadler, 2009). However, even if Robin is attending a mathematics lesson in the episode, learning mathematics does not seem to be his primary focus and Robin does not use the teacher primarily as a mathematical resource, even though Robin seems to have a huge need for attention from the teacher and contact with her. She provides explanations and actions that are primarily influenced by her analysis of his needs rather than his conclusions of what kind of help that is suitable for him. Thus, the teacher shapes the situation and directs actions towards a suitable and desirable mathematical content and provides potential mathematical resources that she deems appropriate. In this way, the teacher gets a key role to shape the students' actions as learners rather than being a mathematical resource that the student uses for acquiring a specific mathematical learning object.

During this lesson, Robin uses computer software to learn and practice fractions, decimal numbers and percentages. Various tools and aids can help to make mathematics more concrete and understandable for students with learning difficulties. However, in this case Robin does not need to learn how the computer program in itself works. Instead, it is the mathematical content that is a genuine challenge for him. Robin does not use the computer software as a mathematical resource, because the programme does not offer any help or is not used as a tool to facilitate his understanding of the mathematical content at hand. In this case the computer software is just a provider of tasks and plays no virtual role for Robin and his learning of mathematics.

The last category in my theoretical model is students' actions as learners, which captures students' actions, intentions and conceptions in relation to leaning mathematics. Students use different mathematical resources to acquire different mathematical learning objects, and different mathematical learning objects require different mathematical resources. A significant, but maybe not surprising, feature of the dialogue between Robin and his teacher is the rare occurrence of actions that involve mathematical learning objects and mathematical resources. In their conversation, Robin gives priority to maintaining attention and thereby a relation to the teacher, but seems to lack the ability to establish this relation with focus on mathematical issues as a starting point. Instead, he picks subjects from outside school in order to have a conversation with the teacher about topics that are both more interesting and familiar to Robin than mathematics is.

To summarise the analysis of Robin and his work with fractions, decimal numbers and percentages, his learning difficulties in mathematics can be described in terms of difficulties to focus on an appropriate mathematical learning object and to use suitable mathematical resources. For an ordinary student, a mathematical learning object is usually already at hand and the student needs help to accomplish it or the student requests support to use available mathematical resources to attain a specific mathematical learning object. For students with learning disabilities in mathematics, the situation is slightly different. From the episode above, it is clear that the essence of Robin's difficulties seems to lie in his inability to formulate and focus on an appropriate mathematical learning object and to use suitable mathematical resources. This becomes visible in his actions as a learner and students' learning disabilities could be explained or described in terms of the difficulties in focusing or working towards a relevant mathematical learning object and use suitable mathematical resources.

Summary and conclusions

In this paper, I have presented a theoretical model for analysing students' learning of mathematics. I have argued that this framework also can be used for students with learning difficulties and special needs in mathematics. I have conducted a tentative analysis of a student called Robin, where the model has been used as a methodological approach for researching a student with learning disabilities. My conclusion is that the model can be used for this purpose. However, in order to use the three categories for analysing students with learning disabilities in mathematics, the focus needs to move from mathematical learning objects and the use of mathematical resources towards a stronger emphasis on the students' actions as learners. Instead of defining specific mathematical learning objects, the students need to be steered to the right area of finding a mathematical learning object by explicit guidance for how students should act as learners. In this case, the teacher as a potential mathematical resource has a key role because he or she must be able to clearly define mathematical learning objects and mathematical resources in terms of what a student as a learner should do. It seems that learning disabilities and difficulties can be viewed in terms of the category of the students' actions as learners so the focus for analysing observations and episodes should be directed exactly towards these actions. Therefore, when dealing with students with learning disabilities in mathematics, an initial effort to look for mathematical learning objects and mathematical resources might be impracticable. On the other hand, if students' actions as learners are studied, suitable mathematical learning objects and appropriate mathematical resources may be identified. This corresponds to using the model to design favourable learning situations for students. My conclusion is that the three category model can be used both in practice, for example for establishing action programmes, but also for further research about the causes of and consequences for students with learning disabilities and special needs in mathematics education.

References

Charmaz, K. (2006). Constructing grounded theory – A Practical guide through qualitative analysis. London: SAGE Publications.

Miller, S. P. & Mercer, C. D. (1997). Educational aspects of mathematics disabilities. Journal of Learning Disabilities, 30(1), 47-56.

Persson, B. (1998). Den motsägelsefulla specialpedagogiken. Motiveringar, genomförande och konsekvenser. Specialpedagogiska rapporter, nr 11. Göteborg: Göteborgs universitet, Institutionen för specialpedagogik.

Stadler, E. (2009). Stadieövergången mellan gymnasiet och universitetet: matematik och lärande ur ett studerandeperspektiv. Doktorsavhandling. Växjö: Acta Wexion

erika.stadler@lnu.se