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An algorithm to construct Lyapunov functions for nonlinear switched systems

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Abstract: In this thesis a converse theorem on arbitrary switched nonlinear, nonautonomous, continuous systems is proved. Then an algorithm to construct Lyapunov functions for such systems is presented and it is proved, that the algorithm always succeeds in generating a Lyapunov function if one exists. These results together imply, that the algorithm always succeeds if the equilibrium is uniformly asymptotically stable. The size of the domain of the Lyapunov function generated is only limited by the size of the equilibrium's region of attraction. Note, that the systems $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$, where $\mathbf{f}$ is not necessarily linear, fall under the class of arbitrary switched systems.

Keywords: Lyapunov function; Switched system; Uniform asymptotic stability; Converse theorem; Algorithm.

(Útdráttur: næsta síða)

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Algrím til smíði Lyapunov-falla fyrir ólínuleg skiptikerfi

Sigurður Freyr Hafstein

Tækni- og verkfræðdeild – Tækni- og verkfræðideild
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Útdráttur: Í þessari ritgerð er fyrst sannað, að aðfellustöðugleiki jafnvægispunkts ólínulegs skiptikerfis er jafngildur tilvist Lyapunov falls fyrir kerfið. Síðan er algrím til smíði slíkra Lyapunov falla sett fram og sannað, að algríminu tekst ávallt að stika Lyapunov fall fyrir kerfið ef slíkt fall er til. Saman þýðir þetta að algríminu tekst alltaf að finna Lyapunov fall fyrir kerfið ef jafnvægispunktur þess er aðfellustöðugur. Stærð skylgreiningarmengis Lyapunov fallsis er einungis takmörkuð af stærð aðdráttarmengis jafnvægispunktisins. Athuga ber, að $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ og $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$, þar sem $\mathbf{f}$ er ekki nauðsynlega línulegt fall, falla undir flokk skiptikerfa.

Lykilorð: Lyapunov fall; Skiptikerfi; Aðfellustöðugleiki; Andhverfusetning; Algrím.

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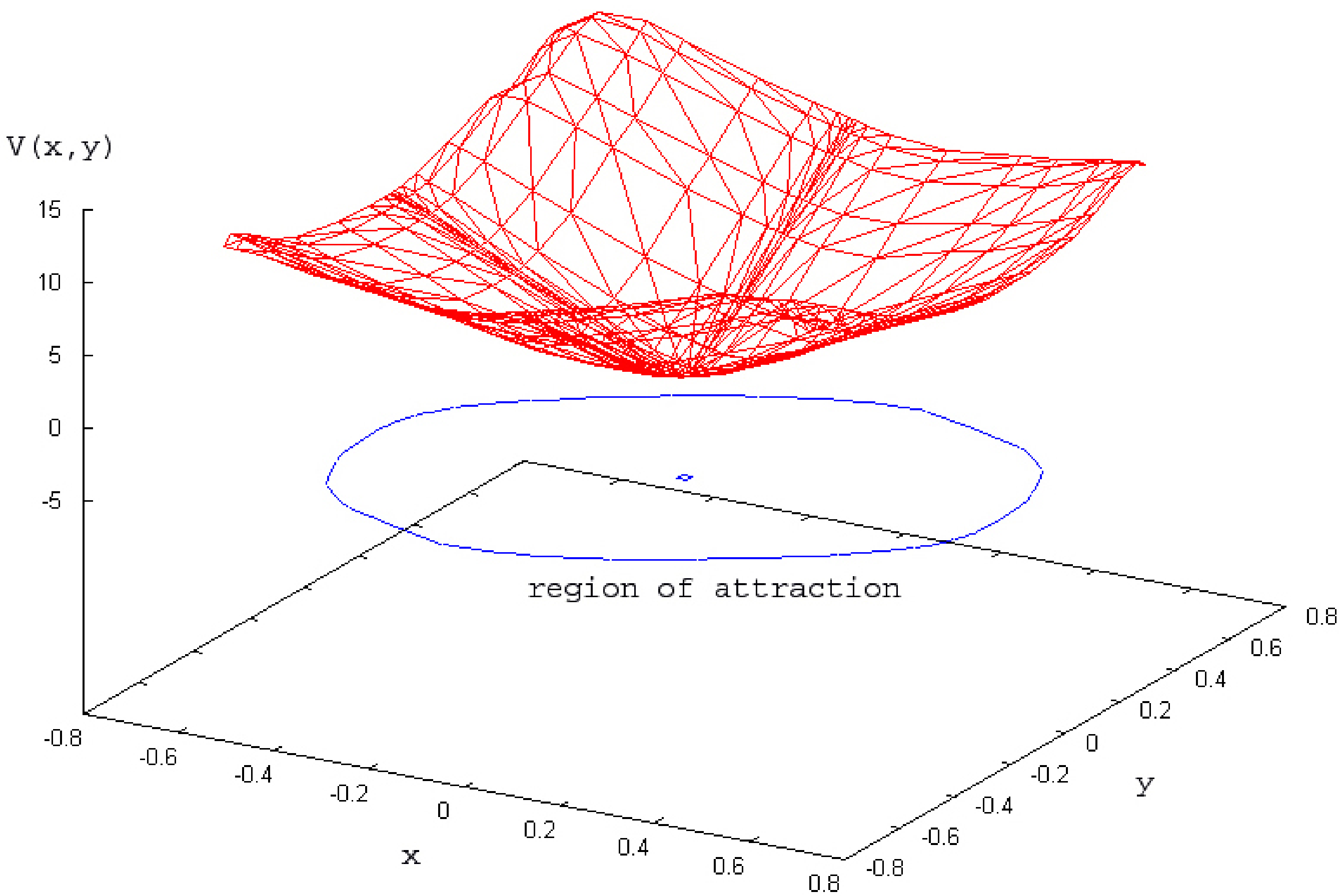
An algorithm to construct Lyapunov functions for nonlinear switched systems



$$\dot{\mathbf{x}} = \mathbf{f}_p(\mathbf{x}), \quad p \in \{1,2,3\}, \qquad \mathbf{f}_1(x,y) := \begin{pmatrix} -y \\ x - y(1 - x^2 + 0.1x^4) \end{pmatrix},$$

$$\mathbf{f}_2(x,y) := \begin{pmatrix} -y + x(x^2+y^2-1) \\ x + y(x^2+y^2-1) \end{pmatrix}, \quad \mathbf{f}_3(x,y) := \begin{pmatrix} -1.5y \\ \frac{x}{1.5} + y\left(\left(\frac{x}{1.5}\right)^2 + y^2 - 1\right) \end{pmatrix}$$

Sigurdur Freyr Hafstein

Contents

Symbols

$\mathbb{R}$	the real-numbers
$\mathbb{R}_{\geq 0}$	the real-numbers larger than or equal to zero
$\mathbb{R}_{>0}$	the real-numbers larger than zero
$\mathbb{Z}$	the integers
$\mathbb{Z}_{\geq 0}$	the integers larger than or equal to zero
$\mathbb{Z}_{>0}$	the integers larger than zero
$\mathcal{A}^n$	set of n-tuples of elements belonging to a set $\mathcal{A}$
$\mathbb{R}^n$	the n-dimensional Euclidean space, $n \in \mathbb{N}_{>0}$
$\overline{\mathcal{A}}$	the topological closure of a set $\mathcal{A} \subset \mathbb{R}^n$
$\overline{\mathbb{R}}$	$:= \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$
$\partial\mathcal{A}$	the boundary of a set $\mathcal{A}$
$\mathrm{dom}(f)$	the domain of a function f
$f(\mathcal{U})$	the image of a set $\mathcal{U}$ under a mapping f
$f^{-1}(\mathcal{U})$	the preimage of a set $\mathcal{U}$ with respect to a mapping f
$\mathcal{C}(\mathcal{U})$	continuous real-valued functions with domain $\mathcal{U}$
$\mathcal{C}^k(\mathcal{U})$	k-times continuously differentiable real-valued functions with domain $\mathcal{U}$
$[\mathcal{C}^k(\mathcal{U})]^n$	vector fields $\mathbf{f} = (f_1, f_2, \dots, f_n)$ of which $f_i \in \mathcal{C}^k(\mathcal{U})$ for $i = 1, 2, \dots, n$
$\mathcal{K}$	strictly monotonically increasing functions on $[0, +\infty[$ vanishing at the origin
$\mathcal{L}$	strictly monotonically decreasing functions on $[0, +\infty[$ approaching zero at infinity
$\mathfrak{P}(\mathcal{A})$	the power set of a set $\mathcal{A}$
$\mathrm{Perm}[\mathcal{A}]$	the permutation group of $\mathcal{A}$, i.e., the set of all bijective functions $\mathcal{A} \longrightarrow \mathcal{A}$
$\mathrm{con}\,\mathcal{A}$	the convex hull of a set $\mathcal{A}$
$\mathrm{graph}(f)$	the graph of a function f
$\mathbf{e}_i$	the i-th unit vector
$\mathbf{x} \cdot \mathbf{y}$	the inner product of vectors $\mathbf{x}$ and $\mathbf{y}$
$\\|\mathbf{x}\\|_p$	p-norm of a vector $\mathbf{x}$, $\\|\mathbf{x}\\|_p := \left(\sum_i \lvert x_i\rvert^p\right)^{\frac{1}{p}}$ if $1 \leq p < +\infty$ and $\\|\mathbf{x}\\|_\infty := \max_i \lvert x_i\rvert$
f'	the derivative of a function f
$\dot{\mathbf{x}}$	the time-derivative of a vector-valued function $\mathbf{x}$
∇f	the gradient of a scalar field $f : \mathbb{R}^n \longrightarrow \mathbb{R}$
$\nabla \mathbf{f}$	the Jacobian of a vector field $\mathbf{f} : \mathbb{R}^m \longrightarrow \mathbb{R}^n$
$\chi_{\mathcal{A}}$	the characteristic function of a set $\mathcal{A}$
δ_{ij}	the Kronecker delta, equal to 1 if $i = j$ and equal to 0 if $i \neq j$
$[a, b]$	$:= \{x \in \mathbb{R} \mid a \leq x \leq b\}$
$]a, b]$	$:= \{x \in \mathbb{R} \mid a < x \leq b\}$
$[a, b[$	$:= \{x \in \mathbb{R} \mid a \leq x < b\}$
$]a, b[$	$:= \{x \in \mathbb{R} \mid a < x < b\}$
$\mathrm{supp}(f)$	$:= \overline{\{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) \neq 0\}}$
$\mathcal{A}\Delta\mathcal{B}$	$:= (\mathcal{A} \cup \mathcal{B}) \setminus (\mathcal{A} \cap \mathcal{B})$
$\mathbf{R}^{\mathcal{J}}$	reflection function with respect to the set $\mathcal{J} \subset \{1, 2, \dots, n\}$, see Definition 4.9
$\mathbf{PS}$	piecewise scaling function, see Definition 4.12
$\mathfrak{S}[\mathbf{PS}, \mathcal{N}]$	a simplicial partition of a the set $\mathbf{PS}(\mathcal{N}) \subset \mathbb{R}^n$, see Definition 4.13
$f_{p,i}$	the i-th component of the vector field $\mathbf{f}_p$
$\mathcal{B}_{\\|\cdot\\|, R}$	$:= \{\mathbf{x} \in \mathbb{R}^n \mid \\|\mathbf{x}\\| < R\}$

To Valborg, Maria, and Frederick

Historical background

[The universe] cannot be be read until we have learnt the language and become familiar with the characters in which it is written. It is written in the mathematical language, and the letters are triangles, circles and other geometrical figures, without which means it is humanly impossible to comprehend a single word.

Galileo Galilei (1564 - 1642)

Mathematics is not only real, but it is the only reality. That is that entire universe is made of matter, obviously. And matter is made of particles. It's made of electrons and neutrons and protons. So the entire universe is made out of particles. Now what are the particles made out of? They're not made out of anything. The only thing you can say about the reality of an electron is to cite its mathematical properties. So there's a sense in which matter has completely dissolved and what is left is just a mathematical structure.

Martin Gardner (born 1914)

In classical mechanics every state of a physical system is associated with a real-number known as its *energy*. Energy of a system is an extremely powerful concept, just like momentum and angular momentum, because it obeys a conservation law. However, this was not discovered until at the end of the 17th century trough Sir Isaac Newton's (1642 - 1727) monumental work *Philosophiae naturalis principia mathematica* and to some extent due to Galileo Galilei's (1564 - 1642) and Renés Descartes' (1596 - 1650) earlier work. Before this time it was assumed, in accordance with Aristotelian (384 BC - 322 BC) physics, that a physical system without external forces would come to rest. The late insight in the true nature of the energy is not particularly surprising because every-day experience indicates that terrestrial objects actually do eventually come to rest. A century later this discrepancy was satisfactory resolved by Rudolf Clausius (1822 - 1888), who adopted the concept of entropy. In physical processes energy does not disappear, but as the entropy of the system inevitable increases, the quality of the energy decreases. For example, it is a simple task to transform mechanical work into heat with 100% efficiency, however, gaining work from heat is much more difficult, a temperature difference is needed, and the efficiency has clear theoretical bounds. Thus, the useful energy (for example kinetic energy, potential energy, or electrical energy) of a physical system, that is, the energy assessable by the system to move macroscopic objects or to do some useful work, actually does decrease, whenever the state of the system changes. If we identify physical systems states with points in their state-spaces, this means that the useful energy of a particular system decreases along every trajectory of it in its state-space. This implies that such a system must come to rest at a local minimum of the useful energy and this fact can be very useful for the analysis of physical systems because it is in general a formidable task to integrate the equations of motion.

In the 19th century Sir William Hamilton (1805 - 1865) developed a new mathematical framework for mechanical systems. In the so-called Hamiltonian dynamics the equations of motion are first-order ordinary differential equations $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$, which are often better suited for mathematical analysis than previous attempts by Joseph Lagrange (1736 - 1813) and others. If a physical system $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$ is at rest at the state-space point $\mathbf{y}$ the dynamical variables do not change in the course of time, which obviously implies that $\mathbf{f}(t, \mathbf{y}) = \mathbf{0}$ for all $t \geq 0$. Such a state-space point is called an equilibrium of the differential equation $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$. Thus, a necessary condition for a state $\mathbf{y}$ in the state-space of the system $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$ to asymptotically attract trajectories of a physical system is that it is an equilibrium and that the useful energy of the system has a local minimum at $\mathbf{y}$. Further, one can conclude that an equilibrium is stable, in the sense that small perturbations are ironed out by the dynamics of the system, if and only if the equilibrium is at a local minimum of the useful energy of the system.

In 1892 the Russian mathematician and engineer Alexandr Mikhailovich Lyapunov (1857 - 1918, also found in the literature as Liapunov, Liapunoff, Lyapunoff, Ljapunov, and Ljapunoff) published a revolutionary work on the stability of motion. For an English translation of his work we refer to [32]. The ingenuity of his work was to generalize the concept of energy to arbitrary differential equations of the form $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$. Thus, if $\mathbf{y}$ is an equilibrium of the system $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$ and, additionally, there exists a continuous function V that has a local minimum at $\mathbf{y}$ and that is strictly monotonically decreasing along any trajectory of the system, then every trajectory of the system that starts close enough to the equilibrium is asymptotically attracted to it [1]. The function V is then said to be a Lyapunov function for the system. At first glance it might seem that the Lyapunov theory is purely of theoretical interest, because one seemingly still needs to know the trajectories of the system $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$ to verify that the Lyapunov function is decreasing along the trajectories. It is not! The reason is that by the use of the chain rule one can easily verify that if the Lyapunov function V and the function $\mathbf{f}$ are differentiable, then V is decreasing along every trajectory of the system if

$$[\nabla_{\mathbf{x}} V](t, \mathbf{x}) \cdot \mathbf{f}(t, \mathbf{x}) + \frac{\partial V}{\partial t}(t, \mathbf{x}) < 0$$

for all $\mathbf{x} \neq \mathbf{y}$ in a neighborhood of $\mathbf{y}$, and this can be verified without solving the differential equation.

A Lyapunov function is traditionally denoted by the same alphabet as the potential energy of a physical system, namely V, which is somewhat confusing because a Lyapunov function is a generalization of the energy, traditionally denoted by E or H for Hamiltonian. Maybe this inconsistence is due to the fact, proved by Dirichlet in 1848, that an equilibrium $\mathbf{y}$ of a conservative, holonomic-skleronomic [2] mechanical system is stable, if and only if the potential energy V of the system, traditionally denoted by V or U, has a local minimum at $\mathbf{y}$. However, it has become widely accepted to denote a Lyapunov function by the alphabet V and we will follow that tradition in most cases.

A Lyapunov function V for a differential equation $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$ not only guaranties the stability, as a local property, of an equilibrium $\mathbf{y}$, but we can also take advantage of the fact that the system, initially in a state $(t_0, \boldsymbol{\xi}_0)$, can never reach a state $(t_1, \boldsymbol{\xi}_1)$, in which $V(t_1, \boldsymbol{\xi}_1) > V(t_0, \boldsymbol{\xi}_0)$. Thus, the state of the system is bound to stay in the set $\{(t, \boldsymbol{\xi}) \big| V(t, \boldsymbol{\xi}) \leq V(t_0, \boldsymbol{\xi}_0)\}$ and this can be used to derive a lower bound on the region (domain, basin) of attraction of the equilibrium $\mathbf{y}$, that is, the set of those $\boldsymbol{\xi}$ that are attracted to $\mathbf{y}$ in the course of time by the dynamics of the system.

The original Lyapunov theory did not secure the existence of Lyapunov functions for nonlinear systems with uniformly asymptotically stable equilibrium points, the arguably most important stability concept. The first results on this subject are due to K. Perdeskii in 1933 [14]. The general case was resolved somewhat later, mainly by J. Massera in 1949 and 1956 [38, 39] and I. Malkin in 1977 [33]. Theorems, which secure the existence of a Lyapunov or a Lyapunov-like function for a system possessing an equilibrium, stable in some sense, are called converse theorems in the theory

[1] Those familiar with the Lyapunov theory might note that we are oversimplifying. However, we do this so we can concentrate on the essentials of the theory in this historical discussion.

[2] Holonomic-skleronomic means that the dynamical variables $\mathbf{x}$ obey constraints of the form $g(\mathbf{x}) = 0$ for some smooth function g.

of dynamical systems. The converse theorems have in the past been proved by constructing by a finite or a transfinite procedure a Lyapunov(-like) function using the trajectories of the respective equations of motion. Because the trajectories are, of course, not known, these converse theorems are pure existence theorems and do not help in the search for a Lyapunov function. This has been the main drawback of the Lyapunov theory so far and is probably the main reason why it has proved so difficult to extend the practical use of Lyapunov functions to systems, that do not feature a canonical energy function, like systems in economics, biology, or basically any non-physical system. The aim of this thesis is to present methods to automatically generate Lyapunov functions with computers for arbitrary ordinary differential equations.

The contributions of this work

It is a profoundly erroneous truism, repeated by all copy books and by eminent people when they are making speeches, that we should cultivate the habit of thinking of what we are doing. The precise opposite is the case. Civilization advances by extending the number of important operations which we can perform without thinking about them.

Alfred Whitehead (1861 - 1947)

As the title of this work suggests, we will establish an algorithm that is able to generate Lyapunov functions for switched systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}_p(t, \mathbf{x}), \quad p \in \mathcal{P}, \quad \mathcal{P} \neq \emptyset \text{ finite},$$

under arbitrary switching. Further, we will prove that the algorithm is always able to construct a Lyapunov function for a particular system, whenever the system possesses a Lyapunov function at all, and we will prove that this is exactly the case, when the system has an uniformly asymptotically stable equilibrium. Without loss of generality, we can assume that the equilibrium in question is at the origin.

The author (Hafstein / Marinósson) has recently used non-constructive converse Lyapunov theorems to prove that his linear programming problem, presented 2002 in [35, 36], is capable of parameterizing Lyapunov functions for autonomous systems $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with an exponentially stable equilibrium [11], and more generally, with asymptotically stable equilibrium [10, 13]. These results, however, will turn out to be special cases of a much more general theorem, namely Theorem 6.1 in this work. In this novel theorem the assumption that the system is autonomous is no more needed and further we even prove that we can always generate numerically a common Lyapunov function for a finite set of ordinary differential equations $\dot{\mathbf{x}} = \mathbf{f}_p(t, \mathbf{x})$, $p \in \mathcal{P}$, $\mathcal{P} \neq \emptyset$ finite, whenever such a Lyapunov function exists. Further, in Theorem 3.10, we prove that if an arbitrary switched system possesses a uniformly asymptotically stable equilibrium, then there exists a common Lyapunov function for the systems. Thus, because the existence of a common Lyapunov function is a sufficient condition for the uniform asymptotic stability of an equilibrium, we have proved that an arbitrary switched nonlinear, nonautonomous system possesses a uniformly asymptotically stable equilibrium, exactly then when there exist a common Lyapunov functions for the systems $\dot{\mathbf{x}} = \mathbf{f}_p(t, \mathbf{x})$, $p \in \mathcal{P}$, and we can construct such a Lyapunov function numerically whenever the set $\mathcal{P} \neq \emptyset$ is finite and the functions $\mathbf{f}_p$, $p \in \mathcal{P}$ are two-times continuously differentiable.

Switched systems will be defined and discussed in Section 1.2. Here we let it suffice to point out, that a function V is a Lyapunov function for the switched system, if and only if it is, for each $p \in \mathcal{P}$, a Lyapunov function for the system $\dot{\mathbf{x}} = \mathbf{f}_p(t, \mathbf{x})$. Especially, if $\mathcal{P}$ contains exactly one element p^*, then V is a Lyapunov function for the ordinary differential equation $\dot{\mathbf{x}} = \mathbf{f}_{p^*}(t, \mathbf{x})$. The concept of a Lyapunov function for a switched system under arbitrary switching is thus a non-trivial extension of the concept of a Lyapunov function for an ordinary differential equation.

The functions $\mathbf{f}_p$ are assumed to be mappings from $\mathbb{R}_{\geq 0} \times \mathcal{U}$ into $\mathbb{R}^n$, where $\mathcal{U}$ is a domain in $\mathbb{R}^n$ that contains the origin. The components $f_{p,i}$ of the functions $\mathbf{f}_p$ are assumed to be two-times continuously differentiable and we assume that we can give local upper bounds on their derivatives up to the second order. These bounds do not need to be tight (every upper bound will do the job) so these assumptions must be considered very weak. No further assumptions about the systems $\dot{\mathbf{x}} = \mathbf{f}_p(t, \mathbf{x})$ are needed, in particular, we do not assume that the functions $\mathbf{f}_p$ are of some specific structural form, like linear, piecewise affine, rational, algebraic, real holomorphic, etc.

We will present a linear programming problem (Definition 5.1) and give an algorithmic scheme of how to construct it from the original data, that is, the functions $\mathbf{f}_p$. We will prove that if the linear programming problem possesses a feasible solution, then this solution parameterizes a Lyapunov function $V \in \mathcal{C}([T', T''] \times \mathcal{V})$ for the switched system (Theorem 5.2). The constants $0 \leq T' < T'' < +\infty$ and the compact set $\mathcal{V} \subset \mathcal{U}$ can be chosen at will.

We will prove that if the switched system $\dot{\mathbf{x}} = \mathbf{f}_p(t, \mathbf{x})$ possesses a two-times continuously differentiable Lyapunov function $W : \mathbb{R}_{\geq 0} \times \mathcal{W} \longrightarrow \mathbb{R}$, where $\{\mathbf{0}\} \subset \mathcal{W} \subset \mathcal{U}$, then, for every $0 \leq T' < T'' < +\infty$ and any compact set $\mathcal{V} \subset \mathcal{W} \setminus \{\mathbf{0}\}$, we can use the linear programming problem to generate a Lyapunov function $V \in \mathcal{C}([T', T''] \times \mathcal{V})$ for the system. The condition that $\mathcal{V} \subset \mathcal{W} \setminus \{\mathbf{0}\}$ is necessary, because if the Lyapunov function V is defined in a neighborhood of the origin, then the equilibrium at the origin is uniformly exponentially stable. Because the existence of W merely secures uniform asymptotic stability of the origin, which is a weaker property than exponential stability, this would lead to a contradiction. Therefore, we have to cut out an open neighborhood of the origin from the domain of the Lyapunov function V. However, this neighborhood can be as small as one wishes, so this cannot be considered to be a major drawback.

Those, who merely have a rudimentary knowledge of the Lyapunov stability theory, might think that the construction of Lyapunov functions was in essence a solved problem. This is entirely wrong! The only case that can be considered to have been completely solved is for autonomous linear systems, that is, systems of the form $\dot{\mathbf{x}} = A\mathbf{x}$. By linearizing exponentially stable nonlinear systems at the equilibrium at the origin, one can actually construct a quadratic function that is a Lyapunov function for the nonlinear system in some neighborhood of the origin. However, this neighborhood is in general small so the only implication such a local Lyapunov function secures, is that there are perturbations that are so small that they cannot affect the asymptotic behavior of the system. The Lyapunov functions we will be discussing in this work are not just defined in some arbitrary small neighborhood of the equilibrium, but are defined on a much larger set, whose size is only limited by the size of the equilibriums region of attraction. Our Lyapunov functions can therefore give us a reasonable estimate on the size of the equilibriums region of attraction and then, of course, the magnitude of the disturbances the system can withstand. The fact that we even prove our results for nonautonomous switched systems can be considered a substantial added bonus.

Practically everyone will assert that her / his work is of greatest importance. Because of this it seems quite pointless to the author to expatiate upon the value of his own work. Quoting others is more trustworthy and because there is a passage in almost every book on differential equations, dynamical systems, or stability, that confirms the relevance of the results presented in this work, we leave the affirmation to others.

J. Willems writes on page 38 in his book *Stability Theory of Dynamical Systems* [61]:

> The main drawback of Liapunov's method for the study of stability of dynamic systems is that there exists no systematic procedure for constructing Liapunov functions.

M. Vidyasagar writes some concluding remarks on the Lyapunov theory on page 190 in his book *Nonlinear System Analysis* [56]:

> In this section, several theorems in Lyapunov stability theory have been presented. The favorable aspects of these theorems are:

1. They enable one to draw conclusions about the stability status of an equilibrium *without solving the system equations.*
2. Especially in the theorems on stability and asymptotic stability, the Lyapunov function V has an intuitive appeal as the total energy of the system.

The unfavorable aspects of these theorems are:

1. They represent only sufficient conditions for the various forms of stability. Thus, if a particular Lyapunov function candidate fails to satisfy the hypothesis on $\dot{V}$ [3], then no conclusions can be drawn, and one has to begin anew with another Lyapunov function candidate.
2. In a general system of nonlinear equations, which do not have the structure of Hamiltonian equations of motion or some other such structure, there is no *systematic* procedure for generating Lyapunov function candidates.

and further he writes on pages 235-236 on the converse theorems:

> Since the function V is constructed in terms of the solution trajectories of the system, the converse theorems cannot really be used to construct an explicit formula for the Lyapunov function, except in special cases (e.g., linear systems; see Section 5.4).

H. Khalil writes on page 104 in his book *Nonlinear systems* [24]:

> Lyapunov's theorem can be applied without solving the differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. On the other hand, there is no systematic method for finding Lyapunov functions. In some cases, there are natural Lyapunov function candidates, like energy functions in electrical or mechanical systems. In other cases, it is basically a matter of trial and error.

and on page 180 he writes on the converse theorems:

> Most of these converse theorems are provided by actually constructing auxiliary functions that satisfy the conditions of the respective theorems. Unfortunately, almost always this construction assumes the knowledge of the solutions of the differential equation. Therefore, these theorems do not help in the practical search for an auxiliary function. The mere knowledge that a function exists is, however, better than nothing. At least we know that our search is not hopeless.

J. Slotline and W. Li write on page 120 in their book *Applied Nonlinear Control* [54]:

> A number of interesting results concerning the existence of Lyapunov functions, called *converse Lyapunov theorems*, have been obtained in this regard. For many years, these theorems were thought to be of no practical value because, like the previously described theorems, they do not tell us how to generate Lyapunov functions for a system to be analyzed.

W. Walter writes on page 320 in his book *Ordinary Differential Equations* [59]:

[3] That the Lyapunov function V is decreasing along system trajectories.

> There is no general recipe for constructing Lyapunov functions. In specific cases one may relay on experience and examples; some imagination is also helpful.

and on page 325 he writes:

> Determining, or at least estimating, the domain of attraction is a problem of great practical importance.

M. Hirsch, S. Smale, and R. Devaney write on page 195 in their book *Differential Equations, Dynamical Systems, and an Introduction to Chaos* [15]:

> Note that Liapunov's theorem can be applied without solving the differential equation; all we need to compute is $DL_X(F(X))$ [4]. This is a real plus! On the other hand, there is no cut-and-dried method of finding Liapunov functions; it is usually a matter of ingenuity or trial and error in each case.

The author hopes to have convinced the potential readers of this work, that studying the theory that is laid down in this thesis is worth the efforts. It should further be pointed out, that it is of course possible to use the algorithm to construct Lyapunov functions without understanding in detail how it works. Those that are not interested in the mathematics, but still want to use the results to construct Lyapunov functions, for example for engineering applications, can move directly to Chapter 5.

[4] Here $DL_X(F(X))$ denotes the derivative of the Lyapunov function L along trajectories of the ordinary differential equation $\dot{X} = F(X)$.

Outline of this thesis

Science is built up with facts, as a house is with stones. But a collection of facts is no more a science than a heap of stones is a house.

Jules Pointcaré (1854 - 1912)

This work is divided into three parts. In Part I, which consists of the chapters 1, 2, and 3, we develop a stability theory for arbitrary switched systems. In Chapter 1 we introduce switched dynamical systems $\dot{\mathbf{x}} = \mathbf{f}_p(t, \mathbf{x})$, $p \in \mathcal{P}$, and discuss some elementary properties of their solutions. Switched systems have gained much interest recently. For an overview see, for example, [29] and [27].

In Chapter 2 we consider the stability of isolated equilibria of arbitrary switched systems and we prove that if such a system possesses a Lyapunov function, then the equilibrium is uniformly asymptotically stable. These results are quite straightforward if one is familiar with the Lyapunov stability theory for ordinary differential equations, but, because we consider Lyapunov functions that are merely continuous and not necessarily differentiable in this work, we establish the most important results.

In Chapter 3 we prove a converse theorem on uniform asymptotic stability for arbitrary switched nonlinear, nonautonomous, continuous systems. In the literature there are numerous results regarding the existence of Lyapunov functions for switched systems. A short non-exhaustive overview follows: In [44] K. Narendra and J. Balakrishnan consider the problem of common quadratic Lyapunov functions for a set of autonomous linear systems, in [9], in [42], in [28], and in [2] the results were considerably improved by I. Gurvits; Y. Mori, T. Mori, and Y. Kuroe; D. Liberzon, J. Hespanha, and A. Morse; and A. Agrachev and D. Liberzon respectively. H. Shim, D. Noh, and J. Seo in [53] and L. Vu and D. Liberzon in [57] generalized the approach to commuting autonomous nonlinear systems. The resulting Lyapunov function is not necessarily quadratic. W. Dayawansa and C. Martin proved in [37] that a set of linear autonomous systems possesses a common Lyapunov function, whenever the corresponding arbitrary switched system is asymptotically stable, and they proved that even in this simple case there might not exist any quadratic Lyapunov function. The same authors generalized their approach to exponentially stable nonlinear, autonomous systems in [5]. J. Mancilla-Aguilar and R. García used results from Y. Lin, E. Sontag, and Y. Wang in [31] to prove a converse Lyapunov theorem on asymptotically stable nonlinear, autonomous switched systems in [34].

In this work we prove a converse Lyapunov theorem for uniformly asymptotically stable nonlinear switched systems and we allow the systems to depend explicitly on the time t, that is, we work the nonautonomous case out. We proceed as follows: If the functions $\mathbf{f}_p$, of the switched system $\dot{\mathbf{x}} = \mathbf{f}_p(t, \mathbf{x})$, $p \in \mathcal{P}$, are locally Lipschitz in the state-space argument with a common Lipschitz constant $L > 0$, that is, $\|\mathbf{f}_p(t, \mathbf{x}) - \mathbf{f}_p(t, \mathbf{y})\|_2 \leq L\|\mathbf{x} - \mathbf{y}\|_2$ for all $p \in \mathcal{P}$, all $t \geq 0$, and all $\mathbf{x}, \mathbf{y}$ in some compact neighborhood of the origin. Then, by combining a Lyapunov function construction method by J. Massera for ordinary differential equations, see, for example, [38] or Section 5.7 in [56], with the construction method presented by W. Dayawansa and C. Martin in [5], it is possible to construct a Lyapunov function V that is Lipschitz in the state-space. However, we need V to be smooth so we prove that if the functions $\mathbf{f}_p$, $p \in \mathcal{P}$, are Lipschitz with a common Lipschitz constant L, that is $\|\mathbf{f}_p(t, \mathbf{x}) - \mathbf{f}_p(s, \mathbf{y})\|_2 \leq L(|t - s| + \|\mathbf{x} - \mathbf{y}\|_2)$ for all $p \in \mathcal{P}$, all $s, t \geq 0$, and all $\mathbf{x}, \mathbf{y}$ in some compact neighborhood of the origin in the state-space, then we can smooth the Lipschitz Lyapunov function to be an infinitely differentiable Lyapunov function. A proof of this needs a lot of efforts, partially because we have to prove that if $\mathcal{N} \subset \mathbb{R}^n$ is a neighborhood of the origin and $W \in \mathcal{C}(\mathcal{N}) \cap \mathcal{C}^\infty(\mathcal{N} \setminus \{\mathbf{0}\})$ is Lipschitz at the origin, then $W \exp(-W^{-1}) \in \mathcal{C}^\infty(\mathcal{N})$. This fact was used by F. Wilson in [62] without any justification. In [43] T. Nadzieja repairs some other parts of Wilson's proof and notices this problem too. He argues that this must hold true because $W \exp(-W^{-1})$ converges faster to zero than any polynomial. However, this argument does not seem satisfactory to the author because some arbitrary derivative of W might still diverge to fast at the origin. Therefore we deliver a rigid proof of this fact. Surprisingly enough it is about 6 pages long of quite technical mathematics.

In Part II of this work, which consists of the chapters 4, 5, 6, and 7, we give an algorithmic construction scheme of a linear programming problem for the switched system $\dot{\mathbf{x}} = \mathbf{f}_p(t, \mathbf{x})$, $p \in \mathcal{P} \neq \emptyset$ finite, where the $\mathbf{f}_p$ are assumed to be $\mathcal{C}^2$ functions. Further, we prove that if this linear programming problem possesses a feasible solution, then such a solution can be used to parameterize a function that is a Lyapunov function for all of the individual systems and then a Lyapunov function for the arbitrary switched system. We then use this fact to derive an algorithm to construct Lyapunov functions for nonlinear, nonautonomous, arbitrary switched systems that possess a uniformly asymptotically stable equilibrium.

In Chapter 4 we introduce the function space CPWA, a set of continuous functions $\mathbb{R}^n \longrightarrow \mathbb{R}$ that are piecewise affine (often called piecewise linear in the literature) with a certain simplicial boundary configuration. The spaces CPWA are essentially the function spaces PWL, presented by P. Julian, A. Desages, and O. Agamennoni in [22], P. Julian, J. Guivant, and A. Desages in [23], and by P. Julian in [21], with variable grid sizes. A function space CPWA defined on a compact domain is a finite dimensional vector space over $\mathbb{R}$, which makes it particularly well suited as the foundation for the search of a parameterized Lyapunov function. Another property which renders them appropriate as a search space, is that a function in $\mathcal{C}^2$ can be neatly approximated by a function in CPWA as shown in Lemma 4.14.

In Chapter 5 we define our linear programming problem in Definition 5.1 and then we show how to use a feasible solution to it to parameterize a Lyapunov function for the corresponding switched system. We discuss the autonomous case separately, because in this case it is possible to parameterize an autonomous Lyapunov function with a simpler linear programming problem, which is defined in Definition 5.5. These results are generalizations of former results by the author, presented in [35, 36, 11, 10, 12, 13].

In Chapter 6 we prove, that if we construct a linear programming problem as in Definition 5.1 for a switched system that possesses a uniformly asymptotically stable equilibrium, then, if the boundary configuration of the function space CPWA is sufficiently closely meshed, there exist feasible solutions to the linear programming problem. There are algorithms, for example the simplex algorithm, that always find a feasible solution to a linear programming problem, provided there exists at least one. This implies that we have reduced the problem of constructing a Lyapunov function for the arbitrary switched system to a simpler problem of choosing an appropriate boundary configuration for the CPWA space. If the systems $\dot{\mathbf{x}} = \mathbf{f}_p(t, \mathbf{x})$, $p \in \mathcal{P}$, are autonomous and exponentially stable, then it follows by [11] that it is even possible to calculate the mesh-sizes directly from the original data, that is, the functions $\mathbf{f}_p$. This, however, is much more restrictive than necessary, because a systematic scan of boundary configurations is considerably more effective, will lead to success for merely uniformly asymptotically stable nonautonomous systems, and delivers a boundary configuration that is more coarsely meshed. Just as in Chapter 5 we consider the more simple autonomous case separately.

In Chapter 7 we use the results from Chapter 6 to define our algorithm in Procedure 7.1 to construct Lyapunov functions and we prove in Theorem 7.2 that it always delivers a Lyapunov function, if the arbitrary switched system possesses a uniformly asymptotically stable equilibrium. For autonomous systems we do the same in Procedure 7.3 and Theorem 7.4. These procedures and theorems are generalizations of results presented by the author in [11, 10, 13].

In the last decades there have been several proposals of how to numerically construct Lyapunov functions. For comparison to the construction method presented in this work some of these are listed below. This list is by no means exhaustive.

In [55] L. Vandenberghe and S. Boyd present an interior-point algorithm to construct a common quadratic Lyapunov function for a finite set of autonomous linear systems and in [30] D. Liberzon and R. Tempo took a somewhat different approach to do the same and introduced a gradient decreasing algorithm. Booth methods are numerically efficient, but unfortunately, limited by the fact that there might exist Lyapunov functions for the system, non of which is quadratic. In [19], [20], and [18] M. Johansson and A. Rantzer proposed construction methods for piecewise quadratic Lyapunov functions for piecewise affine autonomous systems. Their construction scheme in based on continuity matrices for the partition of the respective state-space. The generation of these continuity matrices remains, to the best knowledge of the author, an open problem. R. Brayton and C. Tong in [3] and [4], Y. Ohta, H. Imanishi, L. Gong, and H. Haneda in [45], A. Michel, N. Sarabudla, and R. Miller in [41], and A. Michel, B. Nam, and V. Vittal in [40] reduced the Lyapunov function construction for a set of autonomous linear systems to the design of a balanced polytope fulfilling certain invariance properties. A. Polanski in [48] and X. Koutsoukos and P. Antsaklis in [26] consider the construction of a Lyapunov function of the form $V(\mathbf{x}) := \|W\mathbf{x}\|_\infty$, where W is a matrix, for autonomous linear systems by linear programming. P. Julian, J. Guivant, and A. Desages in [23] and P. Julian in [21] presented a linear programming problem to construct piecewise affine Lyapunov functions for autonomous piecewise affine systems. This method can be used for autonomous, nonlinear systems if some a posteriori analysis of the generated Lyapunov function is done. The difference between this method and our (autonomous) method is described in Section 6.2 in [36]. In [17] T. Johansen uses linear programming to parameterize Lyapunov functions for autonomous nonlinear systems. His results are, however, only valid within an approximation error, which is difficult to determine. P. Parrilo in [47] and A. Papachristodoulou and S. Prajna in [46] consider the numerical construction of Lyapunov functions that are presentable as sums of squares for autonomous polynomial systems under polynomial constraints.

In Part III of this thesis, which consists of the chapters 8-14, we give several examples of Lyapunov functions that we generated by the linear programming problems from Definition 5.1 (nonautonomous) and Definition 5.5 (autonomous). Further, in Chapter 14, we give some final words. The examples are as follows: In Chapter 8 we generate a Lyapunov function for a two-dimensional, autonomous, nonlinear ordinary differential equation, of which the equilibrium is asymptotically stable but not exponentially stable. In Chapter 9 we consider three different two-dimensional, autonomous, nonlinear ordinary differential equations and we generate a Lyapunov function for each of them. Then we generate a Lyapunov function for the corresponding arbitrary switched system. In Chapter 10 we generate a Lyapunov function for a two-dimensional, autonomous, piecewise linear variable structure system without sliding modes. Variable structure systems are switched systems, where the switching is not arbitrary but is performed in dependence of the current state-space position of the system. Such systems are not discussed in Part I and Part II of this work (the theoretical part), but as explained in the example such an extension is straight forward. For variable structure systems, however, one cannot use the theorems that guarantee the success of the linear programming problem in parameterizing a Lyapunov function. In Chapter 11 we generate a Lyapunov function for a two-dimensional, autonomous, piecewise affine variable structure system with sliding modes. In Chapter 12 we generate Lyapunov functions for two different one-dimensional, nonautonomous, nonlinear systems. We then parameterize a Lyapunov function for the corresponding arbitrary switched system. Finally, in

Chapter 13, we parameterize Lyapunov functions for two different two-dimensional, nonautonomous, nonlinear systems. Then we generate a Lyapunov function for the corresponding arbitrary switched system.

Part I

Stability theory of dynamical systems under arbitrary switching

In this first part of the thesis we introduce arbitrary switched dynamical systems and prove some elementary properties of their solutions. Thereafter we derive the direct method of Lyapunov for these systems. The Lyapunov function is required to be continuous but not necessarily differentiable. Finally, the main contribution of this part is presented, namely Theorem 3.10, a converse Lyapunov theorem on uniform asymptotic stability for arbitrary switched nonautonomous, nonlinear, continuous dynamical systems. These results are more general than the converse theorems found in the literature (see *Outline of this work*) and because these results are needed later on we give a rigid proof.

Chapter 1

Switched dynamical systems

In this thesis we will consider arbitrary switched nonautonomous, nonlinear, continuous dynamical systems. In order to define exactly what is meant by these concepts and to introduce the notations used throughout this thesis, we will start by discussing continuous dynamical systems and some elementary, but useful, properties of them and their solutions. Then we introduce switching signals and switched systems.

1.1 Continuous dynamical systems

A continuous dynamical system is a system, of which the dynamics can be modeled by an ordinary differential equation of the form

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}). \tag{1.1}$$

This equation is called the *state equation* of the dynamical system. We refer to $\mathbf{x}$ as the *state* of the system and to the set of all possible states as the *state-space* of the system. Further, we refer to t as the *time* of the system. If the mapping $\mathbf{f}$ in (1.1) does not depend explicitly on the time t, that is $\mathbf{f}(t_1, \mathbf{x}) = \mathbf{f}(t_2, \mathbf{x})$ for all times t_1 and t_2 and all states $\mathbf{x}$, then the system (1.1) is said to be *autonomous*. A system that is not autonomous it is said to be *nonautonomous*.

In order to define the solution to a continuous dynamical system, we first need to define what we mean by a solution to an initial value problem of the form

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x}(t_0) = \boldsymbol{\xi},$$

and we have to assure, that a unique solution exists for every $\boldsymbol{\xi}$ in the state-space. In order to define a solution to such an initial value problem, it is advantageous to assume that the state-space of the system is a *domain* in $\mathbb{R}^n$, that is, an open and connected subset[1]. The set $\mathcal{U} \subset \mathbb{R}^n$ is said to be *connected*, if and only if for every points $\mathbf{a}, \mathbf{b} \in \mathcal{U}$ there is a continuous mapping $\boldsymbol{\gamma} : [0, 1] \longrightarrow \mathcal{U}$ such that $\boldsymbol{\gamma}(0) = \mathbf{a}$ and $\boldsymbol{\gamma}(1) = \mathbf{b}$.

The definition of a solution to an initial value problem is somewhat involved because we want it to be a solution, of which the domain cannot be extended in a self-evident way. By a solution to an initial value problem we exactly mean:

[1]The term domain has several meanings in mathematics and it has to be deduced from the context what is meant. If we, for example, say that a function g has the domain $\mathcal{D} \subset \mathbb{R}^n$, it means that the function value of g is properly defined at $\mathbf{x} \in \mathbb{R}^n$, if and only if $\mathbf{x} \in \mathcal{D}$. The set $\mathcal{D}$ does not necessarily have to be a domain in $\mathbb{R}^n$.

Definition 1.1 (Solution to an initial value problem) *Let $\mathcal{U}$ be a domain in $\mathbb{R}^n$, let $\boldsymbol{\xi} \in \mathcal{U}$, let $-\infty < a < b < +\infty$, let $\mathcal{I} \subset \mathbb{R}$ be an interval, of which the interior is the interval $]a, b[$, let $t_0 \in \mathcal{I}$, and let $\mathbf{f} : \mathcal{I} \times \mathcal{U} \longrightarrow \mathbb{R}^n$ be a function.*

We call $\mathbf{y} :]a', b'[\longrightarrow \mathbb{R}^n$, $a \leq a' < b' \leq b$, a solution to the initial value problem

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x}(t_0) = \boldsymbol{\xi},$$

if and only if $\mathbf{y}(t_0) = \boldsymbol{\xi}$, $\text{graph}(\mathbf{y}) \subset]a', b'[\times \mathcal{U}$, $\dot{\mathbf{y}}(t) = \mathbf{f}(t, \mathbf{y}(t))$ for all $t \in]a', b'[$, and neither the closure of $\text{graph}\, \mathbf{y}_{[t_0, b'[}$ nor the closure of $\text{graph}\, \mathbf{y}_{]a', t_0]}$ is a compact subset of $]a, b[\times \mathcal{U}$.

Further, if $a \in \mathcal{I}$ and the limit $\lim_{t \to a+} \mathbf{y}(t)$ exists and is in $\mathcal{U}$, then we define $\mathbf{y}(a)$ to be this limit (continuous extension), and, if $b \in \mathcal{I}$ and the limit $\lim_{t \to b-} \mathbf{y}(t)$ exists and is in $\mathcal{U}$, then we define $\mathbf{y}(b)$ to be this limit

□

The condition that the closure of $\text{graph}\, \mathbf{y}_{[t_0, b'[}$ is not a compact subset of $]a, b[\times \mathcal{U}$ implies that one of the following applies:

i) $b' = +\infty$ so the solution $\mathbf{y}$ exists for all $t \geq t_0$.

ii) $b' < +\infty$ and $\limsup_{t \to b'-} \|\mathbf{y}(t)\|_2 = +\infty$.

iii) $b' < +\infty$ and $\lim_{t \to b'-} \left(\inf_{\mathbf{z} \in \mathbb{R}^n \setminus \mathcal{U}} \|\mathbf{z} - \mathbf{y}(t)\|_2 \right) = 0$.

For $\mathbf{y}$ restricted to $]a', t_0]$ analogous propositions apply. For a proof of this fact see, for example, II.§6.VII in [59]. Together these requirements secure, that $\mathbf{y}$ is a solution, of which the domain cannot be extended in any sensible way. For a more detailed discussion on this we refer to [59] again.

One possibility to secure the existence and uniqueness of a solution for any initial state $\boldsymbol{\xi}$ in the state-space of a system, is given by the Lipschitz condition.

The function $\mathbf{f} : \mathbb{R}_{\geq 0} \times \mathcal{U} \longrightarrow \mathbb{R}^n$, where $\mathcal{U} \subset \mathbb{R}^m$, is said to be *Lipschitz*, with a *Lipschitz constant* $L > 0$ with respect to the norms $\|\cdot\|$ on $\mathbb{R} \times \mathbb{R}^m$ and $\|\cdot\|_*$ on $\mathbb{R}^n$, if and only if the *Lipschitz condition*

$$\|\mathbf{f}(s, \mathbf{x}) - \mathbf{f}(t, \mathbf{y})\|_* \leq L \|(s, \mathbf{x}) - (t, \mathbf{y})\|$$

holds true for all $(s, \mathbf{x}), (t, \mathbf{y}) \in \mathbb{R}_{\geq 0} \times \mathcal{U}$.

It is said to be *Lipschitz* on $\mathcal{U}$, with a *Lipschitz constant* $L > 0$, if and only if the *Lipschitz condition*

$$\|\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{y})\|_* \leq L \|\mathbf{x} - \mathbf{y}\|$$

holds true for all $\mathbf{x}, \mathbf{y} \in \mathcal{U}$ and all $t \in \mathbb{R}_{\geq 0}$.

It is said to be *locally Lipschitz* on $\mathcal{U}$, if and only if its restriction $\mathbf{f}|_{[t_1, t_2] \times \mathcal{C}}$ to any compact set $[t_1, t_2] \times \mathcal{C} \subset \mathbb{R}_{\geq 0} \times \mathcal{U}$ is Lipschitz on $[t_1, t_2] \times \mathcal{C}$.

Still another concept that we will use in this work is *locally Lipschitz uniformly in the first argument*. It is defined as follows: the function $\mathbf{f}$ is said to be *locally Lipschitz uniformly in the first argument* on $\mathcal{U}$, if and only if its restriction $\mathbf{f}|_{\mathbb{R}_{\geq 0} \times \mathcal{C}}$ to any set $\mathbb{R}_{\geq 0} \times \mathcal{C}$, where $\mathcal{C}$ is a compact subset of $\mathcal{U}$, is Lipschitz on $\mathbb{R}_{\geq 0} \times \mathcal{C}$. That is, there is a constant $L > 0$ such that

$$\|\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{y})\|_* \leq L \|\mathbf{x} - \mathbf{y}\|$$

for all $\mathbf{x}, \mathbf{y} \in \mathcal{C}$ and all $t \in \mathbb{R}_{\geq 0}$.

Note, that because all norms on $\mathbb{R}^k$ are equivalent, the properties *Lipschitz*, *locally Lipschitz*, and *locally Lipschitz uniformly in the first argument* do not depend on the norms. However, the numerical values of the Lipschitz constants might.

Further note, that if the components of $\mathbf{f} : \mathbb{R}_{\geq 0} \times \mathcal{U} \longrightarrow \mathbb{R}^n$, where $\mathcal{U} \subset \mathbb{R}^m$, are all differentiable, then

$$\sup_{\substack{i=1,2,\ldots,n \\ t\geq 0, \mathbf{x}\in\mathcal{U}}} \left|\frac{\partial f_i}{\partial t}(t,\mathbf{x})\right| < +\infty \quad \text{and} \quad \sup_{\substack{i=1,2,\ldots,n \\ j=1,2,\ldots,m \\ t\geq 0, \mathbf{x}\in\mathcal{U}}} \left|\frac{\partial f_i}{\partial x_j}(t,\mathbf{x})\right| < +\infty$$

implies that $\mathbf{f}$ is Lipschitz,

$$\sup_{\substack{i=1,2,\ldots,n \\ j=1,2,\ldots,m \\ t\geq 0, \mathbf{x}\in\mathcal{U}}} \left|\frac{\partial f_i}{\partial x_j}(t,\mathbf{x})\right| < +\infty$$

implies that $\mathbf{f}$ is Lipschitz on $\mathcal{U}$,

$$\sup_{\substack{i=1,2,\ldots,n \\ j=1,2,\ldots,m \\ t\in\mathcal{I}, \mathbf{x}\in\mathcal{C}}} \left|\frac{\partial f_i}{\partial x_j}(t,\mathbf{x})\right| < +\infty \qquad \text{for every compact } \mathcal{I}\times\mathcal{C} \subset \mathbb{R}_{\geq 0}\times\mathcal{U},$$

implies that $\mathbf{f}$ is locally Lipschitz on $\mathcal{U}$, and

$$\sup_{\substack{i=1,2,\ldots,n \\ j=1,2,\ldots,m \\ t\geq 0, \mathbf{x}\in\mathcal{C}}} \left|\frac{\partial f_i}{\partial x_j}(t,\mathbf{x})\right| < +\infty \qquad \text{for every compact } \mathcal{C} \subset \mathcal{U},$$

implies that $\mathbf{f}$ is locally Lipschitz on $\mathcal{U}$ uniformly in the first argument.

The next theorem states the arguably two most important results in the theory of ordinary differential equations. It gives sufficient conditions for the existence and the uniqueness of solutions of initial value problems. These conditions are not the most general known, but they are intuitive and usually more easy to affirm than more abstract ones. For more general conditions see, for example, III.§12.VII in [59].

Theorem 1.2 (Peano / Picard-Lindelöf) *Let $\mathcal{U} \subset \mathbb{R}^n$ be a domain, $\mathbf{f} : \mathbb{R}_{\geq 0} \times \mathcal{U} \longrightarrow \mathbb{R}^n$ be a continuous function, $t_0 \geq 0$ be a constant, and $\boldsymbol{\xi} \in \mathcal{U}$. Then there exists a solution to the initial value problem*

$$\dot{\mathbf{x}} = \mathbf{f}(t,\mathbf{x}), \quad \mathbf{x}(t_0) = \boldsymbol{\xi}.$$

If $\mathbf{f}$ is locally Lipschitz on $\mathcal{U}$, then there are no further solutions.

PROOF:
See, for example, Theorems VI and IX in III.§10 in [59].

■

In this thesis we will only consider continuous dynamical systems, of which the dynamics are modeled by an ordinary differential equation

$$\dot{\mathbf{x}} = \mathbf{f}(t,\mathbf{x}), \tag{1.2}$$

where $\mathbf{f} : \mathbb{R}_{\geq 0} \times \mathcal{U} \longrightarrow \mathbb{R}^n$ is a locally Lipschitz function on the domain $\mathcal{U} \subset \mathbb{R}^n$. The last theorem allows us to define the solution to the state equation of such a dynamical system.

Definition 1.3 (Solution to an ordinary differential equation) *Let $\mathcal{U} \subset \mathbb{R}^n$ be a domain and let $\mathbf{f} : \mathbb{R}_{\geq 0} \times \mathcal{U} \longrightarrow \mathbb{R}^n$ be locally Lipschitz on $\mathcal{U}$. For every $\boldsymbol{\xi} \in \mathcal{U}$ and every $t_0 \in \mathbb{R}_{\geq 0}$ let $\mathbf{y}_{(t_0,\boldsymbol{\xi})}$ be the solution to the initial value problem*

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x}(t_0) = \boldsymbol{\xi}.$$

Let the function

$$\boldsymbol{\phi} : \{(t, t_0, \boldsymbol{\xi}) \big| t_0 \in \mathbb{R}_{\geq 0}\,, \ \boldsymbol{\xi} \in \mathcal{U}, \ \textit{and } t \in \operatorname{dom}(\mathbf{y}_{(t_0,\boldsymbol{\xi})})\} \longrightarrow \mathbb{R}^n$$

be defined by $\boldsymbol{\phi}(t, t_0, \boldsymbol{\xi}) := \mathbf{y}_{(t_0,\boldsymbol{\xi})}(t)$ for all $t_0 \in \mathbb{R}_{\geq 0}$, all $\boldsymbol{\xi} \in \mathcal{U}$, and all $t \in \operatorname{dom}(\mathbf{y}_{(t_0,\boldsymbol{\xi})})$. The function $\boldsymbol{\phi}$ is called the solution to the state equation

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}).$$

□

In this thesis we will use the functions spaces $\mathcal{C}^m$ of m-times continuously differentiable functions, $m = 0, 1, \ldots, \infty$. The characterization of these spaces is most conveniently done by the use of so-called multiindices.

Definition 1.4 (Multiindex) *An n-dimensional multiindex $\boldsymbol{\beta}$ is an n-tuple of nonnegative integers, $\boldsymbol{\beta} := (\beta_1, \beta_2, \ldots, \beta_n)$. The length $|\boldsymbol{\beta}|$ of the multiindex $\boldsymbol{\beta}$ is defined as the sum of the β_i, that is*

$$|\boldsymbol{\beta}| := \sum_{i=1}^{n} \beta_i.$$

The differential operator $\partial^{\boldsymbol{\beta}}$ is defined for sufficiently smooth functions $f : \mathcal{U} \longrightarrow \mathbb{R}$, where $\mathcal{U} \subset \mathbb{R}^n$ is a domain, by

$$\partial^{\boldsymbol{\beta}} f(\mathbf{x}) := \frac{\partial^{|\boldsymbol{\beta}|} f}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \cdots \partial x_n^{\beta_n}}(\mathbf{x}).$$

□

It is a well known fact, that if $\partial^{\boldsymbol{\beta}} f$ is properly defined and continuous on $\mathcal{U}$, then the order of the differentiations does not matter.

Let $m \in \mathbb{N}_{\geq 0}$. A function $f : \mathcal{U} \longrightarrow \mathbb{R}$, such that $\partial^{\boldsymbol{\beta}} f$ is properly defined and a continuous function on $\mathcal{U}$ for all multiindices $\boldsymbol{\beta}$ satisfying $|\boldsymbol{\beta}| \leq m$, is said to be of class $\mathcal{C}^m(\mathcal{U})$. We usually write $\mathcal{C}(\mathcal{U})$ instead of $\mathcal{C}^0(\mathcal{U})$ and a function $f : \mathcal{U} \longrightarrow \mathbb{R}$ that is of class $\mathcal{C}^m(\mathcal{U})$ for all $m \in \mathbb{N}_{\geq 0}$ is said to be of class $\mathcal{C}^\infty(\mathcal{U})$.

A vector valued mapping $\mathbf{f} : \mathcal{U} \longrightarrow \mathbb{R}^k$, $\mathbf{f} := (f_1, f_2, \ldots, f_k)$, of which each component f_i is of class $\mathcal{C}^m(\mathcal{U})$, is said to be of class $\mathcal{C}^m(\mathcal{U})$ too and we write $\mathbf{f} \in [\mathcal{C}^m(\mathcal{U})]^k$.

It is a remarkable fact, that if $\mathbf{f}$ in (1.2) is a $[\mathcal{C}^m(\mathbb{R}_{\geq 0} \times \mathcal{U})]^n$ function for some $m \in \mathbb{N}_{\geq 0}$, then its solution $\boldsymbol{\phi}$ and the time-derivative $\dot{\boldsymbol{\phi}}$ of the solution are $\mathcal{C}^m$ functions on their domains as well. This follows, for example, by the corollary at the end of III.§13 in [59]. We need this fact later so we state it as a theorem.

Theorem 1.5 *Let $\mathcal{U} \subset \mathbb{R}^n$ be a domain, $\mathbf{f} : \mathbb{R}_{\geq 0} \times \mathcal{U} \longrightarrow \mathbb{R}^n$ be locally Lipschitz on $\mathcal{U}$, and $\boldsymbol{\phi}$ be the solution to the state equation $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$. Let $m \in \mathbb{N}_{\geq 0}$ and assume that $\mathbf{f} \in [\mathcal{C}^m(\mathbb{R}_{\geq 0} \times \mathcal{U})]^n$, then $\boldsymbol{\phi}, \dot{\boldsymbol{\phi}} \in [\mathcal{C}^m(\operatorname{dom}(\boldsymbol{\phi}))]^n$.*

■

We end this discussion on continuous dynamical systems with a well know theorem, that gives an upper bound on the difference of solutions to differential equations.

Theorem 1.6 *Let $\mathcal{I} \subset \mathbb{R}$ be a nonempty interval and let $\mathcal{U}$ be a domain in $\mathbb{R}^n$. Let $\mathbf{f}, \mathbf{g} : \mathcal{I} \times \mathcal{U} \longrightarrow \mathbb{R}^n$ be continuous mappings and assume that $\mathbf{f}$ is Lipschitz on $\mathcal{U}$ with a Lipschitz constant $L > 0$ with respect to the norm $\|\cdot\|$ on $\mathbb{R}^n$. Let $t_0 \in \mathcal{I}$ and let $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathcal{U}$ and denote the unique solution to the initial value problem*

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x}(t_0) := \boldsymbol{\xi},$$

by $\mathbf{y} : \mathcal{I}_{\mathbf{y}} \longrightarrow \mathbb{R}^n$ and let $\mathbf{z} : \mathcal{I}_{\mathbf{z}} \longrightarrow \mathbb{R}^n$ be any solution to the initial value problem

$$\dot{\mathbf{x}} = \mathbf{g}(t, \mathbf{x}), \quad \mathbf{x}(t_0) = \boldsymbol{\eta}.$$

Set $\mathcal{J} := \mathcal{I}_{\mathbf{y}} \cap \mathcal{I}_{\mathbf{z}}$ and let γ and δ be constants, such that

$$\|\boldsymbol{\xi} - \boldsymbol{\eta}\| \leq \gamma \quad \text{and} \quad \|\mathbf{f}(t, \mathbf{z}(t)) - \mathbf{g}(t, \mathbf{z}(t))\| \leq \delta$$

for all $t \in \mathcal{J}$. Then the inequality

$$\|\mathbf{y}(t) - \mathbf{z}(t)\| \leq \gamma e^{L|t-t_0|} + \frac{\delta}{L}(e^{L|t-t_0|} - 1)$$

holds true for all $t \in \mathcal{J}$.

PROOF:
Follows, for example, by Theorem III.§12.V in [59].

■

1.2 Switched systems

A switched system is basically a family of dynamical systems and a switching signal, where the switching signal determines which system in the family describes the dynamics at what times or states. As we will be concerned with the stability of switched systems under arbitrary switchings, the following definition of a switching signal is sufficient for our needs.

Definition 1.7 (Switching signal) *Let $\mathcal{P}$ be a nonempty set and equip it with the discrete metric $d(p, q) := 1$ if $p \neq q$. A switching signal $\sigma : \mathbb{R}_{\geq 0} \longrightarrow \mathcal{P}$ is a right-continuous function, of which the discontinuity-points are a discrete subset of $\mathbb{R}_{\geq 0}$. The discontinuity-points are called switching times. For technical reasons it is convenient to count zero with the switching times, so we agree upon that zero is a switching time as well. We denote the set of all switching signals $\mathbb{R}_{\geq 0} \longrightarrow \mathcal{P}$ by $\mathcal{S}_{\mathcal{P}}$.*

□

With the help of the switching signal in the last definition we can define the concept of a switched system and its solution.

Definition 1.8 (Solution to a switched system) *Let $\mathcal{U} \subset \mathbb{R}^n$ be a domain, let $\mathcal{P}$ be a nonempty set, and let*

$$\{\mathbf{f}_p : \mathbb{R}_{\geq 0} \times \mathcal{U} \longrightarrow \mathbb{R}^n \big| p \in \mathcal{P}\}$$

be a family of mappings, each locally Lipschitz on $\mathcal{U}$. For every switching signal $\sigma \in \mathcal{S}_\mathcal{P}$ we define the solution $t \mapsto \boldsymbol{\phi}_\sigma(t, s, \boldsymbol{\xi})$ to the initial value problem

$$\dot{\mathbf{x}} = \mathbf{f}_\sigma(t, \mathbf{x}), \quad \mathbf{x}(s) = \boldsymbol{\xi}, \tag{1.3}$$

in the following way:

Denote by $t_0, t_1, t_2, \ldots$ the switching points of σ in an increasing order. If there is a largest switching point t_k we set $t_{k+1} := +\infty$ and if there is no switching time besides zero we set $t_1 := +\infty$. Let $s \in \mathbb{R}_{\geq 0}$ and let $k \in \mathbb{N}_{\geq 0}$ be such that $t_k \leq s < t_{k+1}$. Then $\boldsymbol{\phi}_\sigma$ is defined by gluing together the trajectories of the systems

$$\dot{\mathbf{x}} = \mathbf{f}_p(t, \mathbf{x}), \quad p \in \mathcal{P},$$

using $p := \sigma(s)$ between s and t_{k+1}, $p := \sigma(t_{k+1})$ between t_{k+1} and t_{k+2}, and in general $p := \sigma(t_i)$ between t_i and t_{i+1}, $i \geq k+1$. Mathematically this can be expressed inductively as follows:

Forward solution:

i) $\boldsymbol{\phi}_\sigma(s, s, \boldsymbol{\xi}) = \boldsymbol{\xi}$ *for all* $s \in \mathbb{R}_{\geq 0}$ *and all* $\boldsymbol{\xi} \in \mathcal{U}$.

ii) Denote by $\mathbf{y}$ the solution to the initial value problem

$$\dot{\mathbf{x}} = \mathbf{f}_{\sigma(s)}(t, \mathbf{x}), \quad \mathbf{x}(s) = \boldsymbol{\xi},$$

on the interval $[s, t_{k+1}[$, where $k \in \mathbb{N}_{\geq 0}$ is such that $t_k \leq s < t_{k+1}$. Then we define $\boldsymbol{\phi}_\sigma(t, s, \boldsymbol{\xi})$ on the domain of $t \mapsto \mathbf{y}(t)$ by $\boldsymbol{\phi}_\sigma(t, s, \boldsymbol{\xi}) := \mathbf{y}(t)$. If the closure of $\operatorname{graph}(\mathbf{y})$ is a compact subset of $[s, t_{k+1}] \times \mathcal{U}$, then the limit $\lim_{t \to t_{k+1}-} \mathbf{y}(t)$ exists and is in $\mathcal{U}$ and we define $\boldsymbol{\phi}_\sigma(t_{k+1}, s, \boldsymbol{\xi}) := \lim_{t \to t_{k+1}-} \mathbf{y}(t)$.

iii) Assume $\boldsymbol{\phi}_\sigma(t_i, s, \boldsymbol{\xi}) \in \mathcal{U}$ is defined for some integer $i \geq k+1$ and denote by $\mathbf{y}$ the solution to the initial value problem

$$\dot{\mathbf{x}} = \mathbf{f}_{\sigma(t_i)}(t, \mathbf{x}), \quad \mathbf{x}(t_i) = \boldsymbol{\phi}_\sigma(t_i, s, \boldsymbol{\xi}),$$

on the interval $[t_i, t_{i+1}[$. Then we define $\boldsymbol{\phi}_\sigma(t, s, \boldsymbol{\xi})$ on the domain of $t \mapsto \mathbf{y}(t)$ by $\boldsymbol{\phi}_\sigma(t, s, \boldsymbol{\xi}) := \mathbf{y}(t)$. If the closure of $\operatorname{graph}(\mathbf{y})$ is a compact subset of $[t_i, t_{i+1}] \times \mathcal{U}$, then the limit $\lim_{t \to t_{i+1}-} \mathbf{y}(t)$ exists and is in $\mathcal{U}$ and we define $\boldsymbol{\phi}_\sigma(t_{i+1}, s, \boldsymbol{\xi}) := \lim_{t \to t_{i+1}-} \mathbf{y}(t)$.

Backward solution:

i) $\boldsymbol{\phi}_\sigma(s, s, \boldsymbol{\xi}) = \boldsymbol{\xi}$ *for all* $s \in \mathbb{R}_{\geq 0}$ *and all* $\boldsymbol{\xi} \in \mathcal{U}$.

ii) Denote by $\mathbf{y}$ the solution to the initial value problem

$$\dot{\mathbf{x}} = \mathbf{f}_{\sigma(t_k)}(t, \mathbf{x}), \quad \mathbf{x}(s) = \boldsymbol{\xi},$$

on the interval $]t_k, s]$, where $k \in \mathbb{N}_{\geq 0}$ is such that $t_k \leq s < t_{k+1}$. Then we define $\boldsymbol{\phi}_\sigma(t, s, \boldsymbol{\xi})$ on the domain of $t \mapsto \mathbf{y}(t)$ by $\boldsymbol{\phi}_\sigma(t, s, \boldsymbol{\xi}) := \mathbf{y}(t)$. If the closure of $\operatorname{graph}(\mathbf{y})$ is not empty and a compact subset of $[t_k, s] \times \mathcal{U}$, then the limit $\lim_{t \to t_k+} \mathbf{y}(t)$ exists and is in $\mathcal{U}$ and we define $\boldsymbol{\phi}_\sigma(t_k, s, \boldsymbol{\xi}) := \lim_{t \to t_k+} \mathbf{y}(t)$.

iii) Assume $\boldsymbol{\phi}_\sigma(t_i, s, \boldsymbol{\xi}) \in \mathcal{U}$ is defined for some integer i, $0 < i \leq k$ and denote by $\mathbf{y}$ the solution to the initial value problem

$$\dot{\mathbf{x}} = \mathbf{f}_{\sigma(t_i)}(t, \mathbf{x}), \quad \mathbf{x}(t_i) = \boldsymbol{\phi}_\sigma(t_i, s, \boldsymbol{\xi}),$$

on the interval $]t_{i-1}, t_i]$. Then we define $\boldsymbol{\phi}_\sigma(t, s, \boldsymbol{\xi})$ on the domain of $t \mapsto \mathbf{y}(t)$ by $\boldsymbol{\phi}_\sigma(t, s, \boldsymbol{\xi}) := \mathbf{y}(t)$. If the closure of $\mathrm{graph}(\mathbf{y})$ is a compact subset of $[t_{i-1}, t_i] \times \mathcal{U}$, then the limit $\lim_{t \to t_{i-1}+} \mathbf{y}(t)$ exists and is in $\mathcal{U}$ and we define $\boldsymbol{\phi}_\sigma(t_{i-1}, s, \boldsymbol{\xi}) := \lim_{t \to t_{i-1}+} \mathbf{y}(t)$.

Thus, for every $\sigma \in \mathcal{S}_\mathcal{P}$ we have defined the solution $\boldsymbol{\phi}_\sigma$ to the differential equation

$$\dot{\mathbf{x}} = \mathbf{f}_\sigma(t, \mathbf{x}).$$

□

Now, that we have defined the solution to (1.3) for every $\sigma \in \mathcal{S}_\mathcal{P}$, we can define the switched system and its solution.

Switched System 1.9 *Let $\mathcal{U} \subset \mathbb{R}^n$ be a domain, let $\mathcal{P}$ be a nonempty set, and let*

$$\{\mathbf{f}_p : \mathbb{R}_{\geq 0} \times \mathcal{U} \longrightarrow \mathbb{R}^n \big| p \in \mathcal{P}\}$$

be a family of mappings, each locally Lipschitz on $\mathcal{U}$.

The arbitrary switched system

$$\dot{\mathbf{x}} = \mathbf{f}_\sigma(t, \mathbf{x}), \ \sigma \in \mathcal{S}_\mathcal{P},$$

is simply the collection of all the differential equations

$$\{\dot{\mathbf{x}} = \mathbf{f}_\sigma(t, \mathbf{x}) \big| \sigma \in \mathcal{S}_\mathcal{P}\},$$

whose solutions we defined in Definition 1.8.

The solution $\boldsymbol{\phi}$ of the arbitrary switched system is the collection of all the solutions $\boldsymbol{\phi}_\sigma$ to the individual switched systems.

□

Note, that if $\sigma, \varsigma \in \mathcal{S}_\mathcal{P}$, $\sigma \neq \varsigma$, then in general $\boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi})$ is not equal to $\boldsymbol{\phi}_\varsigma(t, t_0, \boldsymbol{\xi})$.

Because the trajectories of the Switched System 1.9 are defined by gluing together trajectory-pieces of the corresponding continuous systems, the remarks after Definition 1.1 about the solution to a continuous dynamical system apply equally to the switched system. That is, for every $\sigma \in \mathcal{S}_\mathcal{P}$, every $s \in \mathbb{R}_{\geq 0}$, and every $\boldsymbol{\xi} \in \mathcal{U}$ the closure of the graph of $t \mapsto \boldsymbol{\phi}_\sigma(t, s, \boldsymbol{\xi})$, $t \geq s$, is not a compact subset of $\mathbb{R}_{\geq 0} \times \mathcal{U}$ and the closure of the graph of $t \mapsto \boldsymbol{\phi}_\sigma(t, s, \boldsymbol{\xi})$, $t \leq s$, is not a compact subset of $\mathbb{R}_{>0} \times \mathcal{U}$.

Note, that if the system 1.9 is autonomous, that is, $\mathbf{f}_p(t, \mathbf{x}) = \mathbf{f}_p(s, \mathbf{x})$ for all $p \in \mathcal{P}$, all $s, t \geq 0$ and all $\mathbf{x} \in \mathcal{U}$, then

$$\boldsymbol{\phi}_\sigma(t, t', \mathbf{x}) = \boldsymbol{\phi}_\gamma(t - t', 0, \boldsymbol{\xi}), \quad \text{where } \gamma(s) := \sigma(s + t') \text{ for all } s \geq 0,$$

for all $t \geq t' \geq 0$ and all $\boldsymbol{\xi} \in \mathcal{U}$. Therefore, we often suppress the middle argument of the solution to an autonomous system and simply write $\boldsymbol{\phi}_\sigma(t, \boldsymbol{\xi})$.

We end this introduction on switched systems with a generalization of Theorem 1.6 to switched systems.

Theorem 1.10 *Consider the Switched System 1.9 and assume that the functions $\mathbf{f}_p$ are all Lipschitz on $\mathcal{U}$ and have a common Lipschitz constant $L > 0$ with regard to some norm $\|\cdot\|$ on $\mathbb{R}^n$. That is,*

$$\|\mathbf{f}_p(t,\mathbf{x}) - \mathbf{f}_p(t,\mathbf{y})\| \leq L\|\mathbf{x}-\mathbf{y}\|$$

for all $t \geq 0$, all $\mathbf{x},\mathbf{y} \in \mathcal{U}$, and all $p \in \mathcal{P}$. Let $t_0 \geq 0$, let $\boldsymbol{\xi},\boldsymbol{\eta} \in \mathcal{U}$, let $\sigma,\varsigma \in \mathcal{S}_\mathcal{P}$, and assume there is a constant $\delta \geq 0$ such that

$$\|\mathbf{f}_{\sigma(t)}(t,\mathbf{x}) - \mathbf{f}_{\varsigma(t)}(t,\mathbf{x})\| \leq \delta$$

for all $t \geq 0$ and all $\mathbf{x} \in \mathcal{U}$.

Denote the solution to the initial value problem

$$\dot{\mathbf{x}} = \mathbf{f}_\sigma(t,\mathbf{x}), \quad \mathbf{x}(t_0) = \boldsymbol{\xi},$$

by $\mathbf{y} : \mathcal{I}_\mathbf{y} \longrightarrow \mathbb{R}^n$ and denote the solution to the initial value problem

$$\dot{\mathbf{x}} = \mathbf{f}_\varsigma(t,\mathbf{x}), \quad \mathbf{x}(t_0) = \boldsymbol{\eta},$$

by $\mathbf{z} : \mathcal{I}_\mathbf{z} \longrightarrow \mathbb{R}^n$. Set $\mathcal{J} := \mathcal{I}_\mathbf{y} \cap \mathcal{I}_\mathbf{z}$ and set $\gamma := \|\boldsymbol{\xi} - \boldsymbol{\eta}\|$. Then the inequality

$$\|\mathbf{y}(t) - \mathbf{z}(t)\| \leq \gamma e^{L|t-t_0|} + \frac{\delta}{L}(e^{L|t-t_0|} - 1) \tag{1.4}$$

holds true for all $t \in \mathcal{J}$.

PROOF:
We only prove inequality (1.4) for $t \geq t_0$, the case $t < t_0$ follows similarly. Let t_1 be the smallest real number larger than t_0 that is a switching time of σ or a switching time of ς. If there is no such a number, then set $t_1 := \sup_{x\in\mathcal{J}} x$. By Theorem 1.6 inequality (1.4) holds true for all t, $t_0 \leq t < t_1$. If $t_1 = \sup_{x\in\mathcal{J}} x$ we are finished with the proof, otherwise $t_1 \in \mathcal{J}$ and inequality (1.4) holds true for $t = t_1$ too. In the second case, let t_2 be the smallest real number larger than t_1 that is a switching time of σ or a switching time of ς. If there is no such a number, then set $t_2 := \sup_{x\in\mathcal{J}} x$. Then, by Theorem 1.6,

$$\begin{aligned}
\|\mathbf{y}(t) - \mathbf{z}(t)\| &\leq \left(\gamma e^{L(t_1-t_0)} + \frac{\delta}{L}(e^{L(t_1-t_0)} - 1)\right) e^{L(t-t_1)} + \frac{\delta}{L}(e^{L(t-t_1)} - 1) \\
&= \gamma e^{L(t-t_0)} + \frac{\delta}{L}e^{L(t-t_0)} - \frac{\delta}{L}e^{L(t-t_1)} + \frac{\delta}{L}e^{L(t-t_1)} - \frac{\delta}{L} \\
&= \gamma e^{L(t-t_0)} + \frac{\delta}{L}(e^{L(t-t_0)} - 1)
\end{aligned}$$

for all t such that $t_1 \leq t < t_2$.

As this argumentation can, if necessary, be repeated ad infinitum, inequality (1.4) holds true for all $t \geq t_0$ such that $t \in \mathcal{J}$.

■

Chapter 2

Stability and Lyapunov functions for switched systems

In this chapter we will discuss the stability of equilibrium points for switched systems. Further, we will carry over the Lyapunov stability theory of continuous dynamical systems to switched systems under arbitrary switching. Differentiable Lyapunov functions are too restrictive for our purposes so we develop the theory for merely continuous Lyapunov functions.

We will make use of Dini derivatives and we need to smooth functions in our proofs, so a short discussion on these subjects is given for completeness.

2.1 Equilibrium points and stability

The concepts equilibrium point and stability are motivated by the desire to keep a dynamical system in, or at least close to, some desirable state. The term *equilibrium* or *equilibrium point* of a dynamical system, is used for a state of the system that does not change in the course of time, that is, if the system is at an equilibrium at time $t = 0$, then it will stay there for all times $t > 0$.

Definition 2.1 (Equilibrium point) *A state* $\mathbf{y}$ *in the state-space of the Switched System 1.9 is called an equilibrium or an equilibrium point of the system, if and only if* $\mathbf{f}_p(t, \mathbf{y}) = \mathbf{0}$ *for all* $p \in \mathcal{P}$ *and all* $t \geq 0$.

□

If $\mathbf{y}$ is an equilibrium point of Switched System 1.9, then obviously the initial value problem

$$\dot{\mathbf{x}} = \mathbf{f}_\sigma(t, \mathbf{x}), \quad \mathbf{x}(0) = \mathbf{y}$$

has the solution $\mathbf{x}(t) = \mathbf{y}$ for all $t \geq 0$ regardless of the switching signal $\sigma \in \mathcal{S}$. The solution with $\mathbf{y}$ as an initial value in the state-space is thus a constant vector and the state does not change in the course of time. By a translation in the state-space one can always reach that $\mathbf{y} = \mathbf{0}$ without affecting the dynamics. Hence, there is no loss of generality in assuming that a particular equilibrium point is at the origin.

A real-world system is always subject to some fluctuations in the state. There are some external effects that are unpredictable and cannot be modeled, some dynamics that have (hopefully) very little impact on the behavior of the system are neglected in the modeling, etc. Even if the mathematical model of a physical system would be perfect, which hardly seems possible, the system state would still be subject to quantum mechanical fluctuations. The concept of local stability in the theory of dynamical systems is motivated by the desire, that the system state stays at least close to an equilibrium point after small fluctuations in the state. Any system that is expected to do something useful must have a predictable behavior to some degree. This excludes all equilibria that are not locally stable as usable working points for a dynamical system. Local stability is thus a minimum requirement for an equilibrium. It is, however, not a very strong property. It merely states, that there are disturbances that are so small, that they do not have a great effect on the system in the long run. In this thesis we will concentrate on *uniform asymptotic stability on a set* containing the equilibrium. This means that we are demanding that the *uniform asymptotic stability* property of the equilibrium is not merely valid for some, possibly arbitrary small, neighborhood of the origin, but this property must hold on a a priori defined neighborhood of the origin. This is a much more robust and powerful concept. It denotes, that all disturbances up to a certain known degree are ironed out by the dynamics of the system, and, because the domain of the Lyapunov functions is only limited by the size of the equilibriums' region of attraction, that we can get a reasonable lower bound on the region of attraction.

The common stability concepts are most practically characterized by the use of so-called $\mathcal{K}$, $\mathcal{L}$, and $\mathcal{KL}$ functions.

Definition 2.2 (Comparison functions $\mathcal{K}$, $\mathcal{L}$, and $\mathcal{KL}$) *The function classes $\mathcal{K}$, $\mathcal{L}$, and $\mathcal{KL}$ of comparison functions are defined as follows:*

i) A continuous function $\alpha : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$, if and only if $\alpha(0) = 0$, it is strictly monotonically increasing, and $\lim_{r \to +\infty} \alpha(r) = +\infty$.

ii) A continuous function $\beta : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{L}$, if and only if it is strictly monotonically decreasing and $\lim_{s \to +\infty} \beta(s) = 0$.

iii) A continuous function $\varsigma : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$, if and only if for every fixed $s \in \mathbb{R}_{\geq 0}$ the mapping $r \mapsto \varsigma(r, s)$ is of class $\mathcal{K}$ and for every fixed $r \in \mathbb{R}_{\geq 0}$ the mapping $s \mapsto \varsigma(r, s)$ is of class $\mathcal{L}$. □

Note that some experts make a difference between strictly monotonically increasing functions that vanish at the origin and strictly monotonically increasing functions that vanish at the origin and additionally asymptotically approach infinity at infinity. They usually denote the functions of the former type as class $\mathcal{K}$ functions and the functions of the latter type as class $\mathcal{K}_\infty$ functions. We are not interested in functions of the former type and in this work $\alpha \in \mathcal{K}$ always implies $\lim_{r \to +\infty} \alpha(r) = +\infty$.

We now define various stability concepts for equilibrium points of switched dynamical systems with help of the comparison functions.

Definition 2.3 (Stability concepts for equilibria) *Assume that the origin is an equilibrium point of the Switched System 1.9, denote by $\boldsymbol{\phi}$ the solution to the system, and let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$.*

i) The origin is said to be a uniformly stable equilibrium point of the Switched System 1.9 on a neighborhood $\mathcal{N} \subset \mathcal{U}$ of the origin, if and only if there exists an $\alpha \in \mathcal{K}$ such that for every $\sigma \in \mathcal{S}_\mathcal{P}$, every $t \geq t_0 \geq 0$, and every $\boldsymbol{\xi} \in \mathcal{N}$ the inequality

$$\|\boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi})\| \leq \alpha(\|\boldsymbol{\xi}\|)$$

holds true.

ii) The origin is said to be a uniformly asymptotically stable equilibrium point of the Switched System 1.9 on the neighborhood $\mathcal{N} \subset \mathcal{U}$ of the origin, if and only if there exists a $\varsigma \in \mathcal{KL}$ such that for every $\sigma \in \mathcal{S}_\mathcal{P}$, every $t \geq t_0 \geq 0$, and every $\boldsymbol{\xi} \in \mathcal{N}$ the inequality

$$\|\boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi})\| \leq \varsigma(\|\boldsymbol{\xi}\|, t - t_0) \tag{2.1}$$

holds true.

iii) The origin is said to be a uniformly exponentially stable equilibrium point of the Switched System 1.9 on the neighborhood $\mathcal{N} \subset \mathcal{U}$ of the origin, if and only if there exist constants $k > 0$ and $\gamma > 0$, such that for every $\sigma \in \mathcal{S}_\mathcal{P}$, every $t \geq t_0 \geq 0$, and every $\boldsymbol{\xi} \in \mathcal{N}$ the inequality

$$\|\boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi})\| \leq k e^{-\gamma(t-t_0)} \|\boldsymbol{\xi}\|$$

holds true.

□

The stability definitions above imply, that if the origin is a uniformly exponentially stable equilibrium of the Switched System 1.9 on the neighborhood $\mathcal{N}$, then the origin is a uniformly asymptotically stable equilibrium on $\mathcal{N}$ as well, and, if the origin is a uniformly asymptotically stable equilibrium of the Switched System 1.9 on the neighborhood $\mathcal{N}$, then the origin is a uniformly stable equilibrium on $\mathcal{N}$.

If the Switched System 1.9 is autonomous, then the stability concepts presented above for the systems equilibria are *uniform* in a canonical way, that is, independent of t_0, and the definitions are somewhat simpler.

Definition 2.4 (Stability concepts for equilibria of autonomous systems) *Assume that the origin is an equilibrium point of the Switched System 1.9, denote by $\boldsymbol{\phi}$ the solution to the system, let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$, and assume that the system is autonomous.*

i) The origin is said to be a stable equilibrium point of the autonomous Switched System 1.9 on a neighborhood $\mathcal{N} \subset \mathcal{U}$ of the origin, if and only if there exists an $\alpha \in \mathcal{K}$ such that for every $\sigma \in \mathcal{S}_\mathcal{P}$ and every $\boldsymbol{\xi} \in \mathcal{N}$ the inequality

$$\|\boldsymbol{\phi}_\sigma(t, \boldsymbol{\xi})\| \leq \alpha(\|\boldsymbol{\xi}\|)$$

holds true.

ii) The origin is said to be an asymptotically stable equilibrium point of the autonomous Switched System 1.9 on the neighborhood $\mathcal{N} \subset \mathcal{U}$ of the origin, if and only if there exists a $\varsigma \in \mathcal{KL}$ such that for every $\sigma \in \mathcal{S}_\mathcal{P}$ and every $\boldsymbol{\xi} \in \mathcal{N}$ the inequality

$$\|\boldsymbol{\phi}_\sigma(t, \boldsymbol{\xi})\| \leq \varsigma(\|\boldsymbol{\xi}\|, t)$$

holds true.

iii) The origin is said to be an exponentially stable equilibrium point of the Switched System 1.9 on the neighborhood $\mathcal{N} \subset \mathcal{U}$ of the origin, if and only if there exist constants $k > 0$ and $\gamma > 0$, such that for every $\sigma \in \mathcal{S}_\mathcal{P}$ and every $\boldsymbol{\xi} \in \mathcal{N}$ the inequality

$$\|\boldsymbol{\phi}_\sigma(t, \boldsymbol{\xi})\| \leq k e^{-\gamma t} \|\boldsymbol{\xi}\|$$

holds true.

□

The set of those points in the state-space of a dynamical system, that are attracted to an equilibrium point by the dynamics of the system, is of great importance. It is called the *region of attraction* of the equilibrium. Some experts prefer *domain*, *basin*, or even *bassin* instead of *region*. For nonautonomous systems it might depend on the initial time.

Definition 2.5 (Region of attraction) *Assume that $\mathbf{y} = \mathbf{0}$ is an equilibrium point of the Switched System 1.9 and let $\boldsymbol{\phi}$ be the solution to the system. For every $t_0 \in \mathbb{R}_{\geq 0}$ the set*

$$\mathcal{R}^{t_0}_{Att} := \{\boldsymbol{\xi} \in \mathcal{U} \big| \limsup_{t \to +\infty} \boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi}) = \mathbf{0} \quad \text{for all } \sigma \in \mathcal{S}_\mathcal{P}\}$$

is called the region of attraction with respect to t_0 of the equilibrium at the origin.

The region of attraction $\mathcal{R}_{Att}$ of the equilibrium at the origin is defined by

$$\mathcal{R}_{Att} := \bigcap_{t_0 \geq 0} \mathcal{R}^{t_0}_{Att}.$$

□

Thus, for the Switched System 1.9 $\boldsymbol{\xi} \in \mathcal{R}_{Att}$ implies $\lim_{t \to +\infty} \boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi}) = \mathbf{0}$ for all $\sigma \in \mathcal{S}_\mathcal{P}$ and all $t_0 \geq 0$. The next theorem confirms the usefulness of the definition of uniform asymptotic stability from Definition 2.3.

Theorem 2.6 *Assume that the origin is a uniformly asymptotically stable equilibrium point of the Switched System 1.9 on the neighborhood $\mathcal{N} \subset \mathcal{U}$ of the origin and let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. Then $\mathcal{N} \subset \mathcal{R}_{Att}$. Further, if $\mathcal{N}$ is bounded, there exists a constant $C > 0$ such that*

$$\sup_{\substack{\sigma \in \mathcal{S}_\mathcal{P} \\ t \geq t_0 \geq 0 \\ \boldsymbol{\xi} \in \mathcal{N}}} \|\boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi})\| \leq C.$$

Proof:
Obvious from the definition of the set $\mathcal{R}_{Att}$ and the definition of uniform asymptotic stability, just set

$$C := \sup_{\mathbf{y} \in \mathcal{N}} \varsigma(\|\mathbf{y}\|, 0),$$

where $\varsigma \in \mathcal{KL}$ is a comparison function as in Definition 2.3 *ii)* with respect to the norm $\|\cdot\|$.

■

Before we come to the Lyapunov stability theory of switched dynamical systems we shortly discuss Dini derivatives and the smoothing of functions by convolutions with $\mathcal{C}^\infty$ functions having a compact support as a preparation for the forthcoming.

2.2 Dini derivatives

The Italian mathematician Ulisse Dini introduced in 1878 in his textbook [6] on analysis the so-called Dini derivatives. They are a generalization of the classical derivative and inherit some important properties from it. Because the Dini derivatives are point-wise defined, they are more suited for our needs than some more modern approaches to generalize the concept of a derivative like Sobolev Spaces (see, for example, [1]) or distributions (see, for example, [60]). The Dini derivatives are defined as follows:

Definition 2.7 (Dini derivatives) *Let $\mathcal{I} \subset \mathbb{R}$, $g : \mathcal{I} \longrightarrow \mathbb{R}$ be a function, and $y \in \mathcal{I}$.*

i) Assume y is a limit point of $\mathcal{I} \cap]y, +\infty[$. Then the right-hand upper Dini derivative D^+ of g at the point y is defined by

$$D^+g(y) := \limsup_{x \to y+} \frac{g(x) - g(y)}{x - y} := \lim_{\varepsilon \to 0+} \left(\sup_{\substack{x \in \mathcal{I} \cap]y,+\infty[\\ 0 < x - y \leq \varepsilon}} \frac{g(x) - g(y)}{x - y} \right)$$

and the right-hand lower Dini derivative D_+ of g at the point y is defined by

$$D_+g(y) := \liminf_{x \to y+} \frac{g(x) - g(y)}{x - y} := \lim_{\varepsilon \to 0+} \left(\inf_{\substack{x \in \mathcal{I} \cap]y,+\infty[\\ 0 < x - y \leq \varepsilon}} \frac{g(x) - g(y)}{x - y} \right).$$

ii) Assume y is a limit point of $\mathcal{I} \cap]-\infty, y[$. Then the left-hand upper Dini derivative D^- of g at the point y is defined by

$$D^-g(y) := \limsup_{x \to y-} := \lim_{\varepsilon \to 0-} \left(\sup_{\substack{x \in \mathcal{I} \cap]-\infty,y[\\ \varepsilon \leq x - y < 0}} \frac{g(x) - g(y)}{x - y} \right)$$

and the left-hand lower Dini derivative D_- of g at the point y is defined by

$$D_-g(y) := \liminf_{x \to y-} \frac{g(x) - g(y)}{x - y} := \lim_{\varepsilon \to 0-} \left(\inf_{\substack{x \in \mathcal{I} \cap]-\infty,y[\\ \varepsilon \leq x - y < 0}} \frac{g(x) - g(y)}{x - y} \right).$$

□

The four Dini derivatives defined in Definition 2.7 are sometimes called the *derived numbers* of g at y, or more exactly the *right-hand upper derived number*, the *right-hand lower derived number*, the *left-hand upper derived number*, and the *left-hand lower derived number* respectively.

It is clear from elementary calculus, that if $g : \mathcal{I} \longrightarrow \mathbb{R}$ is a function from a nonempty open subset $\mathcal{I} \subset \mathbb{R}$ into $\mathbb{R}$ and $y \in \mathcal{I}$, then all four Dini derivatives $D^+g(y)$, $D_+g(y)$, $D^-g(y)$, and $D_-g(y)$ of g at the point y exist. This means that if $\mathcal{I}$ is a nonempty open interval, then the functions $D^+g, D_+g, D^-g, D_-g : \mathcal{I} \longrightarrow \overline{\mathbb{R}}$ defined in the canonical way, are all properly defined. It is not difficult to see that if this is the case, then the classical derivative $g' : \mathcal{I} \longrightarrow \mathbb{R}$ of g exists, if and only if the Dini derivatives are all real-valued and $D^+g = D_+g = D^-g = D_-g$.

Using *lim sup* and *lim inf* instead of the usual limit in the definition of a derivative has the advantage, that they are always properly defined. The disadvantage is, that because of the elementary

$$\limsup_{x \to y+} [g(x) + h(x)] \leq \limsup_{x \to y+} g(x) + \limsup_{x \to y+} h(x),$$

a derivative defined in this way is not a linear operation anymore. However, when the right-hand limit of the function h exists, then it is easy to see that

$$\limsup_{x \to y+}[g(x) + h(x)] = \limsup_{x \to y+} g(x) + \lim_{x \to y+} h(x).$$

This leads to the following lemma, which we will need later.

Lemma 2.8 *Let g and h be real-valued functions, the domains of which are subsets of $\mathbb{R}$, and let $D^* \in \{D^+, D_+, D^-, D_-\}$ be a Dini derivative. Let $y \in \mathbb{R}$ be such, that the Dini derivative $D^*g(y)$ is properly defined and h is differentiable at y in the classical sense. Then*

$$D^*[g + h](y) = D^*g(y) + h'(y).$$

■

The reason why Dini derivatives are so useful for the applications in this thesis, is the following generalization of the Mean-value theorem of differential calculus and its corollary.

Theorem 2.9 (Mean-value theorem for Dini derivatives) *Let $\mathcal{I}$ be an interval of strictly positive measure in $\mathbb{R}$, let $\mathcal{C}$ be a countable subset of $\mathcal{I}$, and let $g : \mathcal{I} \longrightarrow \mathbb{R}$ be a continuous function. Let $D^* \in \{D^+, D_+, D^-, D_-\}$ be a Dini derivative and let $\mathcal{J}$ be an interval, such that $D^*f(x) \in \mathcal{J}$ for all $x \in \mathcal{I} \setminus \mathcal{C}$. Then*

$$\frac{g(x) - g(y)}{x - y} \in \mathcal{J}$$

for all $x, y \in \mathcal{I}$, $x \neq y$.

Proof:
See, for example, Theorem 12.24 in [58].

■

This theorem has an obvious corollary.

Corollary 2.10 *Let $\mathcal{I}$ be an interval of strictly positive measure in $\mathbb{R}$, let $\mathcal{C}$ be a countable subset of $\mathcal{I}$, let $g : \mathcal{I} \longrightarrow \mathbb{R}$ be a continuous function, and let $D^* \in \{D^+, D_+, D^-, D_-\}$ be a Dini derivative. Then:*

$$\begin{aligned}
&D^*g(x) \geq 0 \ \text{ for all } x \in \mathcal{I} \setminus \mathcal{C} \ \text{ implies that } g \text{ is monotonically increasing on } \mathcal{I}.\\
&D^*g(x) > 0 \ \text{ for all } x \in \mathcal{I} \setminus \mathcal{C} \ \text{ implies that } g \text{ is strictly monotonically increasing on } \mathcal{I}.\\
&D^*g(x) \leq 0 \ \text{ for all } x \in \mathcal{I} \setminus \mathcal{C} \ \text{ implies that } g \text{ is monotonically decreasing on } \mathcal{I}.\\
&D^*g(x) < 0 \ \text{ for all } x \in \mathcal{I} \setminus \mathcal{C} \ \text{ implies that } g \text{ is strictly monotonically decreasing on } \mathcal{I}.
\end{aligned}$$

■

Theses results on Dini derivatives will suffice for our needs.

2.3 Use of convolution to smooth functions

The support of a function $g : \mathbb{R}^n \longrightarrow \mathbb{R}$, denoted by $\mathrm{supp}(g)$, is defined as the closure of the set $\{\mathbf{x} \in \mathbb{R}^n \big| g(\mathbf{x}) \neq 0\}$. A common and convenient method to smooth a locally integrable function $\mathbf{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is given by a convolution with a nonnegative $\mathcal{C}^\infty$ function g, that has a compact support and whose integral is equal to one. The so constructed function

$$\widetilde{\mathbf{f}}(\mathbf{x}) := \int_{\mathbb{R}^n} g(\mathbf{x}-\mathbf{y})\mathbf{f}(\mathbf{y})d^n y = \int_{\mathbb{R}^n} g(\mathbf{y})\mathbf{f}(\mathbf{x}-\mathbf{y})d^n y$$

inherits the smoothness from the function g. If, additionally, $\mathbf{f}$ is continuous, we have

$$\widetilde{f}_i(\mathbf{x}) = f_i(\mathbf{z}_i)$$

for some $\mathbf{z}_i \in \mathbf{x} + \mathrm{supp}(g)$ for every $i = 1, 2, \ldots, n$, so $\widetilde{\mathbf{f}}$ has similar function values to $\mathbf{f}$ if the support of g is small enough. A basic example of such a nonnegative $\mathcal{C}^\infty$ function with a compact support is the Sobolev function.

Definition 2.11 (Sobolev function) *On $\mathbb{R}^n$ the Sobolev function $\rho_n : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ is defined by*

$$\rho_n(\mathbf{x}) := \begin{cases} C_n^{-1} \exp\left(\frac{-1}{1-\|\mathbf{x}\|_2^2}\right), & \text{if } \|\mathbf{x}\|_2 < 1, \\ 0, & \text{else,} \end{cases}$$

where

$$C_n := \int_{\|\mathbf{x}\|_2<1} \exp\left(\frac{-1}{1-\|\mathbf{x}\|_2^2}\right) d^n x.$$

□

Because

$$\lim_{y\to 1-} \frac{\exp\left(\frac{-1}{1-y^2}\right)}{(1-y^2)^k} = 0$$

for any $k \in \mathbb{N}_{\geq 0}$, we have $\rho_n \in \mathcal{C}^\infty(\mathbb{R}^n)$ and obviously $\mathrm{supp}(\rho_n) = \overline{B_{\|\cdot\|_2,1}(\mathbf{0})}$. Further, by the definition of the constant C_n, we clearly have $\int_{\mathbb{R}^n} \rho_n(\mathbf{x})d^n x = 1$.

We will use convolutions with nonnegative $\mathcal{C}^\infty$ functions that have a compact support later on to smooth functions. The first results of this kind, which we will use later in this thesis, are delivered by the next lemma.

Lemma 2.12 *Let $f : \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{\geq 0}$ be a monotonically decreasing function. Then there exists a function $g : \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{>0}$ with the following properties:*

i) $g \in \mathcal{C}^\infty(\mathbb{R}_{>0})$.

ii) $g(x) > f(x)$ *for all* $x \in \mathbb{R}_{>0}$.

iii) *g is strictly monotonically decreasing.*

iv) $\lim_{x\to 0+} g(x) = +\infty$ *and* $\lim_{x\to+\infty} g(x) = \lim_{x\to+\infty} f(x)$.

v) *g is invertible and* $g^{-1} \in \mathcal{C}^\infty(g(\mathbb{R}_{>0}))$.

Proof:
We define the function $\widetilde{h} : \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{>0}$ by,

$$\widetilde{h}(x) := \begin{cases} f\left(\frac{1}{n+1}\right) + \frac{1}{x}, & \text{if } x \in [\frac{1}{n+1}, \frac{1}{n}[\text{ for some } n \in \mathbb{N}_{>0}, \\ f(n) + \frac{1}{x}, & \text{if } x \in [n, n+1[\text{ for some } n \in \mathbb{N}_{>0}, \end{cases}$$

and the function $h : \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{>0}$ by

$$h(x) := \widetilde{h}(x - \tanh(x)).$$

Then h is a strictly monotonically decreasing measurable function and because $\widetilde{h}$ is, by its definition, strictly monotonically decreasing and larger than f, we have

$$h(x + \tanh(x)) = \widetilde{h}(x + \tanh(x) - \tanh(x + \tanh(x))) > \widetilde{h}(x) > f(x)$$

for all $x \in \mathbb{R}_{>0}$.
We claim that the function $g : \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{>0}$,

$$g(x) := \int_{x-\tanh(x)}^{x+\tanh(x)} \rho_1\left(\frac{x-y}{\tanh(x)}\right)\frac{h(y)}{\tanh(x)}dy = \int_{-1}^{1} \rho_1(y)h(x - y\tanh(x))dy,$$

where ρ_1 is the one-dimensional Sobolev function, fulfills the properties *i) - v)*.
Proposition *i)* follows from elementary Lebesgue integration theory. Proposition *ii)* follows from

$$g(x) = \int_{-1}^{1} \rho_1(y)h(x - y\tanh(x))dy > \int_{-1}^{1} \rho_1(y)h(x + \tanh(x))dy > \int_{-1}^{1} \rho_1(y)f(x)dy = f(x).$$

To see that g is strictly monotonically decreasing let $t > s > 0$ and consider that

$$t - y\tanh(t) > s - y\tanh(s) \tag{2.2}$$

for all y in the interval $[-1, 1]$. Inequality (2.2) follows from

$$\begin{aligned} t - y\tanh(t) - [s - y\tanh(s)] &= t - s - y[\tanh(t) - \tanh(s)] \\ &= t - s - y(t-s)(1 - \tanh^2(s + \vartheta_{t,s}(t-s))) \\ &> 0, \end{aligned}$$

for some $\vartheta_{t,s} \in [0, 1]$, where we used the Mean-value theorem. But then

$$h(t - y\tanh(t)) < h(s - y\tanh(s))$$

for all $y \in [-1, 1]$ and the definition of g implies that $g(t) < g(s)$. Thus, proposition *iii)* is fulfilled.
Proposition *iv)* is obvious from the definition of g. Clearly g is invertible and by the chain rule

$$[g^{-1}]'(x) = \frac{1}{g'(g^{-1}(x))},$$

so it follows by mathematical induction that $g^{-1} \in \mathcal{C}^\infty(g(\mathbb{R}_{>0}))$, that is, proposition *v)*.

■

We will use the last lemma to give a relatively short proof of Massera's lemma later on (Lemma 3.1). Another useful results implied by Lemma 2.12 is given by the next lemma.

Lemma 2.13 *Let $\alpha \in \mathcal{K}$. Then, for every $R > 0$, there is a function $\beta_R \in \mathcal{K}$, such that:*

i) β_R is a convex function.

ii) β_R restricted to $\mathbb{R}_{>0}$ is infinitely differentiable.

iii) For all $0 \leq x \leq R$ we have $\beta_R(x) \leq \alpha(x)$.

PROOF:
By Lemma 2.12 there is a function g, such that $g \in \mathcal{C}^\infty(\mathbb{R}_{>0})$, $g(x) > 1/\alpha(x)$ for all $x > 0$, $\lim_{t\to 0+} g(x) = +\infty$, and g is strictly monotonically decreasing. Then $\beta_R : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$, defined through

$$\beta_R(x) := \frac{1}{R}\int_0^x \frac{d\tau}{g(\tau)},$$

is the desired function.

First, $\beta_R(0) = \alpha(0) = 0$ and for every $0 < x \leq R$ we have

$$\beta_R(x) = \frac{1}{R}\int_0^x \frac{d\tau}{g(\tau)} \leq \frac{1}{g(x)} < \alpha(x).$$

Second, to prove that β_R is a convex $\mathcal{K}$ function is suffices to prove that the second derivative of β_R is strictly positive. But this follows immediately because for every $x > 0$ we have $g'(x) < 0$, which implies

$$\frac{d^2\beta_R}{dx^2}(x) = \frac{-g'(x)}{R[g(x)]^2} > 0.$$

■

Another results we will take advantage of later on is given in the following lemma.

Lemma 2.14 *Let $m \geq 2$ be an integer and let $-\infty =: t_1 < t_2 < \ldots < t_{m-1} < t_m := +\infty$ be elements of the compactified real-line. Let $\mathbf{a}_i, \mathbf{b}_i$, $i = 1, 2, \ldots, m-1$, be vectors in $\mathbb{R}^n$ such that the piecewise affine function*

$$\mathbf{p}(t) := \mathbf{a}_i t + \mathbf{b}_i \quad \textit{if} \;\; t_i < t \leq t_{i+1}$$

for all $i = 1, 2, \ldots, m-2$, and

$$\mathbf{p}(t) := \mathbf{a}_{m-1} t + \mathbf{b}_{m-1} \quad \textit{if} \;\; t_{m-1} < t < t_m$$

is continuous. Then, for every nonnegative function $\rho \in \mathcal{C}^\infty(\mathbb{R})$ with a compact support and such that $\int_\mathbb{R} \rho(\tau)d\tau = 1$, we have for any norm $\|\cdot\|$ on $\mathbb{R}^n$ that

$$\|\frac{d}{dt}\int_\mathbb{R} \rho(t-\tau)\mathbf{p}(\tau)d\tau\| \leq \sum_{i=1}^{m-1} \|\mathbf{a}_i\|.$$

Proof:
By partial integration

$$\frac{d}{dt}\int_{\mathbb{R}}\rho(t-\tau)\mathbf{p}(\tau)d\tau = \sum_{i=1}^{m-1}\int_{t_i}^{t_{i+1}}\rho'(t-\tau)(\mathbf{a}_i\tau+\mathbf{b}_i)d\tau$$
$$= \sum_{i=1}^{m-1}\left(-\rho(t-\tau)(\mathbf{a}_i\tau+\mathbf{b}_i)\Big|_{\tau=t_i}^{t_{i+1}} + \int_{t_i}^{t_{i+1}}\rho(t-\tau)\mathbf{a}_i d\tau\right).$$

Because $\mathbf{p}$ is continuous we have $\mathbf{a}_i t_i + \mathbf{b}_i = \mathbf{a}_{i+1}t_i + \mathbf{b}_{i+1}$ for all $i = 2, 3, \ldots, m-2$ and because ρ has a compact support $\rho(t-t_1) = \rho(t-t_m) = 0$. Therefore

$$\|\frac{d}{dt}\int_{\mathbb{R}}\rho(t-\tau)\mathbf{p}(\tau)d\tau\| \leq \sum_{i=1}^{m-1}\|\mathbf{a}_i\|\int_{t_i}^{t_{i+1}}\rho(t-\tau)d\tau \leq \sum_{i=1}^{m-1}\|\mathbf{a}_i\|.$$

■

2.4 Direct method of Lyapunov

The Russian mathematician and engineer Alexandr Mikhailovich Lyapunov published a revolutionary work in 1892 on the stability of motion, where he introduced two methods to study the stability of general continuous dynamical systems. An English translation of this work can be found in [32].

The more important of these two methods, known as *Lyapunov's second method* or *Lyapunov's direct method*, enables one to prove the stability of an equilibrium of (1.2) without integrating the differential equation. It states, that if $\mathbf{y} = \mathbf{0}$ is an equilibrium point of the system, $V \in \mathcal{C}^1(\mathbb{R}_{\geq 0} \times \mathcal{U})$ is a *positive definite function*, that is, there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}$ such that

$$\alpha_1(\|\mathbf{x}\|_2) \leq V(t,\mathbf{x}) \leq \alpha_2(\|\mathbf{x}\|_2)$$

for all $\mathbf{x} \in \mathcal{U}$ and all $t \in \mathbb{R}_{\geq 0}$, and $\boldsymbol{\phi}$ is the solution to the ordinary differential equation (1.2). Then the equilibrium is uniformly asymptotically stable, if there is an $\omega \in \mathcal{K}$ such that the inequality

$$\frac{d}{dt}V(t,\boldsymbol{\phi}(t,t_0,\boldsymbol{\xi})) = [\nabla_{\mathbf{x}}V](t,\boldsymbol{\phi}(t,t_0,\boldsymbol{\xi}))\cdot\mathbf{f}(t,\boldsymbol{\phi}(t,t_0,\boldsymbol{\xi})) + \frac{\partial V}{\partial t}(t,\boldsymbol{\phi}(t,t_0,\boldsymbol{\xi})) \leq -\omega(\|\boldsymbol{\phi}(t,t_0,\boldsymbol{\xi})\|_2) \tag{2.3}$$

holds true for all $\boldsymbol{\phi}(t,t_0,\boldsymbol{\xi})$ in an open neighborhood $\mathcal{N} \subset \mathcal{U}$ of the equilibrium $\mathbf{y}$. In this case the equilibrium is uniformly asymptotically stable on a neighborhood, which depends on V, of the origin. The function V satisfying (2.3) is said to be a *Lyapunov function* for (1.2). The direct method of Lyapunov is covered in practically all modern textbooks on nonlinear systems and control theory. Some good examples are [14], [15], [24], [50], [56], and [61].

In this section we are going to prove, that if the time-derivative in the inequalities above is replaced with a Dini derivative with respect to t, then the assumption $V \in \mathcal{C}^1(\mathbb{R}_{\geq 0} \times \mathcal{U})$ can be replaced with the less restrictive assumption, that V is merely continuous. The same is done in Theorem 42.5 in [14], but a lot of details are left out. Further, we generalize the results to arbitrary switched systems of the form Switched System 1.9.

Before we state and proof the direct method of Lyapunov for switched systems, we proof a lemma that we use in its proof.

Lemma 2.15 *Assume that the origin is an equilibrium of the Switched System 1.9 and let* $\|\cdot\|$ *be a norm on* $\mathbb{R}^n$*. Further, assume that there is a function* $\alpha \in \mathcal{K}$*, such that for all* $\sigma \in \mathcal{S_P}$ *and all* $t \geq t_0 \geq 0$ *the inequality*

$$\|\boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi})\| \leq \alpha(\|\boldsymbol{\xi}\|) \tag{2.4}$$

holds true for all $\boldsymbol{\xi}$ *in some bounded neighborhood* $\mathcal{N} \subset \mathcal{U}$ *of the origin.*

Under these assumptions the following two propositions are equivalent:

i) There exists a function $\beta \in \mathcal{L}$*, such that*

$$\|\boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi})\| \leq \sqrt{\alpha(\|\boldsymbol{\xi}\|)}\beta(t - t_0)$$

for all $\sigma \in \mathcal{S_P}$*, all* $t \geq t_0 \geq 0$*, and all* $\boldsymbol{\xi} \in \mathcal{N}$*.*

ii) For every $\varepsilon > 0$ *there exists a* $T > 0$*, such that for every* $t_0 \geq 0$*, every* $\sigma \in \mathcal{S_P}$*, and every* $\boldsymbol{\xi} \in \mathcal{N}$ *the inequality*

$$\|\boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi})\| \leq \varepsilon$$

holds true for all $t \geq T + t_0$*.*

PROOF:
Let $R > 0$ be so large that $\mathcal{N} \subset \mathcal{B}_{\|\cdot\|,R}$ and set $C := \max\{1, \alpha(R)\}$.

Proposition *i)* implies proposition *ii)*:

For every $\varepsilon > 0$ we set $T := \beta^{-1}(\varepsilon/\sqrt{\alpha(R)})$ and proposition *ii)* follows immediately.

Proposition *ii)* implies proposition *i)*:

For every $\varepsilon > 0$ define $\widetilde{T}(\varepsilon)$ as the infimum of all $T > 0$ with the property, that for every $t_0 \geq 0$, every $\sigma \in \mathcal{S_P}$, and every $\boldsymbol{\xi} \in \mathcal{N}$ the inequality

$$\|\boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi})\| \leq \varepsilon$$

holds true for all $t \geq T + t_0$.

Then $\widetilde{T}$ is a monotonically decreasing function $\mathbb{R}_{>0} \longrightarrow \mathbb{R}_{\geq 0}$ and, because of (2.4), $\widetilde{T}(\varepsilon) = 0$ for all $\varepsilon > \alpha(R)$. By Lemma 2.12 there exists a strictly monotonically decreasing $\mathcal{C}^\infty(\mathbb{R}_{>0})$ bijective function $g : \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{>0}$, such that $g(\varepsilon) > \widetilde{T}(\varepsilon)$ for all $\varepsilon > 0$. Now, for every pair $t > t_0 \geq 0$ set $\varepsilon' := g^{-1}(t - t_0)$ and note that because $t = g(\varepsilon') + t_0 \geq \widetilde{T}(\varepsilon') + t_0$ we have

$$g^{-1}(t - t_0) = \varepsilon' \geq \|\boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi})\|.$$

But then

$$\beta(s) := \begin{cases} \sqrt{2C - C/g(1) \cdot s}, & \text{if } s \in [0, g(1)], \\ \sqrt{Cg^{-1}(s)}, & \text{if } s > g(1), \end{cases}$$

is an $\mathcal{L}$ function such that

$$\sqrt{\|\boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi})\|} \leq \beta(t - t_0),$$

for all $t \geq t_0 \geq 0$ and all $\boldsymbol{\xi} \in \mathcal{N}$, and therefore

$$\|\boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi})\| \leq \sqrt{\alpha(\|\boldsymbol{\xi}\|)}\beta(t - t_0).$$

■

We come to the main theorem of this chapter, the direct method of Lyapunov for arbitrary switched systems.

Theorem 2.16 (Direct Method of Lyapunov for arbitrary switched systems) *Assume that the Switched System 1.9 has an equilibrium at the origin. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and let $R > 0$ be a constant such that the closure of the ball $\mathcal{B}_{\|\cdot\|,R}$ is a subset of $\mathcal{U}$. Let $V : \mathbb{R}_{\geq 0} \times \mathcal{B}_{\|\cdot\|,R} \longrightarrow \mathbb{R}$ be a continuous function and assume that there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}$ such that*

$$\alpha_1(\|\boldsymbol{\xi}\|) \leq V(t, \boldsymbol{\xi}) \leq \alpha_2(\|\boldsymbol{\xi}\|)$$

for all $t \geq 0$ and all $\boldsymbol{\xi} \in \mathcal{U}$. Denote the solution to the Switched System 1.9 by $\boldsymbol{\phi}$ and set $d := \alpha_2^{-1}(\alpha_1(R))$. Finally, let $D^* \in \{D^+, D_+, D^-, D_-\}$ be a Dini derivative with respect to the time t, which means, for example with $D^* = D^+$, that

$$D^+[V(t, \boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi}))] := \limsup_{h \to 0+} \frac{V(t+h, \boldsymbol{\phi}_\sigma(t+h, t_0, \boldsymbol{\xi})) - V(t, \boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi}))}{h}.$$

Then the following propositions are true:

i) If for every $\sigma \in \mathcal{S}_\mathcal{P}$, every $\boldsymbol{\xi} \in \mathcal{U}$, and every $t \geq t_0 \geq 0$, such that $\boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi}) \in \mathcal{B}_{\|\cdot\|,R}$, the inequality

$$D^*[V(t, \boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi}))] \leq 0 \tag{2.5}$$

holds true, then the origin is a uniformly stable equilibrium of the Switched System 1.9 on $\mathcal{B}_{\|\cdot\|,d}$.

ii) If there exists a function $\psi \in \mathcal{K}$, with the property that for every $\sigma \in \mathcal{S}_\mathcal{P}$, every $\boldsymbol{\xi} \in \mathcal{U}$, and every $t \geq t_0 \geq 0$, such that $\boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi}) \in \mathcal{B}_{\|\cdot\|,R}$, the inequality

$$D^*[V(t, \boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi}))] \leq -\psi(\|\boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi})\|) \tag{2.6}$$

holds true, then the origin is a uniformly asymptotically stable equilibrium of the Switched System 1.9 on $\mathcal{B}_{\|\cdot\|,d}$.

Proof:
Proposition *i)*:
Let $t_0 \geq 0$, $\boldsymbol{\xi} \in \mathcal{B}_{\|\cdot\|,d}$, and $\sigma \in \mathcal{S}_\mathcal{P}$ all be arbitrary but fixed. By the note after Definition 1.8 either $\boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi}) \in \mathcal{B}_{\|\cdot\|,R}$ for all $t \geq t_0$ or there is a $t^* > t_0$ such that $\boldsymbol{\phi}_\sigma(s, t_0, \boldsymbol{\xi}) \in \mathcal{B}_{\|\cdot\|,R}$ for all $s \in [t_0, t^*[$ and $\boldsymbol{\phi}_\sigma(t^*, t_0, \boldsymbol{\xi}) \in \partial\mathcal{B}_{\|\cdot\|,R}$. Assume that the second possibility applies. Then, by inequality (2.5) and Corollary 2.10

$$\alpha_1(R) \leq V(t^*, \boldsymbol{\phi}_\sigma(t^*, t_0, \boldsymbol{\xi})) \leq V(t_0, \boldsymbol{\xi}) \leq \alpha_2(\|\boldsymbol{\xi}\|) < \alpha_2(d),$$

which is contradictory to $d = \alpha_2^{-1}(\alpha_1(R))$. Therefore $\boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi}) \in \mathcal{B}_{\|\cdot\|,R}$ for all $t \geq t_0$.

But then it follows by inequality (2.5) and Corollary 2.10 that

$$\alpha_1(\|\boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi})\|) \leq V(t, \boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi})) \leq V(t_0, \boldsymbol{\xi}) \leq \alpha_2(\|\boldsymbol{\xi}\|),$$

for all $t \geq t_0$, so

$$\|\boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi})\| \leq \alpha_1^{-1}(\alpha_2(\|\boldsymbol{\xi}\|))$$

for all $t \geq t_0$. Because $\alpha_1^{-1} \circ \alpha_2$ is a class $\mathcal{K}$ function, it follows, because $t_0 \geq 0$, $\boldsymbol{\xi} \in \mathcal{B}_{\|\cdot\|,d}$, and $\sigma \in \mathcal{S}_\mathcal{P}$ were arbitrary, that the equilibrium at the origin is a uniformly stable equilibrium point of the Switched System 1.9 on $\mathcal{B}_{\|\cdot\|,d}$.

Proposition *ii)*:
Inequality (2.6) implies inequality (2.5) so Lemma 2.15 applies and it suffices to show that for every $\varepsilon > 0$ there is a finite $T > 0$, such that

$$t \geq T + t_0 \ \text{ implies } \ \|\boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi})\| \leq \varepsilon \tag{2.7}$$

for all $t_0 \geq 0$, all $\boldsymbol{\xi} \in \mathcal{B}_{\|\cdot\|,d}$, and all $\sigma \in \mathcal{S}_\mathcal{P}$. To prove this choose an arbitrary $\varepsilon > 0$ and set

$$\delta^* := \min\{d, \alpha_2^{-1}(\alpha_1(\varepsilon))\} \ \text{ and } \ T := \frac{\alpha_2(d)}{\psi(\delta^*)}.$$

We first prove that for every $\sigma \in \mathcal{S}_\mathcal{P}$,

$$\boldsymbol{\xi} \in \mathcal{B}_{\|\cdot\|,d} \ \text{ and } \ t_0 \geq 0 \ \text{ implies } \ \|\boldsymbol{\phi}_\sigma(t^*, t_0, \boldsymbol{\xi})\| < \delta^* \tag{2.8}$$

for some $t^* \in [t_0, T + t_0]$. We prove (2.8) by contradiction. Assume that

$$\|\boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi})\| \geq \delta^* \tag{2.9}$$

for all $t \in [t_0, T + t_0]$. Then

$$0 < \alpha_1(\delta^*) \leq \alpha_1(\|\boldsymbol{\phi}_\sigma(T + t_0, t_0, \boldsymbol{\xi})\|) \leq V(T + t_0, \boldsymbol{\phi}_\sigma(T + t_0, t_0, \boldsymbol{\xi})). \tag{2.10}$$

By Theorem 2.9 and the assumption (2.9), there is an $s \in [t_0, T + t_0]$, such that

$$\begin{aligned}\frac{V(T + t_0, \boldsymbol{\phi}_\sigma(T + t_0, t_0, \boldsymbol{\xi})) - V(t_0, \boldsymbol{\xi})}{T} &\leq [D^*V](s, \boldsymbol{\phi}(s, t_0, \boldsymbol{\xi}))] \\ &\leq -\psi(\|\boldsymbol{\phi}_\sigma(s, t_0, \boldsymbol{\xi})\|) \\ &\leq -\psi(\delta^*),\end{aligned}$$

that is

$$\begin{aligned}V(T + t_0, \boldsymbol{\phi}_\sigma(T + t_0, t_0, \boldsymbol{\xi})) &\leq V(t_0, \boldsymbol{\xi}) - T\psi(\delta^*) \\ &\leq \alpha_2(\|\boldsymbol{\xi}\|) - T\psi(\delta^*) \\ &< \alpha_2(d) - T\psi(\delta^*) \\ &= \alpha_2(d) - \frac{\alpha_2(d)}{\psi(\delta^*)}\psi(\delta^*) \\ &= 0,\end{aligned}$$

which is contradictory to (2.10). Therefore proposition (2.8) is true.
Now, let t^* be as in (2.8) and let $t > T + t_0$ be arbitrary. Then, because

$$s \mapsto V(s, \boldsymbol{\phi}_\sigma(s, t_0, \boldsymbol{\xi})), \quad s \geq t_0,$$

is strictly monotonically decreasing by inequality (2.6) and Corollary 2.10, we get by (2.8), that

$$\begin{aligned}\alpha_1(\|\boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi})\|) &\leq V(t, \boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi})) \\ &\leq V(t^*, \boldsymbol{\phi}_\sigma(t^*, t_0, \boldsymbol{\xi})) \\ &\leq \alpha_2(\|\boldsymbol{\phi}_\sigma(t^*, t_0, \boldsymbol{\xi})\|) \\ &< \alpha_2(\delta^*) \\ &= \min\{\alpha_2(d), \alpha_1(\varepsilon)\} \\ &\leq \alpha_1(\varepsilon),\end{aligned}$$

and we have proved (2.7). The proposition *ii)* follows.

■

The function V in the last theorem is called a Lyapunov function for the Switched System 1.9. Because of its importance, we spend it a definition.

Definition 2.17 (Lyapunov function) *Assume that the Switched System 1.9 has an equilibrium at the origin. Denote the solution to the Switched System 1.9 by* $\boldsymbol{\phi}$ *and let* $\|\cdot\|$ *be a norm on* $\mathbb{R}^n$*. Let* $R > 0$ *be a constant such that the closure of the ball* $\mathcal{B}_{\|\cdot\|,R}$ *is a subset of* $\mathcal{U}$*. A continuous function* $V : \mathbb{R}_{\geq 0} \times \mathcal{B}_{\|\cdot\|,R} \longrightarrow \mathbb{R}$ *is called a Lyapunov function for the Switched System 1.9 on* $\mathcal{B}_{\|\cdot\|,R}$*, if and only if there exists a Dini derivative* $D^* \in \{D^+, D_+, D^-, D_-\}$ *with respect to the time* t *and functions* $\alpha_1, \alpha_2, \psi \in \mathcal{K}$ *with the properties that:*

(L1)

$$\alpha_1(\|\boldsymbol{\xi}\|) \leq V(t, \boldsymbol{\xi}) \leq \alpha_2(\|\boldsymbol{\xi}\|)$$

for all $t \geq 0$ *and all* $\boldsymbol{\xi} \in \mathcal{B}_{\|\cdot\|,R}$*.*

(L2)

$$D^*[V(t, \boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi}))] \leq -\psi(\|\boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi})\|)$$

for every $\sigma \in \mathcal{S}_\mathcal{P}$*, every* $\boldsymbol{\xi} \in \mathcal{U}$*, and every* $t \geq t_0 \geq 0$*, such that* $\boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi}) \in \mathcal{B}_{\|\cdot\|,R}$*.*

□

The Direct Method of Lyapunov (Theorem 2.16) can thus, by Definition 2.17, be rephrased as follows:

> Assume that the Switched System 1.9 has an equilibrium point at the origin and that there exists a Lyapunov function defined on the ball $\mathcal{B}_{\|\cdot\|,R}$, of which the closure is a subset of $\mathcal{U}$, for the system. Then there is a d, $0 < d < R$, such that the origin is a uniformly asymptotically stable equilibrium point of the system on the ball $\mathcal{B}_{\|\cdot\|,d}$. If the comparison functions α_1 and α_2 in the condition **(L1)** for a Lyapunov function are known, then we can take $d = \alpha_2^{-1}(\alpha_1(R))$.

Chapter 3

A converse theorem for switched systems

In the last chapter we proved, that the existence of a Lyapunov function V for the Switched System 1.9 is a sufficient condition for the uniform asymptotic stability of an equilibrium at the origin. In this chapter we prove the converse of this theorem. That is, if the origin of the Switched System 1.9 is uniformly asymptotically stable, then there exists a Lyapunov function for the system. Because the algorithm we present in Chapter 7 to actually construct Lyapunov functions for switched systems demands the second-order derivatives of the Lyapunov functions to be bounded, we have to prove that there exists a Lyapunov function that complies with these requirements. As an added bonus we even prove that there exists a Lyapunov function for the switched system that is continuously differentiable up to any order.

3.1 Some notes on converse theorems

There are several theorems known, similar to Theorem 2.16, where one either uses more or less restrictive assumptions regarding the Lyapunov function than in Theorem 2.16. Such theorems are often called *Lyapunov-like* theorems. An example for less restrictive assumptions is Theorem 46.5 in [14] or equivalently Theorem 4.10 in [24], where the solution to a continuous system is shown to be uniformly bounded, and an example for more restrictive assumptions is Theorem 5.17 in [50], where an equilibrium is proved to be uniformly exponentially stable. The Lyapunov-like theorems all have the form :

> If one can find a function V for a dynamical system, such that V satisfies the properties X, then the system has the stability property Y.

A natural question awakened by any Lyapunov-like theorem is whether its converse is true or not, that is, if there is a corresponding theorem of the form :

> If a dynamical system has the stability property Y, then there exists a function V for the dynamical system, such that V satisfies the properties X.

Such theorems are called *converse theorems* in the Lyapunov stability theory. For nonlinear systems they are more complicated than the direct method of Lyapunov and the results came rather late and did not stem from Lyapunov himself. The converse theorems are covered quite thoroughly in

Chapter VI in [14]. Some further general references are Section 5.7 in [56] and Section 4.3 in [24]. More specific references were given in *Outline of this thesis*.

About the techniques to prove such theorems Hahn writes on page 225 in his book *Stability of Motion* [14] :

> In the converse theorems the stability behavior of a family of motions $\mathbf{p}(t, \mathbf{a}, t_0)$[1] is assumed to be known. For example, it might be assumed that the expression $\|\mathbf{p}(t, \mathbf{a}, t_0)\|$ is estimated by known comparison functions (secs. 35 and 36). Then one attempts to construct by means of a finite or transfinite procedure, a Lyapunov function which satisfies the conditions of the stability theorem under consideration.

In this chapter we will proof a converse theorem on uniform asymptotic stability of an arbitrary switched system's equilibrium, where the functions $\mathbf{f}_p$, $p \in \mathcal{P}$, of the systems $\dot{\mathbf{x}} = \mathbf{f}_p(t, \mathbf{x})$, are Lipschitz on $\mathbb{R}_{\geq 0} \times \mathcal{B}_{\|\cdot\|,R}$ with a common Lipschitz constant $L > 0$. To construct a Lyapunov function that is merely Lipschitz in its state-space argument, it suffices that the functions $\mathbf{f}_p$, $p \in \mathcal{P}$, are locally Lipschitz in the state-space argument uniformly in their first arguments (time) with a common Lipschitz constant $L_{\mathbf{x}} > 0$ on $\mathcal{B}_{\|\cdot\|,R}$, as shown in Theorem 3.3. Our procedure to smooth it to a $\mathcal{C}^\infty$ function, as done in Theorem 3.10, does not necessarily work if $(t, s) \mapsto \|\mathbf{f}_p(t, \mathbf{x}) - \mathbf{f}_p(s, \mathbf{x})\|/|t - s|$, $s \neq t$, is unbounded. Therefore we additionally have to assume that there exists a constant $L_t > 0$ such that $\|\mathbf{f}_p(t, \mathbf{x}) - \mathbf{f}_p(s, \mathbf{x})\| \leq L_t|t - s|$ for all $p \in \mathcal{P}$, all $t, s \in \mathbb{R}_{\geq 0}$ and all $\mathbf{x} \in \mathcal{B}_{\|\cdot\|,R}$. Note, that this additional assumption does not affect the "growth" of the functions $\mathbf{f}_p$, $p \in \mathcal{P}$, but merely excludes "infinitely fast oscillations" in the temporal domain. The "locally Lipschitz in the state-space argument uniformly in the time argument" assumption already takes care of that $\|\mathbf{f}_p(t, \mathbf{x})\|_p \leq L_{\mathbf{x}}\|\mathbf{x}\| \leq L_{\mathbf{x}}R$ for all $t \geq 0$ and all $\mathbf{x} \in \mathcal{B}_{\|\cdot\|,R}$ because $\mathbf{f}_p(t, \mathbf{0}) = \mathbf{0}$.

3.2 A converse theorem for arbitrary switched systems

The construction here of a smooth Lyapunov function for the Switched System 1.9 is long, complicated, and technical. We will therefore arrange the proof in a series of lemmas and theorems. As a preparation for the construction of a concrete Lipschitz Lyapunov function (Definition 3.2) for the Switched System 1.9 (it is proved in Theorem 3.3 and Lemma 3.4 that it actually is a Lipschitz Lyapunov function) we state and prove the next lemma.

Lemma 3.1 (Massera's lemma) *Let $f \in \mathcal{L}$ and $\lambda \in \mathbb{R}_{>0}$. Then there is a function $g \in \mathcal{C}^1(\mathbb{R}_{\geq 0})$, such that $g, g' \in \mathcal{K}$, g restricted to $\mathbb{R}_{>0}$ is a $\mathcal{C}^\infty(\mathbb{R}_{>0})$ function,*

$$\int_0^{+\infty} g(f(t))dt < +\infty, \quad \text{and} \quad \int_0^{+\infty} g'(f(t))e^{\lambda t}dt < +\infty.$$

PROOF:
By Lemma 2.12 there is a strictly monotonically decreasing $\mathcal{C}^\infty(\mathbb{R}_{>0})$ bijective function $h : \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{>0}$ such that $h(x) > f(x)$ for all $x > 0$ and $h^{-1} \in \mathcal{C}^\infty(\mathbb{R}_{>0})$. We define the function $g : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ by

$$g(t) := \int_0^t e^{-(1+\lambda)h^{-1}(\tau)}d\tau$$

and prove that it fulfills all the claimed properties.

[1] In our notation $\mathbf{p}(t, \mathbf{a}, t_0) = \boldsymbol{\phi}(t, t_0, \mathbf{a})$.

i) Clearly $g \in \mathcal{C}^1(\mathbb{R}_{\geq 0})$, $g, g' \in \mathcal{K}$, and g is continuously differentiable up to any order at every point larger than zero.

ii) We have

$$\int_1^{+\infty} g(f(t))dt = \int_1^{+\infty} \int_0^{f(t)} e^{-(1+\lambda)h^{-1}(\tau)} d\tau dt \leq \int_1^{+\infty} \int_0^{h(t)} e^{-(1+\lambda)h^{-1}(\tau)} d\tau dt$$

and because for $0 < \tau \leq h(t)$ we have $h^{-1}(\tau) \geq t$, we get

$$\int_0^{h(t)} e^{-(1+\lambda)h^{-1}(\tau)} d\tau \leq \int_0^{h(t)} e^{-(1+\lambda)t} d\tau = h(t)e^{-(1+\lambda)t} \leq h(1)e^{-(1+\lambda)t}$$

for all $t \geq 1$. Hence

$$\int_0^{+\infty} g(f(t))dt \leq \int_0^1 g(f(t))dt + \int_1^{+\infty} h(1)e^{-(1+\lambda)t} dt < +\infty$$

and the the first integral of the lemma is bounded.

iii) Because $h^{-1}(f(t)) > h^{-1}(h(t)) = t$ for all $t > 0$, we have

$$g'(f(t)) = e^{-(1+\lambda)h^{-1}(f(t))} < e^{-(1+\lambda)t}$$

for all $t \geq 0$. Hence,

$$\int_0^{+\infty} g'(f(t))e^{\lambda t} dt < \int_0^{+\infty} e^{-t} dt < +\infty,$$

so the second integral of the lemma is bounded too.

■

Note, that because $g, g' \in \mathcal{K}$ in Massera's lemma above, we have for every measurable function $u : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$, such that $u(t) \leq f(t)$ for all $t \in \mathbb{R}_{\geq 0}$, that

$$\int_0^{+\infty} g(u(t))dt \leq \int_0^{+\infty} g(f(t))dt \quad \text{and} \quad \int_0^{+\infty} g'(u(t))e^{\lambda t} dt \leq \int_0^{+\infty} g'(f(t))e^{\lambda t} dt.$$

We now use the results from Massera's lemma (Lemma 3.1) to define the functions W_σ, $\sigma \in \mathcal{S}_\mathcal{P}$, and them in turn to define the function W, and after that we prove that the function W is a Lipschitz Lyapunov function for the Switched System 1.9.

Definition 3.2 (The functions W_σ and W) *Assume that the origin is a uniformly asymptotically stable equilibrium point of the Switched System 1.9 on the ball $\mathcal{B}_{\|\cdot\|,R} \subset \mathcal{U}$, where $\|\cdot\|$ is a norm on $\mathbb{R}^n$ and $R > 0$, and let $\varsigma \in \mathcal{KL}$ be such that $\|\boldsymbol{\phi}_\sigma(t, t_0, \boldsymbol{\xi})\| \leq \varsigma(\|\boldsymbol{\xi}\|, t - t_0)$ for all $\sigma \in \mathcal{S}_\mathcal{P}$, all $\boldsymbol{\xi} \in \mathcal{B}_{\|\cdot\|,R}$, and all $t \geq t_0 \geq 0$. Assume further, that there exists a constant $L > 0$ for the functions $\mathbf{f}_p$, such that*

$$\|\mathbf{f}_p(t, \mathbf{x}) - \mathbf{f}_p(t, \mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\|$$

for all $t \geq 0$, all $\mathbf{x}, \mathbf{y} \in \mathcal{B}_{\|\cdot\|,R}$, and all $p \in \mathcal{P}$. By Massera's lemma (Lemma 3.1) there exists a function $g \in \mathcal{C}^1(\mathbb{R}_{\geq 0})$, such that $g, g' \in \mathcal{K}$, g is infinitely differentiable on $\mathbb{R}_{>0}$,

$$\int_0^{+\infty} g(\varsigma(R, \tau))d\tau < +\infty, \quad \textit{and} \quad \int_0^{+\infty} g'(\varsigma(R, \tau))e^{L\tau} d\tau < +\infty.$$

i) For every $\sigma \in \mathcal{S}_\mathcal{P}$ we define the function W_σ for all $t \geq 0$ and all $\boldsymbol{\xi} \in \mathcal{B}_{\|\cdot\|,R}$ by

$$W_\sigma(t,\boldsymbol{\xi}) := \int_t^{+\infty} g(\|\boldsymbol{\phi}_\sigma(\tau,t,\boldsymbol{\xi})\|)d\tau.$$

ii) We define the function W for all $t \geq 0$ and all $\boldsymbol{\xi} \in \mathcal{B}_{\|\cdot\|,R}$ by

$$W(t,\boldsymbol{\xi}) := \sup_{\sigma\in\mathcal{S}_\mathcal{P}} W_\sigma(t,\boldsymbol{\xi}).$$

Note, that if the Switched System 1.9 is autonomous, then W does not depend on t, that is, it is time-invariant.

□

The function W from the definition above (Definition 3.2) is a Lipschitz Lyapunov function for the Switched System 1.9 used for its construction. This is proved in the next theorem.

Theorem 3.3 (W is a Lipschitz Lyapunov function) *The function W in Definition 3.2 is a Lyapunov function for the Switched System 1.9 used for its construction. Further, there exists a constant $L_W > 0$ such that*

$$|W(t,\boldsymbol{\xi}) - W(t,\boldsymbol{\eta})| \leq L_W\|\boldsymbol{\xi}-\boldsymbol{\eta}\| \tag{3.1}$$

for all $t \geq 0$ and all $\boldsymbol{\xi},\boldsymbol{\eta} \in \mathcal{B}_{\|\cdot\|,R}$,

where the norm $\|\cdot\|$ and the constant R are the same as in Definition 3.2.

PROOF:
We have to show that the function W complies to the conditions **(L1)** and **(L2)** of Definition 2.17. Because

$$\boldsymbol{\phi}_\sigma(u,t,\boldsymbol{\xi}) = \boldsymbol{\xi} + \int_t^u \mathbf{f}_{\sigma(\tau)}(\tau,\boldsymbol{\phi}_\sigma(\tau,t,\boldsymbol{\xi}))d\tau,$$

and $\|\mathbf{f}_{\sigma(s)}(s,\mathbf{y})\| \leq LR$ for all $s \geq 0$ and all $\mathbf{y} \in \mathcal{B}_{\|\cdot\|,R}$, we conclude $\|\boldsymbol{\phi}_\sigma(u,t,\boldsymbol{\xi})\| \geq \|\boldsymbol{\xi}\| - (u-t)LR$ for all $u \geq t \geq 0$, $\boldsymbol{\xi} \in \mathcal{B}_{\|\cdot\|,R}$, and all $\sigma \in \mathcal{S}_\mathcal{P}$. Therefore,

$$\|\boldsymbol{\phi}_\sigma(u,t,\boldsymbol{\xi})\| \geq \frac{\|\boldsymbol{\xi}\|}{2} \quad \text{whenever } t \leq u \leq t + \frac{\|\boldsymbol{\xi}\|}{2LR},$$

which implies

$$W_\sigma(t,\boldsymbol{\xi}) := \int_t^{+\infty} g(\|\boldsymbol{\phi}_\sigma(\tau,t,\boldsymbol{\xi})\|)d\tau \geq \frac{\|\boldsymbol{\xi}\|}{2LR} g(\|\boldsymbol{\xi}\|/2)$$

and then $\alpha_1(\|\boldsymbol{\xi}\|) \leq W(t,\boldsymbol{\xi})$ for all $t \geq 0$ and all $\boldsymbol{\xi} \in \mathcal{B}_{\|\cdot\|,R}$, where $\alpha_1(x) := x/(2LR)g(x/2)$ is a $\mathcal{K}$ function.

By the definition of W,

$$W(t,\boldsymbol{\xi}) \geq \int_t^{t+h} g(\|\boldsymbol{\phi}_\sigma(\tau,t,\boldsymbol{\xi})\|)d\tau + W(t+h,\boldsymbol{\phi}_\sigma(t+h,t,\boldsymbol{\xi}))$$

(reads: supremum over all trajectories emerging from $\boldsymbol{\xi}$
at time t is not less than over any particular trajectory
emerging from $\boldsymbol{\xi}$ at time t)

for all $\boldsymbol{\xi} \in \mathcal{B}_{\|\cdot\|,R}$, all $t \geq 0$, all $h > 0$, and all $\sigma \in \mathcal{S}_\mathcal{P}$, from which

$$\limsup_{h \to 0+} \frac{W(t+h, \boldsymbol{\phi}_\sigma(t+h,t,\boldsymbol{\xi})) - W(t,\boldsymbol{\xi})}{h} \leq -g(\|\boldsymbol{\xi}\|)$$

follows. Because $g \in \mathcal{K}$ this implies that the condition **(L2)** from Definition 2.17 holds true for the function W .

Now, assume that there is an $L_W > 0$ such that inequality (3.1) holds true. Then $W(t,\boldsymbol{\xi}) \leq \alpha_2(\|\boldsymbol{\xi}\|)$ for all $t \geq 0$ and all $\boldsymbol{\xi} \in \mathcal{B}_{\|\cdot\|,R}$, where $\alpha_2(x) := L_W x$ is a class $\mathcal{K}$ function. Thus, it only remains to proof inequality (3.1). However, as this inequality is a byproduct of the next lemma, we spare us the proof here.

■

Then results of the next lemma are needed in the proof of our converse theorem on uniform asymptotic stability of a switched system's equilibria and as a convenient side effect it completes the proof of Theorem 3.3.

Lemma 3.4 *The function W in Definition 3.2 satisfies for all $t \geq s \geq 0$, all $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathcal{B}_{\|\cdot\|,R}$, and all $\sigma \in \mathcal{S}_\mathcal{P}$ the inequality*

$$W(t,\boldsymbol{\xi}) - W(s,\boldsymbol{\eta}) \leq C\|\boldsymbol{\xi} - \boldsymbol{\phi}_\sigma(t,s,\boldsymbol{\eta})\| - \int_s^t g(\|\boldsymbol{\phi}_\sigma(\tau,s,\boldsymbol{\eta})\|)d\tau, \tag{3.2}$$

where

$$C := \int_0^{+\infty} g'(\varsigma(R,\tau))e^{L\tau}d\tau < +\infty.$$

Especially,

$$|W(t,\boldsymbol{\xi}) - W(t,\boldsymbol{\eta})| \leq C\|\boldsymbol{\xi} - \boldsymbol{\eta}\| \tag{3.3}$$

for all $t \geq 0$ and all $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathcal{B}_{\|\cdot\|,R}$.

The norm $\|\cdot\|$, the constants R, L, and the functions ς and g are, of course, the same as in Definition 3.2.

Proof:
By the Mean-value theorem and Theorem 1.10 we have

$$\begin{aligned}
W_\sigma(t,\boldsymbol{\xi}) - W_\sigma(s,\boldsymbol{\eta}) &= \int_t^{+\infty} g(\|\boldsymbol{\phi}_\sigma(\tau,t,\boldsymbol{\xi})\|)d\tau - \int_s^{+\infty} g(\|\boldsymbol{\phi}_\sigma(\tau,s,\boldsymbol{\eta})\|)d\tau \qquad (3.4)\\
&\leq \int_t^{+\infty} \big|g(\|\boldsymbol{\phi}_\sigma(\tau,t,\boldsymbol{\xi})\|) - g(\|\boldsymbol{\phi}_\sigma(\tau,s,\boldsymbol{\eta})\|)\big|d\tau - \int_s^t g(\|\boldsymbol{\phi}_\sigma(\tau,s,\boldsymbol{\eta})\|)d\tau\\
&= \int_t^{+\infty} \big|g(\|\boldsymbol{\phi}_\sigma(\tau,t,\boldsymbol{\xi})\|) - g(\|\boldsymbol{\phi}_\sigma(\tau,t,\boldsymbol{\phi}_\sigma(t,s,\boldsymbol{\eta}))\|)\big|d\tau - \int_s^t g(\|\boldsymbol{\phi}_\sigma(\tau,s,\boldsymbol{\eta})\|)d\tau\\
&\leq \int_t^{+\infty} g'(\varsigma(R,\tau-t))\|\boldsymbol{\phi}_\sigma(\tau,t,\boldsymbol{\xi}) - \boldsymbol{\phi}_\sigma(\tau,t,\boldsymbol{\phi}_\sigma(t,s,\boldsymbol{\eta}))\|d\tau - \int_s^t g(\|\boldsymbol{\phi}_\sigma(\tau,s,\boldsymbol{\eta})\|)d\tau\\
&\leq \int_t^{+\infty} g'(\varsigma(R,\tau-t))e^{L(\tau-t)}\|\boldsymbol{\xi} - \boldsymbol{\phi}_\sigma(t,s,\boldsymbol{\eta})\|d\tau - \int_s^t g(\|\boldsymbol{\phi}_\sigma(\tau,s,\boldsymbol{\eta})\|)d\tau\\
&\leq C\|\boldsymbol{\xi} - \boldsymbol{\phi}_\sigma(t,s,\boldsymbol{\eta})\| - \int_s^t g(\|\boldsymbol{\phi}_\sigma(\tau,s,\boldsymbol{\eta})\|)d\tau.
\end{aligned}$$

We now show that we can replace $W_\sigma(t,\boldsymbol{\xi}) - W_\sigma(s,\boldsymbol{\eta})$ with $W(t,\boldsymbol{\xi}) - W(s,\boldsymbol{\eta})$ on the leftmost side of inequality (3.4) without violating the $\leq$ relations. That this is possible might seem a little surprising at first sight. However, a closer look reveals that this is not surprising at all because the rightmost side of inequality (3.4) only depends on the values of $\sigma(z)$ for $s \leq z \leq t$ and because $W_\sigma(t,\boldsymbol{\xi}) - W(s,\boldsymbol{\eta}) \leq W_\sigma(t,\boldsymbol{\xi}) - W_\sigma(s,\boldsymbol{\eta})$, where the left-hand side only depends on the values of $\sigma(z)$ for $z \geq t$, .

To rigidly prove the validity of this replacement let $\delta > 0$ be an arbitrary constant and choose a $\gamma \in \mathcal{S}_\mathcal{P}$, such that

$$W(t,\boldsymbol{\xi}) - W_\gamma(t,\boldsymbol{\xi}) < \frac{\delta}{2}, \tag{3.5}$$

and a $u > 0$ so small that

$$ug(\varsigma(\|\boldsymbol{\xi}\|,0)) + 2CR(e^u - 1) < \frac{\delta}{2}. \tag{3.6}$$

We define $\theta \in \mathcal{S}_\mathcal{P}$ by

$$\theta(\tau) := \begin{cases} \sigma(\tau), & \text{if } 0 \leq \tau < t+u, \\ \gamma(\tau), & \text{if } \tau \geq t+u. \end{cases}$$

Then

$$\begin{aligned} W(t,\boldsymbol{\xi}) - W(s,\boldsymbol{\eta}) &\leq W(t,\boldsymbol{\xi}) - W_\theta(s,\boldsymbol{\eta}) \\ &\leq [W(t,\boldsymbol{\xi}) - W_\gamma(t,\boldsymbol{\xi})] + [W_\gamma(t,\boldsymbol{\xi}) - W_\theta(t,\boldsymbol{\xi})] + [W_\theta(t,\boldsymbol{\xi}) - W_\theta(s,\boldsymbol{\eta})]. \end{aligned} \tag{3.7}$$

By the Mean-value theorem, Theorem 1.10, and inequality (3.6)

$$\begin{aligned} W_\gamma(t,\boldsymbol{\xi}) - W_\theta(t,\boldsymbol{\xi}) &= \int_t^{t+u} [g(\|\boldsymbol{\phi}_\gamma(\tau,t,\boldsymbol{\xi})\|) - g(\|\boldsymbol{\phi}_\theta(\tau,t,\boldsymbol{\xi})\|)]d\tau \\ &\quad + \int_{t+u}^{+\infty} [g(\|\boldsymbol{\phi}_\gamma(\tau,t+u,\boldsymbol{\phi}_\gamma(t+u,t,\boldsymbol{\xi}))\|) - g(\|\boldsymbol{\phi}_\gamma(\tau,t+u,\boldsymbol{\phi}_\theta(t+u,t,\boldsymbol{\xi}))\|)]d\tau \\ &\leq ug(\varsigma(\|\boldsymbol{\xi}\|,0)) \\ &\quad + \int_{t+u}^{+\infty} g'(\varsigma(R,\tau-t-u))\|\boldsymbol{\phi}_\gamma(\tau,t+u,\boldsymbol{\phi}_\gamma(t+u,t,\boldsymbol{\xi})) - \boldsymbol{\phi}_\gamma(\tau,t+u,\boldsymbol{\phi}_\theta(t+u,t,\boldsymbol{\xi}))\|d\tau \\ &\leq ug(\varsigma(\|\boldsymbol{\xi}\|,0)) + \int_{t+u}^{+\infty} g'(\varsigma(R,\tau-t-u))e^{L(\tau-t-u)}\|\boldsymbol{\phi}_\gamma(t+u,t,\boldsymbol{\xi}) - \boldsymbol{\phi}_\theta(t+u,t,\boldsymbol{\xi})\|d\tau \\ &\leq ug(\varsigma(\|\boldsymbol{\xi}\|,0)) + \int_{t+u}^{+\infty} g'(\varsigma(R,\tau-t-u))e^{L(\tau-t-u)}2RL\frac{e^{Lu}-1}{L}d\tau \\ &= ug(\varsigma(\|\boldsymbol{\xi}\|,0)) + 2R(e^u - 1)\int_0^{+\infty} g'(\varsigma(R,\tau))e^{L\tau}d\tau. \\ &< \frac{\delta}{2}. \end{aligned} \tag{3.8}$$

Because θ and σ coincide on $[s,t]$, we get by (3.4), that

$$\begin{aligned} W_\theta(t,\boldsymbol{\xi}) - W_\theta(s,\boldsymbol{\eta}) &\leq C\|\boldsymbol{\xi} - \boldsymbol{\phi}_\theta(t,s,\boldsymbol{\eta})\| - \int_s^t g(\|\boldsymbol{\phi}_\theta(\tau,s,\boldsymbol{\eta})\|)d\tau \\ &= C\|\boldsymbol{\xi} - \boldsymbol{\phi}_\sigma(t,s,\boldsymbol{\eta})\| - \int_s^t g(\|\boldsymbol{\phi}_\sigma(\tau,s,\boldsymbol{\eta})\|)d\tau. \end{aligned} \tag{3.9}$$

Hence, by (3.7), (3.5), (3.8), and (3.9) we conclude that

$$W(t,\boldsymbol{\xi}) - W(s,\boldsymbol{\eta}) < \delta + C\|\boldsymbol{\xi} - \boldsymbol{\phi}_\sigma(t,s,\boldsymbol{\eta})\| - \int_s^t g(\|\boldsymbol{\phi}_\sigma(\tau,s,\boldsymbol{\eta})\|)d\tau$$

and because $\delta > 0$ was arbitrary we have proved inequality (3.2).

Inequality (3.3) is a trivial consequence of inequality (3.2), just set $s = t$ and note that $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ can be reversed.

■

For our proof of the converse theorem on uniform asymptotic stability for the Switched System 1.9 we need the fact, that if $\mathcal{N} \subset \mathbb{R}^n$ is a neighborhood of the origin and $f \in \mathcal{C}(\mathcal{N}) \cap \mathcal{C}^1(\mathcal{N} \setminus \{\mathbf{0}\})$, then

$$\lim_{\mathbf{x}\to\mathbf{0}} \|\mathbf{x}\|_2^2 \|\nabla f(\mathbf{x})\|_2 = 0. \tag{3.10}$$

For $n = 1$ this is very easy to prove by the use of l'Hospital's rule. For $n > 1$ it is possible to prove this in a similar manner, however, we first need an alternate characterization of convergence in $\mathbb{R}^n$, using smooth paths rather that sequences or open sets. The next theorem proves the validity of this alternate characterization of convergence, but first we prove a technical lemma as a preparation for the proof of the theorem and the proof of the limit (3.10).

Lemma 3.5 *Let $(\mathbf{x}_k)_{k\in\mathbb{N}_{>0}}$ and $(\mathbf{y}_k)_{k\in\mathbb{N}_{>0}}$ be sequences in $\mathbb{R}^n$ with the properties that:*

a) $\lim_{k\to+\infty} \mathbf{x}_k = \mathbf{0}$.

b) $\|\mathbf{y}_k\|_2 = 1$ *for every* $k \in \mathbb{N}_{>0}$.

c) $\mathbf{x}_k \cdot \mathbf{y}_k \geq 0$ *for every* $k \in \mathbb{N}_{>0}$.

Then there exists a path $\mathbf{s} :]0, a[\longrightarrow \mathbb{R}^n$, $a > 0$, with the following properties:

i) $\mathbf{s} \in [\mathcal{C}^\infty(]0,a[)]^n$.

ii) $\lim_{t\to 0+} \mathbf{s}(t) = \mathbf{0}$.

iii) $\sup_{t\in]0,a[} \|\mathbf{s}(t)\|_2 \|\mathbf{s}'(t)\|_2 < +\infty$.

iv) The set $\{k \in \mathbb{N}_{>0} \big| \mathbf{s}(\|\mathbf{x}_k\|_2) = \mathbf{x}_k$ *and* $\mathbf{s}'(\|\mathbf{x}_k\|_2) = \mathbf{y}_k\}$ *has an infinite number of elements.*

PROOF:
Set $\mathcal{X} := \{\mathbf{x}_k \big| k \in \mathbb{N}_{>0}\}$, that is, $\mathcal{X}$ is the set that contains exactly all the elements of the sequence $(\mathbf{x}_k)_{k\in\mathbb{N}_{>0}}$.

For all $\mathbf{z} \in \mathbb{R}^n$ with $\|\mathbf{z}\|_2 = 1$ we define

$$\mathcal{K}_\mathbf{z} := \left\{ \mathbf{x} \in \mathbb{R}^n \,\middle|\, \|\mathbf{x} - r\mathbf{z}\|_2 \leq \frac{r}{\sqrt{3}} \text{ for some } r \geq 0 \right\}.$$

Then $\mathcal{K}_\mathbf{z}$ is a right circular cone with the vertex at the origin and the opening angle at the vertex is $\pi/3$. Because there is a finite number $\mathcal{K}_{\mathbf{z}_1}, \mathcal{K}_{\mathbf{z}_2}, \ldots, \mathcal{K}_{\mathbf{z}_m}$ of these cones such that [2]

$$\mathcal{B}_{\|\cdot\|_2,1} \subset \bigcup_{i=1}^{m} \mathcal{K}_{\mathbf{z}_i},$$

there is a $\mathbf{z}^*$ such that $\mathcal{X} \cap \mathcal{K}_{\mathbf{z}^*}$ contains infinitely many elements. Define the sets

$$\mathcal{K}_{\mathbf{z}^*,k} := \left\{ \mathbf{x} \in \mathcal{K}_{\mathbf{z}^*} \,\middle|\, \frac{1}{k+1} < \|\mathbf{x}\|_2 \leq \frac{1}{k} \right\}$$

for all $k \in \mathbb{N}_{>0}$. Then at least one of the sets

$$\mathcal{X} \cap \bigcup_{k=1}^{+\infty} \mathcal{K}_{\mathbf{z}^*,2k} \quad \text{and} \quad \mathcal{X} \cap \bigcup_{k=1}^{+\infty} \mathcal{K}_{\mathbf{z}^*,2k-1}$$

contains an infinite number of elements. Without loss of generality we assume that $\mathcal{X} \cap \bigcup_{k=1}^{+\infty} \mathcal{K}_{\mathbf{z}^*,2k}$ is infinite. The rest of the proof would be almost identical under the alternate assumption.

We construct sequences $(\mathbf{a}_n)_{n\in\mathbb{N}_{>0}}$ and $(\mathbf{b}_n)_{n\in\mathbb{N}_{>0}}$ in $\mathcal{K}_{\mathbf{z}^*}$ in the following way: For every $k \in \mathbb{N}_{>0}$ consider the intersection $\mathcal{X} \cap \mathcal{K}_{\mathbf{y}^*,2k}$. If it is not empty there is an $i \in \mathbb{N}_{>0}$ such that $\mathbf{x}_i$ from the sequence $(\mathbf{x}_n)_{n\in\mathbb{N}_{>0}}$ is in $\mathcal{K}_{\mathbf{y}^*,2k}$ and in this case we set $\mathbf{a}_k := \mathbf{x}_i$ and $\mathbf{b}_k := \mathbf{y}_i$. If the intersection is empty we set $\mathbf{a}_k$ equal to an arbitrary element of $\mathcal{K}_{\mathbf{y}^*,2k}$ and set $\mathbf{b}_k := \mathbf{a}_k/\|\mathbf{a}_k\|_2$. By this construction there are infinitely many $k \in \mathbb{N}_{>0}$ such that $\mathbf{a}_k = \mathbf{x}_i$ and $\mathbf{b}_k = \mathbf{y}_i$ for some $i \in \mathbb{N}_{>0}$.

We now have everything we need to construct the claimed path $\mathbf{s}$. First, we construct a piecewise affine path $\tilde{\mathbf{s}} \in \mathcal{C}(]0, \|\mathbf{a}_1\|_2])$ and then we round out the corners to get $\mathbf{s}$.

For every $k \in \mathbb{N}_{>0}$ we define for $m = 0, 1, 2$ the constants $t_{k,m}$ by

$$t_{k,m} := \|\mathbf{a}_k\|_2 - \frac{m}{3}\left(\frac{1}{2k+1} - \frac{1}{2k+2}\right).$$

We define $\tilde{\mathbf{s}}$ for every $k \in \mathbb{N}_{>0}$ on the interval $]t_{k+1,0}, t_{k,0}] :=]t_{k+1,0}, t_{k,2}] \cup]t_{k,2}, t_{k,1}] \cup]t_{k,1}, t_{k,0}]$, by:

For every $t \in]t_{k+1,0}, t_{k,2}]$ we set

$$\tilde{\mathbf{s}}(t) := \mathbf{a}_{k+1} + (t - t_{k+1,0})\mathbf{b}_{k+1}.$$

For every $t \in]t_{k,2}, t_{k,1}]$ we set

$$\tilde{\mathbf{s}}(t) := \mathbf{a}_{k+1} + (t_{k,2} - t_{k+1,0})\mathbf{b}_{k+1} + (t - t_{k,2})\frac{\mathbf{a}_k - \mathbf{a}_{k+1} + (t_{k,1} - t_{k,0})\mathbf{b}_k - (t_{k,2} - t_{k+1,0})\mathbf{b}_{k+1}}{t_{k,1} - t_{k,2}}$$

and, for every $t \in]t_{k,1}, t_{k,0}]$ we set

$$\tilde{\mathbf{s}}(t) := \mathbf{a}_k + (t - t_{k,0})\mathbf{b}_k.$$

Then

$$\tilde{\mathbf{s}}(\|\mathbf{a}_k\|_2) = \tilde{\mathbf{s}}(t_{k,0}) = \mathbf{a}_k \ \text{ and } \ \tilde{\mathbf{s}}'(\|\mathbf{a}_k\|_2) = \tilde{\mathbf{s}}'(t_{k,0}) = \mathbf{b}_k \ \text{ for all } k \in \mathbb{N}_{>0}. \tag{3.11}$$

[2] Let the set $\mathcal{S}^n := \{\mathbf{x} \in \mathbb{R}^n \big| \|\mathbf{x}\|_2 = 1\}$ be equipped with its usual topology (see, for example, ğ8-8 in [16]). For every $\|\mathbf{z}\|_2 = 1$ define $\mathcal{O}_\mathbf{z}$ to be the interior of $\mathcal{K}_\mathbf{z} \cap \mathcal{S}^n$ in the $\mathcal{S}^n$ topology. Because $\mathcal{S}^n$ is compact, a finite number of the $\mathcal{O}_\mathbf{z}$ s, say $\mathcal{O}_{\mathbf{z}_1}, \mathcal{O}_{\mathbf{z}_2}, \dots, \mathcal{O}_{\mathbf{z}_m}$, suffice to cover $\mathcal{S}^n$. But then, because $\mathcal{K}_{\mathbf{z}_i} = \bigcup_{r\geq 0} r\mathcal{O}_{\mathbf{z}_i}$, the sets $\mathcal{K}_{\mathbf{z}_1}, \mathcal{K}_{\mathbf{z}_2}, \dots, \mathcal{K}_{\mathbf{z}_m}$ cover $\mathbb{R}^n$.

Further, $\tilde{\mathbf{s}}$ is continuous on $]0, t_{1,0}]$ and smooth with a possible exception of the points $t_{k,i}$, $k \in \mathbb{N}_{>0}$ and $i = 1, 2$.

Let $\rho \in \mathcal{C}^\infty(\mathbb{R})$ be a nonnegative function with $\operatorname{supp}(\rho) \subset [-1, 1]$ and $\int_\mathbb{R} \rho(\tau) d\tau = 1$. We define the smooth path $\mathbf{s} :]0, t_{1,0}[\longrightarrow \mathbb{R}^n$ by setting, whenever

$$\frac{t_{k+1,0} + t_{k,2}}{2} \leq t \leq \frac{t_{k,1} + t_{k,0}}{2}$$

for some $k \in \mathbb{N}_{>0}$,

$$\mathbf{s}(t) := \int_{t - \frac{t_{k,0}-t_{k,1}}{4}}^{t + \frac{t_{k,0}-t_{k,1}}{4}} \rho\left(\frac{4(t-\tau)}{t_{k,0} - t_{k,1}}\right) \frac{4\,\tilde{\mathbf{s}}(\tau)}{t_{k,0} - t_{k,1}} d\tau = \int_{-1}^{1} \rho(\tau)\, \tilde{\mathbf{s}}\left(t - \tau \frac{t_{k,0} - t_{k,1}}{4}\right) d\tau,$$

and we set $\mathbf{s}(t) := \tilde{\mathbf{s}}(t)$ otherwise. Then, as discussed in Section 2.3, $\mathbf{s}$ fulfills the claimed property *i)*. Note that because $t_{k,0} > (2k+1)^{-1}$ and $t_{k+1,0} \leq (2k+2)^{-1}$, which implies $t_{k+1,0} - t_{k,2} \geq t_{k,0} - t_{k,1}$, the paths $\mathbf{s}$ and $\tilde{\mathbf{s}}$ coincide for every t such that

$$t_{k+1,0} - \frac{t_{k,0} - t_{k,1}}{4} < t < t_{k+1,0} + \frac{t_{k,0} - t_{k,1}}{4}$$

and this advisement holds true for every $k \in \mathbb{N}_{>0}$. Hence, by (3.11), the path $\mathbf{s}$ fulfills the claimed property *iv)*. For every $k \in \mathbb{N}_{>0}$ and every $0 < t < t_{k,0}$ we have the estimate

$$\|\mathbf{s}(t)\|_2 \leq \sqrt{t_{k,0}^2 + (t_{k,0} - t_{k,1})^2},$$

so, for every $k \in \mathbb{N}_{>0}$ and every $0 < t < (2k)^{-1}$ we have the crude estimate

$$\|\mathbf{s}(t)\|_2 \leq \frac{1}{k} \tag{3.12}$$

and the claimed property *ii)* is fulfilled as well. We come to the claimed property *iii)*. By Lemma 2.14 we have for every $k \in \mathbb{N}_{>0}$ and every $t_{k+1,0} < t \leq t_{k,0}$ the estimate

$$\begin{aligned}
\|\mathbf{s}'(t)\|_2 &\leq \|\mathbf{b}_k\|_2 + \|\mathbf{b}_{k+1}\|_2 + \left\| \frac{\mathbf{a}_k - \mathbf{a}_{k+1} + (t_{k,1} - t_{k,0})\mathbf{b}_k - (t_{k,2} - t_{k+1,0})\mathbf{b}_{k+1}}{t_{k,1} - t_{k,2}} \right\|_2 \\
&\leq 2 + 3(2k+1)(2k+2)\left(\frac{k+1}{k(2k+2)} + \frac{1}{k(2k+2)} - \frac{1}{3(2k+1)(2k+2)}\right) \\
&\leq 28k,
\end{aligned}$$

where we used

$$\|\mathbf{a}_k - \mathbf{a}_{k+1}\|_2 = \sqrt{\|\mathbf{a}_k\|_2^2 + \|\mathbf{a}_{k+1}\|_2^2 - 2\mathbf{a}_k \cdot \mathbf{a}_{k+1}},$$

which has a maximum with $\|\mathbf{a}_k\|_2 = (2k)^{-1}$, $\|\mathbf{a}_{k+1}\|_2 = (2k+2)^{-1}$, and $\mathbf{a}_k \cdot \mathbf{a}_{k+1} = \|\mathbf{a}_k\|_2 \|\mathbf{a}_{k+1}\|_2 / 2$ for all $k \geq 2$ (recall that the opening angle at the vertex of $\mathcal{K}_{\mathbf{z}^*}$ is $\pi/3$). But then

$$\|\mathbf{s}(t)\|_2 \|\mathbf{s}'(t)\|_2 < +\infty$$

for all $t \in]0, t_{1,0}[$ and we have finished the proof. ■

We come to the theorem that proves that the continuity of a function defined on a subset of $\mathbb{R}^n$ can be characterized by the use of a convenient subset of smooth paths in $\mathbb{R}^n$.

Theorem 3.6 *Let $f : \mathcal{N} \longrightarrow \mathbb{R}$ be a real-valued function, defined on a neighborhood $\mathcal{N} \subset \mathbb{R}^n$ of the origin and such that $f(\mathbf{0}) = 0$. Then f is continuous at the origin, if and only if for every path $\mathbf{s} \in [\mathcal{C}^\infty(]0,a[)]^n$, $a > 0$, such that*

$$\lim_{t\to 0+} \mathbf{s}(t) = \mathbf{0} \quad \textit{and} \quad \sup_{t\in\,]0,a[} \|\mathbf{s}(t)\|_2\|\mathbf{s}'(t)\|_2 < +\infty, \tag{3.13}$$

we have

$$\lim_{t\to 0+} f(\mathbf{s}(t)) = 0.$$

Proof:
The *only if* part is obvious. We prove the *if* part by showing that if f is not continuous at the origin, then there is a path $\mathbf{s} \in [\mathcal{C}^\infty(]0,a[)]^n$, $a > 0$, that fulfills the properties (3.13), but for which $\limsup_{t\to 0+} |f(\mathbf{s}(t))| > 0$.

Assume that f is not continuous at the origin. Then there is an $\varepsilon > 0$ and a sequence $\mathbf{x}_k$, $k \in \mathbb{N}_{>0}$, such that $\lim_{k\to+\infty} \mathbf{x}_k = \mathbf{0}$ but $|f(\mathbf{x}_k)| \geq \varepsilon$ for all k. By Lemma 3.5 there is a path $\mathbf{s} \in [\mathcal{C}^\infty(]0,a[)]^n$, $a > 0$, with the properties that $\lim_{t\to 0+} \mathbf{s}(t) = \mathbf{0}$, $\sup_{t\in\,]0,a[} \|\mathbf{s}(t)\|_2\|\mathbf{s}'(t)\|_2 < +\infty$, and $\mathbf{s}(\|\mathbf{x}_k\|_2) = \mathbf{x}_k$ for an infinite number of $k \in \mathbb{N}_{>0}$. But then $|f(\mathbf{s}(\|\mathbf{x}_k\|_2))| \geq \varepsilon$ for an infinite number of $k \in \mathbb{N}_{>0}$, which implies $\limsup_{t\to 0+} |f(\mathbf{s}(t))| \geq \varepsilon$ and we have finished the proof.

■

We now have everything needed to give a reasonably short proof of the limit (3.10). This is done in the next lemma.

Lemma 3.7 *Let $\mathcal{N} \subset \mathbb{R}^n$ be a neighborhood of the origin and let $f \in \mathcal{C}(\mathcal{N})$ be continuously differentiable on $\mathcal{N} \setminus \{\mathbf{0}\}$. Then*

$$\lim_{\mathbf{x}\to\mathbf{0}} \|\mathbf{x}\|_2^2 \|\nabla f(\mathbf{x})\|_2 = 0.$$

Proof:
Obviously, we can assume without loss of generality that $f(\mathbf{x}) = 0$. By Theorem 3.6 we then have

$$\lim_{t\to 0+} f(\mathbf{s}(t)) = 0$$

for every path $\mathbf{s} \in [\mathcal{C}^\infty(]0,a[)]^n$, $a > 0$, that satisfies (3.13). Then, for every such path we obtain by l'Hospital's rule and partial integration

$$\begin{aligned}\lim_{t\to 0+} \|\mathbf{s}(t)\|_2^2 [\nabla f](\mathbf{s}(t)) \cdot \mathbf{s}'(t) &= \lim_{t\to 0+} \frac{1}{t}\int_0^t \|\mathbf{s}(\tau)\|_2^2 \frac{d}{d\tau} f(\mathbf{s}(\tau))d\tau \\ &= \lim_{t\to 0+} \frac{1}{t}\left(\|\mathbf{s}(\tau)\|_2^2 f(\mathbf{s}(\tau))\Big|_{\tau=0}^t - 2\int_0^t \mathbf{s}(\tau)\cdot\mathbf{s}'(\tau) f(\mathbf{s}(\tau))d\tau \right) \\ &= 0.\end{aligned} \tag{3.14}$$

Just note that, with $C \geq \sup_{\tau\in]0,a[} \|\mathbf{s}(\tau)\|_2\|\mathbf{s}'(\tau)\|_2$ and for every $0 < t < a$, we have

$$\|\mathbf{s}(t)\|_2^2 |f(\mathbf{s}(t))| \leq 2Ct|f(\mathbf{s}(t))| \quad \text{and} \quad |\int_0^t \mathbf{s}(\tau)\cdot\mathbf{s}'(\tau)f(\mathbf{s}(\tau))d\tau| \leq Ct \sup_{\tau\in\,]0,t]} |f(\mathbf{s}(\tau))|.$$

Now, assume that $\limsup_{\mathbf{x}\to\mathbf{0}} \|\mathbf{x}\|_2^2\|\nabla f(\mathbf{x})\|_2 > 0$. Then there is a sequence $(\mathbf{x}_k)_{\mathbb{N}_{>0}}$ and an $\varepsilon > 0$, such that $\lim_{k\to+\infty} \mathbf{x}_k = \mathbf{0}$ and $\|\mathbf{x}_k\|_2^2\|\nabla f(\mathbf{x}_k)\|_2 \geq \varepsilon$ for all $k \in \mathbb{N}_{>0}$. By Lemma 3.5 there is a path

$\mathbf{s} \in [\mathcal{C}^\infty(]0,a[)]^n$, $a > 0$, with the properties that $\lim_{t \to 0+} \mathbf{s}(t) = \mathbf{0}$, $\sup_{t \in]0,a[} \|\mathbf{s}(t)\|_2 \|\mathbf{s}'(t)\|_2 < +\infty$, there are infinitely many $k \in \mathbb{N}_{>0}$ such that

$$\|\mathbf{s}(\|\mathbf{x}_k\|_2)\|_2^2 [\nabla f](\mathbf{s}(\|\mathbf{x}_k\|_2)) = \|\mathbf{x}_k\|_2^2 \nabla f(\mathbf{x}_k),$$

and

$$\mathbf{s}'(\|\mathbf{x}_k\|_2) = \pm \frac{\nabla f(\mathbf{x}_k)}{\|\nabla f(\mathbf{x}_k)\|_2},$$

where the sign is chosen for each k such that $\pm \mathbf{x}_k \cdot \nabla f(\mathbf{x}_k) \geq 0$. But then

$$\|\mathbf{s}(\|\mathbf{x}_k\|_2)\|_2^2 [\nabla f](\mathbf{s}(\|\mathbf{x}_k\|_2)) \cdot \mathbf{s}'(\|\mathbf{x}_k\|_2) = \|\mathbf{s}(\|\mathbf{x}_k\|_2)\|_2^2 \|[\nabla f](\mathbf{s}(\|\mathbf{x}_k\|_2))\|_2 \geq \varepsilon$$

for all such k, which is contradictory to (3.14). Therefore $\lim_{\mathbf{x} \to \mathbf{0}} \|\mathbf{x}\|_2^2 \|\nabla f(\mathbf{x})\|_2 = 0$.

■

Even after we have proved the validity of the limit (3.10), it is still advantageous to source out some parts of our main proof. The next lemma states the last result we use to prove the last lemma for our proof of the main theorem, a converse Lyapunov theorem for a uniformly asymptotically stable equilibrium of the arbitrary Switched System 1.9.

Lemma 3.8 *Let $\mathcal{U} \subset \mathbb{R}^n$ be a domain, let $f \in \mathcal{C}^\infty(\mathcal{U})$ be a strictly positive function on $\mathcal{U}$, and let $\boldsymbol{\beta} \in \mathbb{N}_{\geq 0}^n$ be any multiindex with $|\boldsymbol{\beta}| > 0$. Define the set $\Omega_{\boldsymbol{\beta}}$, of which the elements are sets of multiindices, by*

$$\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_k\} \in \Omega_{\boldsymbol{\beta}}, \text{ if and only if } \sum_{i=1}^{k} \boldsymbol{\alpha}_i = \boldsymbol{\beta} \text{ and } |\boldsymbol{\alpha}_i| > 0 \text{ for all } i = 1, 2, \ldots, k.$$

Denote by ω_m, $m = 1, 2, \ldots, |\Omega_{\boldsymbol{\beta}}|$, the elements of $\Omega_{\boldsymbol{\beta}}$. Then there exists a positive integer $d_{\boldsymbol{\beta}}$ and for every $\omega_m := \{\boldsymbol{\alpha}_{m,1}, \boldsymbol{\alpha}_{m,2}, \ldots, \boldsymbol{\alpha}_{m,|\omega_m|}\} \in \Omega_{\boldsymbol{\beta}}$ a polynomial

$$P_m(y) := \sum_{i=0}^{d_{\boldsymbol{\beta}}} a_{m,i} y^i,$$

such that $\mathbf{x} \mapsto \partial^{\boldsymbol{\beta}} \left[f(\mathbf{x}) \exp\left(\frac{-1}{f(\mathbf{x})}\right) \right]$ has the representation

$$\partial^{\boldsymbol{\beta}} \left[f(\mathbf{x}) \exp\left(\frac{-1}{f(\mathbf{x})}\right) \right] = \sum_{m=1}^{|\Omega_{\boldsymbol{\beta}}|} P_m(f(\mathbf{x})) \left(\frac{1}{f(\mathbf{x})}\right)^{2|\boldsymbol{\beta}|} \exp\left(\frac{-1}{f(\mathbf{x})}\right) \prod_{j=1}^{|\omega_m|} [\partial^{\boldsymbol{\alpha}_{m,j}} f](\mathbf{x}) \tag{3.15}$$

PROOF:
The representation (3.15) can be proved by mathematical induction over the length of the multiindex $\boldsymbol{\beta}$. If $|\boldsymbol{\beta}| = 1$, then $\boldsymbol{\beta} = \mathbf{e}_i$ for some $i \in \{1, 2, \ldots, n\}$ and because

$$\frac{d}{dx_i} \left[f(\mathbf{x}) \exp\left(\frac{-1}{f(\mathbf{x})}\right) \right] = \left([f(\mathbf{x})]^2 + f(\mathbf{x})\right) \left(\frac{1}{f(\mathbf{x})}\right)^2 \exp\left(\frac{-1}{f(\mathbf{x})}\right) \frac{df}{dx_i}(\mathbf{x}),$$

the representation (3.15) holds true for all such $\boldsymbol{\beta}$.

The induction-step is simple. For an arbitrary multiindex $\boldsymbol{\beta}'$ with $|\boldsymbol{\beta}'| > 1$ there is an $i \in \{1, 2, \dots, n\}$ such that $\boldsymbol{\beta} := \boldsymbol{\beta}' - \mathbf{e}_i$ is a multiindex with $|\boldsymbol{\beta}| > 0$. For the multiindex $\boldsymbol{\beta}$ the representation (3.15) holds true by the induction-hypothesis and one just has to calculate

$$\frac{d}{dx_i}\left(\sum_{m=1}^{|\Omega_{\boldsymbol{\beta}}|} P_m(f(\mathbf{x}))\left(\frac{1}{f(\mathbf{x})}\right)^{2|\boldsymbol{\beta}|}\exp\left(\frac{-1}{f(\mathbf{x})}\right)\prod_{j=1}^{|\omega_m|}[\partial^{\boldsymbol{\alpha}_{m,j}} f](\mathbf{x})\right)$$

and verify that it is of the expected form. The calculations are trivial and not particularly interesting, however, quite long. Therefore, we omit them and leave the induction-step as an exercise for disbelievers.

■

Now we come to the last technical results needed for the proof of the main theorem.

Lemma 3.9 *Let $\mathcal{U} \subset \mathbb{R}^n$ be a domain containing the origin and assume that $U : \mathbb{R}_{\geq 0} \times \mathcal{U} \longrightarrow \mathbb{R}_{\geq 0}$ is a function, that is infinitely differentiable at every point in the set $\mathbb{R}_{\geq 0} \times [\mathcal{U} \setminus \{\mathbf{0}\}]$. Further, assume that $U(t, \mathbf{0}) = 0$ for all $t \geq 0$, $U(t, \mathbf{x}) > 0$ for all $t \geq 0$ and $\mathbf{x} \neq \mathbf{0}$, and that there exists a constant $L > 0$ such that $U(t, \mathbf{x}) \leq L\|\mathbf{x}\|_2$ for all $t \geq 0$. Then the function $V : \mathbb{R}_{\geq 0} \times \mathcal{U} \longrightarrow \mathbb{R}$,*

$$V(t, \boldsymbol{\xi}) := \begin{cases} 0, & \text{if } \boldsymbol{\xi} = \mathbf{0}, \\ U(t, \boldsymbol{\xi}) \exp\left(\frac{-1}{U(t,\boldsymbol{\xi})}\right), & \text{if } \boldsymbol{\xi} \neq \mathbf{0}, \end{cases}$$

is a $\mathcal{C}^\infty(\mathbb{R}_{\geq 0} \times \mathcal{U})$ function.

Proof:
We first show that

$$\lim_{(t,\boldsymbol{\xi}) \to (s,\mathbf{0})} \frac{\partial^{\boldsymbol{\beta}} V(t, \boldsymbol{\xi})}{\|\boldsymbol{\xi}\|_2^k} = 0 \tag{3.16}$$

for every multiindex $\boldsymbol{\beta} \in \mathbb{N}_{\geq 0}^{n+1}$, every $k \in \mathbb{N}_{\geq 0}$, and every $s \geq 0$.

Trivially we have for every $k \in \mathbb{N}_{\geq 0}$ and every $s \geq 0$ that

$$\lim_{(t,\boldsymbol{\xi}) \to (s,\mathbf{0})} \frac{1}{U(t, \boldsymbol{\xi})^k} \exp\left(\frac{-1}{2U(t, \boldsymbol{\xi})}\right) = 0. \tag{3.17}$$

Further, because

$$\exp\left(\frac{-1}{2U(t, \boldsymbol{\xi})}\right) \leq \exp\left(\frac{-1}{2L\|\boldsymbol{\xi}\|_2}\right),$$

we have for every $k \in \mathbb{N}_{\geq 0}$ and every $s \geq 0$, that

$$\lim_{(t,\boldsymbol{\xi}) \to (s,\mathbf{0})} \frac{1}{\|\boldsymbol{\xi}\|_2^k} \exp\left(\frac{-1}{2U(t, \boldsymbol{\xi})}\right) = 0.$$

Therefore and by the representation (3.15) of

$$\partial^{\boldsymbol{\beta}} V(t, \boldsymbol{\xi}) = \partial^{\boldsymbol{\beta}}\left[U(t, \boldsymbol{\xi}) \exp\left(\frac{-1}{U(t, \boldsymbol{\xi})}\right)\right],$$

proved in Lemma 3.8, it suffices to show that for every multiindex $\boldsymbol{\beta} \in \mathbb{N}_{\geq 0}^{n+1}$ and every $s \geq 0$ we have

$$\lim_{(t,\boldsymbol{\xi})\to(s,\mathbf{0})} \|\boldsymbol{\xi}\|_2^{2|\boldsymbol{\beta}|}\partial^{\boldsymbol{\beta}}U(t,\boldsymbol{\xi}) = 0 \tag{3.18}$$

to prove (3.16).

We prove (3.18) by mathematical induction. Clearly, it holds true for $|\boldsymbol{\beta}| := \sum_{i=1}^{n+1}\beta_i = 0$. But then, by Lemma 3.7, we have

$$\lim_{(t,\boldsymbol{\xi})\to(s,\mathbf{0})}[(t-s)^2+\|\boldsymbol{\xi}\|_2^2]\sqrt{\left(\frac{\partial U}{\partial t}(t,\boldsymbol{\xi})\right)^2+\sum_{i=1}^{n}\left(\frac{\partial U}{\partial \xi_i}(t,\boldsymbol{\xi})\right)^2} = 0,$$

so equation (3.18) holds true for all $s \geq 0$ and all multiindices $\boldsymbol{\beta}$ with $|\boldsymbol{\beta}| = 1$.

Now, assume that for some $k \in \mathbb{N}_{>0}$ equation (3.18) holds true for all $s \geq 0$ and all multiindices $\boldsymbol{\beta}$ with $|\boldsymbol{\beta}| = k$. But then, with $U := U(t,\boldsymbol{\xi})$ to shorten the equation, we obtain

$$\lim_{(t,\boldsymbol{\xi})\to(s,\mathbf{0})}[(t-s)^2+\|\boldsymbol{\xi}\|_2^2]\sqrt{\|\boldsymbol{\xi}\|_2^{4|\boldsymbol{\beta}|^2}[\partial_t\partial^{\boldsymbol{\beta}}U]^2+\|\boldsymbol{\xi}\|_2^{(2|\boldsymbol{\beta}|-2)^2}\sum_{i=1}^{n}[\xi_i\partial^{\boldsymbol{\beta}}U+\|\boldsymbol{\xi}\|_2^2\partial_{\xi_i}\partial^{\boldsymbol{\beta}}U]^2} = 0,$$

by applying Lemma 3.7 on the function $(t,\boldsymbol{\xi}) \mapsto \|\boldsymbol{\xi}\|_2^{2|\boldsymbol{\beta}|}\partial^{\boldsymbol{\beta}}U(t,\boldsymbol{\xi})$. It follows that for every $i = 1,2,\ldots,n$ we have

$$\lim_{(t,\boldsymbol{\xi})\to(s,\mathbf{0})} \|\boldsymbol{\xi}\|_2^{2|\boldsymbol{\beta}|}\left[\xi_i\partial^{\boldsymbol{\beta}}U(t,\boldsymbol{\xi})+\|\boldsymbol{\xi}\|_2^2\partial_{\xi_i}\partial^{\boldsymbol{\beta}}U(t,\boldsymbol{\xi})\right] = 0$$

and because

$$\lim_{(t,\boldsymbol{\xi})\to(s,\mathbf{0})} \|\boldsymbol{\xi}\|_2^{2|\boldsymbol{\beta}|}\xi_i\partial^{\boldsymbol{\beta}}U(t,\boldsymbol{\xi}) = 0$$

by induction hypothesis, equation (3.18) holds true for all multiindices $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| = k+1$. It follows that equation (3.18) holds true for all $s \geq 0$ and every multiindex $\boldsymbol{\beta} \in \mathbb{N}_{\geq 0}^{n+1}$, which in turn implies that equation (3.16) holds true for all $s \geq 0$, every $k \in \mathbb{N}_{\geq 0}$, and every multiindex $\boldsymbol{\beta} \in \mathbb{N}_{\geq 0}^{n+1}$.

Now, by definition, $(t,\boldsymbol{\xi}) \mapsto \partial^{\boldsymbol{\beta}}V(t,\boldsymbol{\xi})$ is differentiable at $(s,\mathbf{0})$, $s \geq 0$, with the total derivative $\mathbf{0} \in \mathbb{R}^{n+1}$, if and only if

$$\lim_{(t,\boldsymbol{\xi})\to(s,\mathbf{0})} \frac{\partial^{\boldsymbol{\beta}}V(t,\boldsymbol{\xi})-\partial^{\boldsymbol{\beta}}V(s,\mathbf{0})}{\sqrt{(t-s)^2+\|\boldsymbol{\xi}\|_2^2}} = 0.$$

But

$$\frac{\left|\partial^{\boldsymbol{\beta}}V(t,\boldsymbol{\xi})-\partial^{\boldsymbol{\beta}}V(s,\mathbf{0})\right|}{\sqrt{(t-s)^2+\|\boldsymbol{\xi}\|_2^2}} \leq \frac{\left|\partial^{\boldsymbol{\beta}}V(t,\boldsymbol{\xi})\right|}{\|\boldsymbol{\xi}\|_2}$$

and it follows by equation (3.16) that $(t,\boldsymbol{\xi}) \mapsto \partial^{\boldsymbol{\beta}}V(t,\boldsymbol{\xi})$ is differentiable at $(s,\mathbf{0})$ for every $s \geq 0$ and every $\boldsymbol{\beta} \in \mathbb{N}_{\geq 0}^{n+1}$. Further, because differentiability implies continuity, we have proved that $V \in \mathcal{C}^\infty(\mathbb{R}_{\geq 0} \times \mathcal{U})$.

■

Finally, we come to the central theorem of this part. It is the promised converse Lyapunov theorem for a uniformly asymptotically stable equilibrium of the Switched System 1.9.

Theorem 3.10 (Converse theorem on uniform asymptotic stability of switched systems) *Assume that the origin is a uniformly asymptotically stable equilibrium point of the Switched System 1.9 on the ball* $\mathcal{B}_{\|\cdot\|,R} \subset \mathcal{U}$*,* $R > 0$*, where* $\|\cdot\|$ *is a norm on* $\mathbb{R}^n$*. Assume further, that the functions* $\mathbf{f}_p$*,* $p \in \mathcal{P}$*, are all Lipschitz and that there exists a common Lipschitz constant for the functions, that is, there exists a constant* $L > 0$ *such that*

$$\|\mathbf{f}_p(t,\mathbf{x}) - \mathbf{f}_p(s,\mathbf{y})\| \le L(|t-s| + \|\mathbf{x}-\mathbf{y}\|) \tag{3.19}$$

for all $t, s \ge 0$*, all* $\mathbf{x}, \mathbf{y} \in \mathcal{B}_{\|\cdot\|,R}$*, and all* $p \in \mathcal{P}$*.*

Then, for every $0 < R^* < R$*, there exists a smooth Lyapunov function* $V \in \mathcal{C}^\infty(\mathbb{R}_{\ge 0} \times \mathcal{B}_{\|\cdot\|,R^*})$ *for the system. Further, if the Switched System 1.9 is autonomous, then there exists a smooth time-invariant Lyapunov function* $V \in \mathcal{C}^\infty(\mathcal{B}_{\|\cdot\|,R^*})$ *for the system.*

Proof:

The proof is long and technical, even after all the preparation we have done, so we split it into three parts. In part I we introduce some constants and functions that we will use in the rest of the proof. In part II we define a function $U \in \mathcal{C}^\infty(\mathbb{R}_{\ge 0} \times [\mathcal{B}_{\|\cdot\|,R^*} \setminus \{\mathbf{0}\}])$ and prove that it is a Lyapunov function for the system. In part III we smooth U out at the origin to obtain the claimed Lyapunov function $V \in \mathcal{C}^\infty(\mathbb{R}_{\ge 0} \times \mathcal{B}_{\|\cdot\|,R^*})$.

Part I:

Because the assumptions of the theorem imply the assumptions made in Definition 3.2, we can define the functions $W_\sigma : \mathcal{B}_{\|\cdot\|,R} \longrightarrow \mathbb{R}_{\ge 0}$, $\sigma \in \mathcal{S}_\mathcal{P}$, and $W : \mathcal{B}_{\|\cdot\|,R} \longrightarrow \mathbb{R}$ just as in the definition. As in Definition 3.2, denote by g be the function from Massera's lemma 3.1 in the definition of the functions W_σ, and set

$$C := \int_0^{+\infty} g'(\varsigma(R,\tau))e^{L\tau}d\tau,$$

where, once again, ς is the same function as in Definition 3.2.

Let $m, M > 0$ be constants such that

$$\|\mathbf{x}\|_2 \le m\|\mathbf{x}\| \quad \text{and} \quad \|\mathbf{x}\| \le M\|\mathbf{x}\|_2$$

for all $\mathbf{x} \in \mathbb{R}^n$ and let a be a constant such that

$$a > 2m \quad \text{and set} \quad y^* := \frac{mR}{a}.$$

Define

$$K := \frac{g(y^*)}{a}L\left(C\left[m(1+M)R + mR\left(\frac{4}{3}LR + M\right)\right] + g(4R/3)mR\right),$$

and set

$$\epsilon := \min\left\{\frac{a}{3g(y^*)}, \frac{a(R-R^*)}{R^*g(y^*)}, \frac{a}{2mRLg(y^*)}, \frac{1}{K}\right\}. \tag{3.20}$$

Note that ϵ is a real-valued constant that is strictly larger than zero.

We define the function $\varepsilon : \mathbb{R}_{\ge 0} \longrightarrow \mathbb{R}_{\ge 0}$ by

$$\varepsilon(x) := \epsilon\int_0^{\frac{x}{a}} g(z)dz. \tag{3.21}$$

The definition of ε implies

$$\varepsilon(x) \leq \epsilon g(x/a)\frac{x}{a} \leq \frac{a}{3g(y^*)} \cdot g(x/a)\frac{x}{a} \leq \frac{x}{3} \tag{3.22}$$

for all $0 \leq x \leq mR$ and

$$\varepsilon'(x) = \frac{\epsilon}{a} g(x/a) \tag{3.23}$$

for all $x \geq 0$.

Define the function ϑ by $\vartheta(x) := g(2x/3) - g(x/2)$ for all $x \geq 0$. Then $\vartheta(0) = 0$ and for every $x > 0$ we have

$$\vartheta'(x) = \frac{2}{3}g'(2x/3) - \frac{1}{2}g'(x/2) > 0$$

because $g' \in \mathcal{K}$, that is $\vartheta \in \mathcal{K}$.

Part II:

Let $\rho \in \mathcal{C}^\infty(\mathbb{R})$ be a nonnegative function with $\mathrm{supp}(\rho) \subset [-1, 0]$ and $\int_\mathbb{R} \rho(x) = 1$ and let $\varrho \in \mathcal{C}^\infty(\mathbb{R}^n)$ be a nonnegative function with $\mathrm{supp}(\varrho) \subset \mathcal{B}_{\|\cdot\|_2,1}$ and $\int_{\mathbb{R}^n} \varrho(\mathbf{x})d^n x = 1$. Extend W on $\mathbb{R} \times \mathbb{R}^n$ by setting it equal to zero outside of $\mathbb{R}_{\geq 0} \times \mathcal{B}_{\|\cdot\|,R}$. We claim that the function $U : \mathbb{R}_{\geq 0} \times \mathcal{B}_{\|\cdot\|,R^*} \longrightarrow \mathbb{R}_{\geq 0}$, $U(t, \mathbf{0}) := 0$ for all $t \geq 0$, and

$$\begin{aligned} U(t, \boldsymbol{\xi}) &:= \int_\mathbb{R} \int_{\mathbb{R}^n} \rho\left(\frac{t-\tau}{\varepsilon(\|\boldsymbol{\xi}\|_2)}\right) \varrho\left(\frac{\boldsymbol{\xi} - \mathbf{y}}{\varepsilon(\|\boldsymbol{\xi}\|_2)}\right) \frac{W[t, \mathbf{y}]}{\varepsilon^{n+1}(\|\boldsymbol{\xi}\|_2)} d^n y d\tau \\ &= \int_\mathbb{R} \int_{\mathbb{R}^n} \rho(\tau)\varrho(\mathbf{y})W[t - \varepsilon(\|\boldsymbol{\xi}\|_2)\tau, \boldsymbol{\xi} - \varepsilon(\|\boldsymbol{\xi}\|_2)\mathbf{y}]d^n y d\tau \end{aligned}$$

for all $t \geq 0$ and all $\boldsymbol{\xi} \in \mathcal{B}_{\|\cdot\|,R^*} \setminus \{\mathbf{0}\}$, is a $\mathcal{C}^\infty(\mathbb{R}_{\geq 0} \times \left[\mathcal{B}_{\|\cdot\|,R^*} \setminus \{\mathbf{0}\}\right])$ Lyapunov function for the switched system. Note, that if the Switched System 1.9 in question is autonomous, then W is time-invariant, which implies that U is time-invariant too.

Because, for every $\|\mathbf{y}\|_2 \leq 1$ and every $\|\boldsymbol{\xi}\| < R^*$, we have by (3.22) and (3.20), that

$$\begin{aligned} \|\boldsymbol{\xi} - \varepsilon(\|\boldsymbol{\xi}\|_2)\mathbf{y}\| &\leq \left(1 + \frac{\varepsilon(\|\boldsymbol{\xi}\|_2)}{\|\boldsymbol{\xi}\|_2}\right)\|\boldsymbol{\xi}\| \\ &\leq \left(1 + \frac{\epsilon g(\|\boldsymbol{\xi}\|_2/a)}{\|\boldsymbol{\xi}\|_2} \cdot \frac{\|\boldsymbol{\xi}\|_2}{a}\right)\|\boldsymbol{\xi}\| \\ &< \left(1 + \frac{a(R-R^*)g(y^*)}{R^* g(y^*)a}\right)R^* \\ &= R, \end{aligned}$$

so U is properly defined on $\mathbb{R}_{\geq 0} \times \mathcal{B}_{\|\cdot\|,R^*}$. But then, by construction, $U \in \mathcal{C}^\infty(\mathbb{R}_{\geq 0} \times \left[\mathcal{B}_{\|\cdot\|,R^*} \setminus \{\mathbf{0}\}\right])$ (see, for example, Section 2.3). It remains to be shown that U fulfills the conditions **(L1)** and **(L2)** in Definition 2.17 of a Lyapunov function.

By Theorem 3.3 and Lemma 3.4 there is a function $\alpha_1 \in \mathcal{K}$ and a constant $L_W > 0$, such that

$$\alpha_1(\|\boldsymbol{\xi}\|) \leq W(t, \boldsymbol{\xi}) \leq L_W\|\boldsymbol{\xi}\|$$

for all $t \geq 0$ and all $\boldsymbol{\xi} \in \mathcal{B}_{\|\cdot\|,R}$. By inequality (3.22) we have for all $\boldsymbol{\xi} \in \mathcal{B}_{\|\cdot\|,R}$ and all $\|\mathbf{y}\|_2 \leq 1$, that

$$\|\boldsymbol{\xi} - \varepsilon(\|\boldsymbol{\xi}\|_2)\mathbf{y}\| \geq \left\|\boldsymbol{\xi} - \frac{\|\boldsymbol{\xi}\|_2}{3}\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|_2}\right\| = \frac{2}{3}\|\boldsymbol{\xi}\| \tag{3.24}$$

and

$$\|\boldsymbol{\xi} - \varepsilon(\|\boldsymbol{\xi}\|_2)\mathbf{y}\| \leq \left\| \boldsymbol{\xi} + \frac{\|\boldsymbol{\xi}\|_2}{3}\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|_2} \right\| = \frac{4}{3}\|\boldsymbol{\xi}\|. \tag{3.25}$$

Hence

$$\begin{aligned} \alpha_1(2\|\boldsymbol{\xi}\|/3) &= \int_{\mathbb{R}}\int_{\mathbb{R}^n} \rho(\tau)\varrho(\mathbf{y})\alpha_1(2\|\boldsymbol{\xi}\|/3)d^n y d\tau \\ &\leq \int_{\mathbb{R}}\int_{\mathbb{R}^n} \rho(\tau)\varrho(\mathbf{y})\alpha_1(\|\boldsymbol{\xi} - \varepsilon(\|\boldsymbol{\xi}\|_2)\mathbf{y}\|)d^n y d\tau \\ &\leq \int_{\mathbb{R}}\int_{\mathbb{R}^n} \rho(\tau)\varrho(\mathbf{y})W[t - \varepsilon(\|\boldsymbol{\xi}\|_2)\tau, \boldsymbol{\xi} - \varepsilon(\|\boldsymbol{\xi}\|_2)\mathbf{y}]d^n y d\tau \\ &= U(t, \boldsymbol{\xi}) \\ &\leq \int_{\mathbb{R}}\int_{\mathbb{R}^n} \rho(\tau)\varrho(\mathbf{y})L_W\|\boldsymbol{\xi} - \varepsilon(\|\boldsymbol{\xi}\|_2)\mathbf{y}\|d^n y d\tau \\ &\leq \frac{4L_W}{3}\|\boldsymbol{\xi}\|, \end{aligned} \tag{3.26}$$

and the function U fulfills the condition **(L1)**.

We now prove that U fulfills the condition **(L2)**. To do this let $t \geq 0$, $\boldsymbol{\xi} \in \mathcal{B}_{\|\cdot\|,R^*}$, and $\sigma \in \mathcal{S}_{\mathcal{P}}$ be arbitrary, but fixed throughout the rest of this part of the proof. Denote by $\mathcal{I}$ the maximum interval in $\mathbb{R}_{\geq 0}$ on which $s \mapsto \boldsymbol{\phi}_\sigma(s, t, \boldsymbol{\xi})$ is defined and set

$$q(s, \tau) := s - \varepsilon(\|\boldsymbol{\phi}_\sigma(s, t, \boldsymbol{\xi})\|_2)\tau$$

for all $s \in \mathcal{I}$ and all $-1 \leq \tau \leq 0$ and define

$$D(h, \mathbf{y}, \tau) := W[q(t + h, \tau), \boldsymbol{\phi}_\sigma(t + h, t, \boldsymbol{\xi}) - \varepsilon(\|\boldsymbol{\phi}_\sigma(t + h, t, \boldsymbol{\xi})\|_2)\mathbf{y}] - W[q(t, \tau), \boldsymbol{\xi} - \varepsilon(\|\boldsymbol{\xi}\|_2)\mathbf{y}]$$

for all h such that $t + h \in \mathcal{I}$, all $\|\mathbf{y}\|_2 \leq 1$, and all $-1 \leq \tau \leq 0$. Then

$$U(t + h, \boldsymbol{\phi}_\sigma(t + h, t, \boldsymbol{\xi})) - U(t, \boldsymbol{\xi}) = \int_{\mathbb{R}}\int_{\mathbb{R}^n} \rho(\tau)\varrho(\mathbf{y})D(h, \mathbf{y}, \tau)d^n y d\tau$$

for all h such that $t + h \in \mathcal{I}$, especially this equality holds true for all h in an interval of the form $[0, h'[$, where $0 < h' \leq +\infty$.

We are going to show that

$$\limsup_{h \to 0+} \frac{D(h, \mathbf{y}, \tau)}{h} \leq -\vartheta(\|\boldsymbol{\xi}\|). \tag{3.27}$$

If we can prove that (3.27) holds true, then, by Fatou's lemma,

$$\limsup_{h \to 0+} \frac{U(t + h, \boldsymbol{\phi}_\sigma(t + h, t, \boldsymbol{\xi})) - U(t, \boldsymbol{\xi})}{h} \leq \int_{\mathbb{R}}\int_{\mathbb{R}^n} \varrho(\tau)\varrho_n(\mathbf{y}) \limsup_{h \to 0+} \frac{D(h, \mathbf{y}, \tau)}{h} d^n y d\tau \leq -\vartheta(\|\boldsymbol{\xi}\|),$$

and we would have proved that the condition **(L2)** is fulfilled.

To proof inequality (3.27) observe that $q(t, \tau) \geq 0$ for all $-1 \leq \tau \leq 0$ and, for every $s > t$ that is smaller than any switching-time (discontinuity-point) of σ larger than t, and because of (3.20) and

(3.19), we have

$$
\begin{aligned}
\frac{dq}{ds}(s,\tau) &= 1 - \frac{\epsilon g(\|\boldsymbol{\phi}_\sigma(s,t,\boldsymbol{\xi})\|_2/a)}{a}\frac{\boldsymbol{\phi}_\sigma(s,t,\boldsymbol{\xi})}{\|\boldsymbol{\phi}_\sigma(s,t,\boldsymbol{\xi})\|_2}\cdot \mathbf{f}_{\sigma(s)}(s,\boldsymbol{\phi}_\sigma(s,t,\boldsymbol{\xi}))\tau \\
&\geq 1 - \epsilon\frac{g(y^*)LmR}{a} \\
&\geq \frac{1}{2},
\end{aligned}
$$

so $q(t+h,\tau) \geq q(t,\tau) \geq 0$ for all small enough $h \geq 0$.

Now, denote by γ the constant switching signal $\sigma(t)$ in $\mathcal{S}_\mathcal{P}$, that is $\gamma(s) := \sigma(t)$ for all $s \geq 0$, and consider that by Lemma 3.4

$$
\begin{aligned}
\frac{D(h,\mathbf{y},\tau)}{h} &\leq \frac{C}{h}\|\boldsymbol{\phi}_\sigma(t+h,t,\boldsymbol{\xi}) - \varepsilon(\|\boldsymbol{\phi}_\sigma(t+h,t,\boldsymbol{\xi})\|_2)\mathbf{y} - \boldsymbol{\phi}_\gamma(q(t+h,\tau),q(t,\tau),\boldsymbol{\xi}-\varepsilon(\|\boldsymbol{\xi}\|_2)\mathbf{y})\| \\
&\quad - \frac{1}{h}\int_{q(t,\tau)}^{q(t+h,\tau)} g(\|\boldsymbol{\phi}_\gamma(s,q(t,\tau),\boldsymbol{\xi}-\varepsilon(\|\boldsymbol{\xi}\|_2)\mathbf{y})\|)ds \\
&= C\left\|\frac{\boldsymbol{\phi}_\sigma(t+h,t,\boldsymbol{\xi})-\boldsymbol{\xi}}{h} - \frac{\boldsymbol{\phi}_\gamma(q(t+h,\tau),q(t,\tau),\boldsymbol{\xi}-\varepsilon(\|\boldsymbol{\xi}\|_2)\mathbf{y}) - [\boldsymbol{\xi}-\varepsilon(\|\boldsymbol{\xi}\|_2)\mathbf{y}]}{h}\right. \\
&\quad \left. - \frac{\varepsilon(\|\boldsymbol{\phi}_\sigma(t+h,t,\boldsymbol{\xi})\|_2)-\varepsilon(\|\boldsymbol{\xi}\|_2)}{h}\mathbf{y}\right\| - \frac{1}{h}\int_{q(t,\tau)}^{q(t+h,\tau)} g(\|\boldsymbol{\phi}_\gamma(s,q(t,\tau),\boldsymbol{\xi}-\varepsilon(\|\boldsymbol{\xi}\|_2)\mathbf{y})\|)ds.
\end{aligned}
$$

For the next calculations we need $s \mapsto q(s,\tau)$ to be differentiable at t. If it is not, which might be the case if t is a switching time of σ, we replace σ with $\sigma^* \in \mathcal{S}_\mathcal{P}$ where

$$
\sigma^*(s) := \begin{cases} \sigma(t), & \text{if } 0 \leq s \leq t, \\ \sigma(s), & \text{if } s \geq t. \end{cases}
$$

Note that this does not affect the numerical value

$$
\limsup_{h\to 0+}\frac{D(h,\mathbf{y},\tau)}{h}
$$

because $\sigma^*(t+h) = \sigma(t+h)$ for all $h \geq 0$. Hence, with $p := \sigma(t)$, and by (3.19), the chain rule, (3.24), and (3.25),

$$\begin{aligned}
&\limsup_{h\to 0+} \frac{D(h,\mathbf{y},\tau)}{h} \\
&\leq C \left\| \mathbf{f}_p(t,\boldsymbol{\xi}) - \mathbf{f}_p(q(t,\tau),\boldsymbol{\xi} - \varepsilon(\|\boldsymbol{\xi}\|_2)\mathbf{y}) \cdot \frac{dq}{dt'}(t',\tau)\Big|_{t'=t} - \varepsilon'(\|\boldsymbol{\xi}\|_2) \cdot \frac{d}{dt'}\|\boldsymbol{\phi}_\sigma(t',t,\boldsymbol{\xi})\|_2\Big|_{t'=t}\mathbf{y} \right\| \\
&\quad - g(\|\boldsymbol{\phi}_\gamma(q(t,\tau),q(t,\tau),\boldsymbol{\xi} - \varepsilon(\|\boldsymbol{\xi}\|_2)\mathbf{y})\|) \cdot \frac{dq}{dt'}(t',\tau)\Big|_{t'=t} \\
&= C \left\| \mathbf{f}_p(t,\boldsymbol{\xi}) - \mathbf{f}_p(q(t,\tau),\boldsymbol{\xi} - \varepsilon(\|\boldsymbol{\xi}\|_2)\mathbf{y}) \left[1 - \varepsilon'(\|\boldsymbol{\xi}\|_2)\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|_2} \cdot \mathbf{f}_p(t,\boldsymbol{\xi})\tau\right] - \varepsilon'(\|\boldsymbol{\xi}\|_2)[\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|_2} \cdot \mathbf{f}_p(t,\boldsymbol{\xi})]\mathbf{y} \right\| \\
&\quad - g(\|\boldsymbol{\xi} - \varepsilon(\|\boldsymbol{\xi}\|_2)\mathbf{y}\|) \left[1 - \varepsilon'(\|\boldsymbol{\xi}\|_2)\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|_2} \cdot \mathbf{f}_p(t,\boldsymbol{\xi})\right] \\
&\leq C\big[\, \|\mathbf{f}_p(t,\boldsymbol{\xi}) - \mathbf{f}_p(q(t,\tau),\boldsymbol{\xi} - \varepsilon(\|\boldsymbol{\xi}\|_2)\mathbf{y})\| + \varepsilon'(\|\boldsymbol{\xi}\|_2)\|\mathbf{f}_p(t,\boldsymbol{\xi})\|_2 \left\{\|\mathbf{f}_p(q(t,\tau),\boldsymbol{\xi} - \varepsilon(\|\boldsymbol{\xi}\|_2)\mathbf{y})\| + \|\mathbf{y}\|\right\} \big] \\
&\quad - g(2\|\boldsymbol{\xi}\|/3) + g(4\|\boldsymbol{\xi}\|/3)\varepsilon'(\|\boldsymbol{\xi}\|_2)\|\mathbf{f}_p(t,\boldsymbol{\xi})\|_2 \\
&\leq C\big[L(|t - q(t,\tau)| + \varepsilon(\|\boldsymbol{\xi}\|_2)\|\mathbf{y}\|) + \varepsilon'(\|\boldsymbol{\xi}\|_2)mLR\left\{L\|\boldsymbol{\xi} - \varepsilon(\|\boldsymbol{\xi}\|_2)\mathbf{y})\| + M\|\mathbf{y}\|_2\right\}\big] \\
&\quad - g(2\|\boldsymbol{\xi}\|/3) + g(4\|\boldsymbol{\xi}\|/3)\varepsilon'(\|\boldsymbol{\xi}\|_2)mLR \\
&\leq C\big[L(1+M)\varepsilon(\|\boldsymbol{\xi}\|_2) + \varepsilon'(\|\boldsymbol{\xi}\|_2)mLR\left\{L\frac{4}{3}\|\boldsymbol{\xi}\| + M\right\}\big] - g(2\|\boldsymbol{\xi}\|/3) + g(4\|\boldsymbol{\xi}\|/3)\varepsilon'(\|\boldsymbol{\xi}\|_2)mLR.
\end{aligned}$$

Therefore, by (3.22), (3.23), and (3.20), and with $x := \|\boldsymbol{\xi}\|$, we can further simplify,

$$\begin{aligned}\limsup_{h\to 0+} \frac{D(h,\mathbf{y},\tau)}{h} &\leq -g(2x/3) + \frac{\epsilon}{a} g(mx/a) L \left(C \left[m(1+M)x + mR\left(\frac{4}{3}Lx + M\right)\right] + g(4x/3)mR \right) \\ &\leq -g(2x/3) + K\epsilon g(x/2) \\ &\leq -\vartheta(x),\end{aligned}$$

and because $t \geq 0$, $\boldsymbol{\xi} \in \mathcal{B}_{\|\cdot\|,R^*}$, and $\sigma \in \mathcal{S}_\mathcal{P}$ were arbitrary, we have proved that U is a Lyapunov function for the system.

Part III:

We define the function $V : \mathbb{R}_{\geq 0} \times \mathcal{B}_{\|\cdot\|,R^*} \longrightarrow \mathbb{R}_{\geq 0}$ by $V(t,\mathbf{0}) = 0$ for all $t \geq 0$ and

$$V(t,\boldsymbol{\xi}) := U(t,\boldsymbol{\xi}) \exp\left(\frac{-1}{U(t,\boldsymbol{\xi})}\right)$$

for all $t \geq 0$ and all $\boldsymbol{\xi} \in \mathcal{B}_{\|\cdot\|,R^*} \setminus \{\mathbf{0}\}$. We claim that the function V is a $\mathcal{C}^\infty(\mathbb{R}_{\geq 0} \times \mathcal{B}_{\|\cdot\|,R^*})$ Lyapunov function for the Switched System 1.9. Note, that if U is time-invariant, which is the case if the Switched System 1.9 is autonomous, then so is V.

That $V \in \mathcal{C}^\infty(\mathbb{R}_{\geq 0} \times \mathcal{B}_{\|\cdot\|,R^*})$ follows by Lemma 3.9 so it only remains to prove that V is a Lyapunov function for the system. By (3.26) we get

$$\alpha_1(2\|\boldsymbol{\xi}\|/3) \exp\left(\frac{-1}{\alpha_1(2\|\boldsymbol{\xi}\|/3)}\right) \leq V(t,\boldsymbol{\xi}) \leq \frac{4}{3} L_W \|\boldsymbol{\xi}\| \exp\left(\frac{-3}{4L_W\|\boldsymbol{\xi}\|}\right)$$

for all $t \geq 0$ and all $\boldsymbol{\xi} \in \mathcal{B}_{\|\cdot\|,R^*}$ so V fulfills the condition **(L1)** in Definition 2.17 of a Lyapunov function. The condition **(L2)** follows by

$$\begin{aligned}\frac{d}{dt} V(t,\boldsymbol{\phi}_\sigma(t,s,\boldsymbol{\xi})) &= \left[\frac{d}{dt} U(t,\boldsymbol{\phi}_\sigma(t,s,\boldsymbol{\xi}))\right] \left(1 + \frac{1}{U(t,\boldsymbol{\phi}_\sigma(t,s,\boldsymbol{\xi}))}\right) \exp\left(\frac{-1}{U(t,\boldsymbol{\phi}_\sigma(t,s,\boldsymbol{\xi}))}\right) \\ &\leq -\vartheta(\|\boldsymbol{\phi}_\sigma(t,s,\boldsymbol{\xi})\|) \exp\left(\frac{-1}{\alpha_1(2\|\boldsymbol{\phi}_\sigma(t,s,\boldsymbol{\xi})\|/3)}\right)\end{aligned}$$

for all $\sigma \in \mathcal{S}_\mathcal{P}$, all $t \geq s \geq 0$, and all $\boldsymbol{\xi} \in \mathcal{U}$, such that $\boldsymbol{\phi}_\sigma(t,s,\boldsymbol{\xi}) \in \mathcal{B}_{\|\cdot\|,R^*}$. ■

Now, we have proved the main theorem of this part of this thesis, our much wanted converse theorem for the arbitrary Switched System 1.9. As a by-product of the proof, we observe, that by combining Lemma 3.9 and the proof of Part III of the last theorem, the following results emerge.

Lemma 3.11 *Let $\mathcal{U} \subset \mathbb{R}^n$ be a domain containing the origin and assume that $U : \mathbb{R}_{\geq 0} \times \mathcal{U} \longrightarrow \mathbb{R}_{\geq 0}$ is a function, that is infinitely differentiable at every point in the set $\mathbb{R}_{\geq 0} \times [\mathcal{U} \setminus \{\mathbf{0}\}]$. Further, assume that $U(t,\mathbf{0}) = 0$ for all $t \geq 0$ and that there exists a constant $L > 0$ such that $U(t,\mathbf{x}) \leq L\|\mathbf{x}\|_2$ for all $t \geq= 0$. Finally, assume that U is a Lyapunov function for the arbitrary Switched System 1.9. Then the function $V : \mathbb{R}_{\geq 0} \times \mathcal{U} \longrightarrow \mathbb{R}$,*

$$V(t,\boldsymbol{\xi}) := \begin{cases} 0, & \text{if } \boldsymbol{\xi} = \mathbf{0}, \\ U(t,\boldsymbol{\xi}) \exp\left(\frac{-1}{U(t,\boldsymbol{\xi})}\right), & \text{if } \boldsymbol{\xi} \neq \mathbf{0}, \end{cases}$$

is a $\mathcal{C}^\infty(\mathbb{R}_{\geq 0} \times \mathcal{U})$ Lyapunov function for the arbitrary Switched System 1.9.

Proof:

Follows immediately by Lemma 3.9 and the proof of Part III of Theorem 3.10. ■

Part II

An algorithm based on linear programming to generate Lyapunov functions for switched systems and a proof that it always succeeds if the equilibrium at the origin is uniformly asymptotically stable

In this part we will present an algorithm to construct Lyapunov functions for the Switched System 1.9. First, we give an algorithmic description of how to derive a linear programming problem from the Switched System 1.9 (Definition 5.1), and we prove that if the linear programming problem possesses a feasible solution, then it can be used to parameterize a Lyapunov function for the system. Then we present an algorithm that systematically generates linear programming problems for the Switched System 1.9 and we prove, that if the switched system possesses a Lyapunov function at all, then the algorithm generates, in a finite number of steps, a linear programming problem that has a feasible solution. Because there are algorithms that always find a feasible solution to a linear programming problem if one exists, this implies that we will have derived an algorithm to construct Lyapunov functions, whenever one exists. Further, we consider the case when the Switched System 1.9 is autonomous separately, because in this case it is possible to parameterize a time-independent Lyapunov function for the system.

For completeness we spend a few words on linear programming problems. A linear programming problem is a set of linear constraints, under which a linear function is to be minimized. There are several equivalent possibilities to state a linear programming problem, one of them is

$$\begin{aligned} \text{minimize} \quad & g(\mathbf{x}) := \mathbf{c} \cdot \mathbf{x}, \\ \text{given} \quad & C\mathbf{x} \leq \mathbf{b}, \ \mathbf{x} \geq \mathbf{0}, \end{aligned} \tag{$*$}$$

where $r, s > 0$ are integers, $C \in \mathbb{R}^{s \times r}$ is a matrix, $\mathbf{b} \in \mathbb{R}^s$ and $\mathbf{c} \in \mathbb{R}^r$ are vectors, and $\mathbf{x} \leq \mathbf{y}$ denotes $x_i \leq y_i$ for all i. The function g is called the objective of the linear programming problem and the conditions $C\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ together are called the constraints. A feasible solution to the linear programming problem is a vector $\mathbf{x}' \in \mathbb{R}^r$ that satisfies the constraints, that is, $\mathbf{x}' \geq \mathbf{0}$ and $C\mathbf{x}' \leq \mathbf{b}$. There are numerous algorithms known to solve linear programming problems, the most commonly used being the simplex method (see, for example, [52]) or interior point algorithms, for example, the primal-dual logarithmic barrier method (see, for example, [49]). Both need a starting feasible solution for initialization. A feasible solution to $(*)$ can be found by introducing slack variables $\mathbf{y} \in \mathbb{R}^s$ and solving the linear programming problem:

$$\begin{aligned} \text{minimize} \quad & g\left(\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}\right) := \sum_{i=1}^{s} y_i, \\ \text{given} \quad & \begin{bmatrix} C & -I_s \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \leq \mathbf{b}, \ \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \geq \mathbf{0}, \end{aligned} \tag{$**$}$$

which has the feasible solution $\mathbf{x} = \mathbf{0}$ and $\mathbf{y} = (|b_1|, |b_2|, \dots, |b_s|)$. If the linear programming problem $(**)$ has the solution $g([\mathbf{x}' \ \mathbf{y}']) = 0$, then $\mathbf{x}'$ is a feasible solution to $(*)$, if the minimum of g is strictly larger than zero, then $(*)$ does not have any feasible solution.

In order to construct a Lyapunov function with a linear programming problem, one needs a class of continuous functions that are easily parameterized. That is, we need a class of functions that is general enough to be used as a search-space for Lyapunov functions, but it has to be a finite-dimensional vector space so that its functions are uniquely characterized by a finite number of real numbers. The class of the continuous piecewise affine [3] functions CPWA is an obvious candidate, as will become clear in the following chapter.

The algorithm to parameterize a Lyapunov function for the Switched System 1.9 consists roughly of the following steps:

[3] Often called piecewise linear in the literature.

i) Partition a neighborhood of the equilibrium under consideration in a family $\mathfrak{S}$ of simplices.

ii) Limit the search for a Lyapunov function V for the system to the class of continuous functions that are affine on any $S \in \mathfrak{S}$.

iii) State linear inequalities for the values of V at the vertices of the simplices in $\mathfrak{S}$, so that if they can be fulfilled, then the function V, which is uniquely determined by its values at the vertices, is a Lyapunov function for the system in the whole area.

In this part we will first partition $\mathbb{R}^n$ into n-simplices and use this partition to define the function spaces CPWA of continuous piecewise affine functions $\mathbb{R}^n \longrightarrow \mathbb{R}$. A function in CPWA is uniquely determined by its values at the vertices of the simplices in $\mathfrak{S}$. Then we will present a linear programming problem, algorithmically derived from the Switched System 1.9, and prove that a CPWA Lyapunov function for the system can be parameterized from any feasible solution to this linear programming problem. Finally, we will prove that if the equilibrium of the Switched System 1.9 is uniformly asymptotically stable, then any simplicial partition with small enough simplices leads to a linear programming problem that does have a feasible solution. Because, by Theorem 2.16 and Theorem 3.10, a Lyapunov function exists for the Switched System 1.9 exactly when the equilibrium is uniformly asymptotically stable, and because it is always possible to algorithmically find a feasible solution if at least one exists, this proves that the algorithm can parameterize a Lyapunov function for the Switched System 1.9 if the system does possess a Lyapunov functions at all.

Chapter 4

Continuous piecewise affine functions

In order to construct a Lyapunov function by a linear programming problem, one needs a class of continuous functions that are easily parameterized. Our approach is a simplicial partition of $\mathbb{R}^n$, on which we define the finite dimensional $\mathbb{R}$-vector space CPWA of continuous functions, that are affine on every of the simplices. In this chapter we are going to derive an appropriate simplicial partition of $\mathbb{R}^n$ and then define the function space CPWA. Further, we will prove some important properties of the space CPWA that we will use in the next chapter to define the linear programming problem, of which every feasible solution parameterizes a CPWA Lyapunov function for the Switched System 1.9 if its equilibrium is uniformly asymptotically stable.

4.1 Preliminaries

Let $\mathcal{U}$ be a nonempty subset of $\mathbb{R}^n$. A function $\mathbf{p} : \mathcal{U} \longrightarrow \mathbb{R}^m$ is said to be an affine function if there is an $m \times n$-matrix P and a vector $\mathbf{c} \in \mathbb{R}^m$ such that $\mathbf{p}(\mathbf{x}) = P\mathbf{x} + \mathbf{c}$ for all $\mathbf{x} \in \mathcal{U}$. A simplex is the convex hull of affinely independent vectors in $\mathbb{R}^n$, more exactly:

Definition 4.1 (Simplex) *Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ be vectors in $\mathbb{R}^n$. A vector $\mathbf{x} \in \mathbb{R}^n$ is said to be a convex combination of the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ if there are numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$, such that $\lambda_i \in [0, 1]$ for all $i = 1, 2, \ldots, k$, $\sum_{i=1}^k \lambda_i = 1$, and*

$$\mathbf{x} = \sum_{i=1}^{k} \lambda_i \mathbf{x}_i.$$

The set of all convex combinations of the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ is denoted by $\operatorname{con}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ and is called the convex hull of the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$.

If the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are affinely independent, which means that the vectors

$$\mathbf{x}_1 - \mathbf{x}_j,\ \mathbf{x}_2 - \mathbf{x}_j,\ \ldots\ ,\ \mathbf{x}_{j-1} - \mathbf{x}_j\ ,\ \mathbf{x}_{j+1} - \mathbf{x}_j\ ,\ \ldots\ ,\ \mathbf{x}_k - \mathbf{x}_j$$

are linearly independent for any $j = 1, 2, \ldots, k$, then the set $\operatorname{con}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ is called a $(k-1)$-simplex and the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are called the vertices of the simplex. □

The convex hull of $n+1$ vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n+1}$ in $\mathbb{R}^n$ has a non-zero volume (n-dimensional Borel measure), if and only if it is an n-simplex. This follows from the well known facts, that the volume of $\operatorname{con}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n+1}\}$ is the absolute value of

$$\det(\mathbf{x}_1 - \mathbf{x}_j, \mathbf{x}_2 - \mathbf{x}_j, \ldots, \mathbf{x}_{j-1} - \mathbf{x}_j, \mathbf{x}_{j+1} - \mathbf{x}_j, \ldots, \mathbf{x}_{n+1} - \mathbf{x}_j)$$

for any $j = 1, 2, \dots, n+1$, and that this determinant is non-zero, if and only if the vectors

$$\mathbf{x}_1 - \mathbf{x}_j,\ \mathbf{x}_2 - \mathbf{x}_j,\ \dots\ ,\ \mathbf{x}_{j-1} - \mathbf{x}_j\ ,\ \mathbf{x}_{j+1} - \mathbf{x}_j\ ,\ \dots\ ,\ \mathbf{x}_{n+1} - \mathbf{x}_j$$

are linearly independent.

It is fairly easy to see that because the vertices of a simplex are affinely independent, an element of a simplex has a unique representations as a convex combination of the vertices. This property implies that a simplex suits exceptionally well as a domain for the definition of affine functions $\mathbb{R}^n \longrightarrow \mathbb{R}^m$. The following lemma confirms this.

Lemma 4.2 *Let* $\mathrm{con}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ *be a* $(k-1)$*-simplex in* $\mathbb{R}^n$ *and let* $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ *be vectors in* $\mathbb{R}^m$*. Then the following propositions about the function* $\mathbf{p} : \mathrm{con}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\} \longrightarrow \mathbb{R}^m$ *are equivalent:*

i) The function $\mathbf{p}$ *is affine and* $\mathbf{p}(\mathbf{x}_i) := \mathbf{a}_i$ *for all* $i = 1, 2, \dots, k$.

ii) For every convex combination $\sum_{i=1}^{k} \lambda_i \mathbf{x}_i$ *of the vertices* $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ *we have*

$$\mathbf{p}(\sum_{i=1}^{k} \lambda_i \mathbf{x}_i) = \sum_{i=1}^{k} \lambda_i \mathbf{a}_i.$$

PROOF:
i) implies *ii)*: Just note that because $\mathbf{p}$ is affine we have

$$\mathbf{p}(\sum_{i=1}^{k} \lambda_i \mathbf{x}_i) = \sum_{i=1}^{k} \lambda_i \mathbf{p}(\mathbf{x}_i) = \sum_{i=1}^{k} \lambda_i \mathbf{a}_i$$

for every convex combination of the vertices $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$.

ii) implies *i)*: Let P be an $m \times n$-matrix such that $P(\mathbf{x}_i - \mathbf{x}_1) = \mathbf{a}_i - \mathbf{a}_1$ for all $i = 2, 3, \dots, k$ and define the vector $\mathbf{c} := \mathbf{a}_1 - P\mathbf{x}_1$. Then $\mathbf{p}(\mathbf{x}) = P\mathbf{x} + \mathbf{c}$ for all $\mathbf{x}$ in the simplex is an affine function and $\mathbf{p}(\mathbf{x}_i) = \mathbf{a}_i$ for all $i = 1, 2, \dots, k$.

■

This lemma has an obvious corollary that is useful for the following, namely:

Corollary 4.3 *Let* $n \in \mathbb{N}_{>0}$ *and let* $S_1 = \mathrm{con}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n+1}\}$ *and* $S_2 = \mathrm{con}\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{n+1}\}$ *be* n*-simplices in* $\mathbb{R}^n$*. Assume that* $S_3 := S_1 \cap S_2 = \mathrm{con}\{\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_{k+1}\}$ *is a* k*-simplex, where* $0 \le k \le n$ *and*

$$\{\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_{k+1}\} \subset \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n+1}\} \cap \{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{n+1}\}.$$

Then, if $\mathbf{p}_1 : S_1 \longrightarrow \mathbb{R}^m$ *and* $\mathbf{p}_2 : S_2 \longrightarrow \mathbb{R}^m$ *are affine functions, the following propositions are equivalent:*

i) The function $\mathbf{q} : S_1 \cup S_2 \longrightarrow \mathbb{R}^m$,

$$\mathbf{q}(\mathbf{x}) := \begin{cases} \mathbf{p}_1(\mathbf{x}), & \text{if } \mathbf{x} \in S_1, \\ \mathbf{p}_2(\mathbf{x}), & \text{if } \mathbf{x} \in S_2, \end{cases}$$

is properly defined and continuous.

ii) For every $i = 1, 2, \ldots, k+1$ we have $\mathbf{p}_1(\mathbf{z}_i) = \mathbf{p}_2(\mathbf{z}_i)$.

PROOF:
Follows immediately by Lemma 4.2. ■

For every $n \in \mathbb{N}_{>0}$, we denote by $\text{Perm}[\{1, 2, \ldots, n\}]$ the *permutation group* of $\{1, 2, \ldots, n\}$, that is, $\text{Perm}[\{1, 2, \ldots, n\}]$ is the set of all one-to-one mappings from $\{1, 2, \ldots, n\}$ onto itself.

The simplices S_σ, where $\sigma \in \text{Perm}[\{1, 2, \ldots, n\}]$, will serve as the atoms of our partition of $\mathbb{R}^n$. They are defined in the following way:

Definition 4.4 (The simplices S_σ) *For every $\sigma \in \text{Perm}[\{1, 2, \ldots, n\}]$ we define the set*

$$S_\sigma := \{\mathbf{y} \in \mathbb{R}^n \big| \; 0 \le y_{\sigma(1)} \le y_{\sigma(2)} \le \ldots \le y_{\sigma(n)} \le 1\},$$

where $y_{\sigma(i)}$ is the $\sigma(i)$-th component of the vector $\mathbf{y}$. □

For every $\sigma \in \text{Perm}[\{1, 2, \ldots, n\}]$ the set S_σ is an n-simplex with the volume $1/n!$. That it is an n-simplex follows by the next theorem. That its volume is $1/n!$ follows from straight forward integration

$$\begin{aligned}
\int \chi_{S_\sigma}(\mathbf{x})\, d^n x &= \int_0^1 \left(\int_0^{x_{\sigma(n)}} \left(\cdots \left(\int_0^{x_{\sigma(2)}} dx_{\sigma(1)} \right) \cdots \right) dx_{\sigma(n-1)} \right) dx_{\sigma(n)} \\
&= \int_0^1 \left(\int_0^{x_{\sigma(n)}} \left(\cdots \left(\int_0^{x_{\sigma(3)}} x_{\sigma(2)} dx_{\sigma(2)} \right) \cdots \right) dx_{\sigma(n-1)} \right) dx_{\sigma(n)} \\
&= \int_0^1 \left(\int_0^{x_{\sigma(n)}} \left(\cdots \left(\int_0^{x_{\sigma(4)}} \frac{1}{2} x_{\sigma(3)}^2 dx_{\sigma(3)} \right) \cdots \right) dx_{\sigma(n-1)} \right) dx_{\sigma(n)} \\
&= \int_0^1 \left(\int_0^{x_{\sigma(n)}} \left(\cdots \left(\int_0^{x_{\sigma(5)}} \frac{1}{2 \cdot 3} x_{\sigma(4)}^3 dx_{\sigma(4)} \right) \cdots \right) dx_{\sigma(n-1)} \right) dx_{\sigma(n)} \\
&\vdots \\
&= \frac{1}{n!}.
\end{aligned}$$

Before stating and proving the next theorem we will first state a very simple, but useful lemma, that will be used in its proof and later on.

Lemma 4.5 *Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^m$ and $\mu_1, \mu_2, \ldots, \mu_n \in \mathbb{R}$. Then*

$$\sum_{i=1}^n \left(\mu_i \sum_{j=i}^n \mathbf{v}_j \right) = \sum_{j=1}^n \left(\mathbf{v}_j \sum_{i=1}^j \mu_i \right).$$

PROOF:
Just calculate

$$\begin{aligned}
\sum_{i=1}^n \mu_i \sum_{j=i}^n \mathbf{v}_j &= \mu_1 \sum_{j=1}^n \mathbf{v}_j + \mu_2 \sum_{j=2}^n \mathbf{v}_j + \; \ldots \; + \mu_{n-1}(\mathbf{v}_{n-1} + \mathbf{v}_n) + \mu_n \mathbf{v}_n \\
&= \mathbf{v}_n \sum_{i=1}^n \mu_i + \mathbf{v}_{n-1} \sum_{i=1}^{n-1} \mu_i + \; \ldots \; + \mathbf{v}_2(\mu_1 + \mu_2) + \mathbf{v}_1 \mu_1 \\
&= \sum_{j=1}^n \mathbf{v}_j \sum_{i=1}^j \mu_i.
\end{aligned}$$

■

The next theorem states that the set S_σ is an n-simplex and provides a formula for its vertices.

Theorem 4.6 *For every $\sigma \in \text{Perm}[\{1,2,\dots,n\}]$ we have*

$$S_\sigma = \text{con}\left\{\sum_{j=1}^{n}\mathbf{e}_{\sigma(j)}, \sum_{j=2}^{n}\mathbf{e}_{\sigma(j)}, \dots, \sum_{j=n+1}^{n}\mathbf{e}_{\sigma(j)}\right\},$$

where $\mathbf{e}_{\sigma(i)}$ is the $\sigma(i)$-th unit vector in $\mathbb{R}^n$.

PROOF:
We first show the inclusion

$$\text{con}\left\{\sum_{j=1}^{n}\mathbf{e}_{\sigma(j)}, \sum_{j=2}^{n}\mathbf{e}_{\sigma(j)}, \dots, \sum_{j=n+1}^{n}\mathbf{e}_{\sigma(j)}\right\} \subset S_\sigma.$$

Let

$$\mathbf{y} \in \text{con}\left\{\sum_{j=1}^{n}\mathbf{e}_{\sigma(j)}, \sum_{j=2}^{n}\mathbf{e}_{\sigma(j)}, \dots, \sum_{j=n+1}^{n}\mathbf{e}_{\sigma(j)}\right\}.$$

Then there are $\lambda_1, \lambda_2, \dots, \lambda_{n+1} \in [0,1]$, such that

$$\mathbf{y} = \sum_{i=1}^{n+1}\lambda_i\sum_{j=i}^{n}\mathbf{e}_{\sigma(j)} \quad \text{and} \quad \sum_{i=1}^{n+1}\lambda_i = 1.$$

Because

$$y_{\sigma(k)} = \mathbf{y}\cdot\mathbf{e}_{\sigma(k)} = \left(\sum_{i=1}^{n+1}\lambda_i\sum_{j=i}^{n}\mathbf{e}_{\sigma(j)}\right)\cdot\mathbf{e}_{\sigma(k)} = \sum_{i=1}^{n+1}\sum_{j=i}^{n}\lambda_i\delta_{jk} = \sum_{i=1}^{k}\lambda_i,$$

it follows that $y_{\sigma(k)} \in [0,1]$ and $y_{\sigma(k)} \le y_{\sigma(l)}$ if $k \le l$, so $\mathbf{y} \in S_\sigma$.
We now show the inclusion

$$S_\sigma \subset \text{con}\left\{\sum_{j=1}^{n}\mathbf{e}_{\sigma(j)}, \sum_{j=2}^{n}\mathbf{e}_{\sigma(j)}, \dots, \sum_{j=n+1}^{n}\mathbf{e}_{\sigma(j)}\right\}.$$

Let $\mathbf{y} \in S_\sigma$. Then $0 \le y_{\sigma(1)} \le y_{\sigma(2)} \le \dots \le y_{\sigma(n)} \le 1$. Set $\lambda_1 := y_{\sigma(1)}$, set $\lambda_i := y_{\sigma(i)} - y_{\sigma(i-1)}$ for $i = 2,3,\dots,n$, and set $\lambda_{n+1} := 1 - y_{\sigma(n)}$. Then obviously $\sum_{i=1}^{n+1}\lambda_i = 1$ and by Lemma 4.5

$$\mathbf{y} = \sum_{j=1}^{n}\sum_{i=1}^{j}\lambda_i\mathbf{e}_{\sigma(j)} = \sum_{i=1}^{n+1}\lambda_i\sum_{j=i}^{n}\mathbf{e}_{\sigma(j)},$$

that is,

$$\mathbf{y} \in \text{con}\left\{\sum_{j=1}^{n}\mathbf{e}_{\sigma(j)}, \sum_{j=2}^{n}\mathbf{e}_{\sigma(j)}, \dots, \sum_{j=n+1}^{n}\mathbf{e}_{\sigma(j)}\right\}.$$

■

In the next theorem we show that for every $\alpha, \beta \in \text{Perm}[\{1,2,\dots,n\}]$ the intersection $S_\alpha \cap S_\beta$ is a simplex, whose vertices are exactly the vertices that are common to S_α and S_β.

Theorem 4.7 *Let $\alpha, \beta \in \text{Perm}[\{1, 2, \ldots, n\}]$. Then*

$$S_\alpha \cap S_\beta = \text{con}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k\},$$

where the $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k$ are the vertices that are common to S_α and S_β, that is,

$$\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k\} := \left\{\mathbf{x} \in \mathbb{R}^n \middle| \mathbf{x} = \sum_{j=a}^{n} \mathbf{e}_{\alpha(j)} = \sum_{j=a}^{n} \mathbf{e}_{\beta(j)} \text{ for some } a \in \{1, 2, \ldots, n+1\}\right\}.$$

Proof:
The inclusion $S_\alpha \cap S_\beta \supset \text{con}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k\}$ is trivial, so we only have to prove

$$S_\alpha \cap S_\beta \subset \text{con}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k\}. \tag{4.1}$$

To do this define $\sigma \in \text{Perm}[\{1, 2, \ldots, n\}]$ by $\sigma := \beta^{-1}\alpha$ and the set

$$\mathcal{A}_\sigma := \{x \in \{1, 2, \ldots, n+1\} \big| \sigma(\{1, 2, \ldots, x-1\}) = \{1, 2, \ldots, x-1\}\}.$$

Clearly

$$\begin{aligned}
\mathcal{A}_\sigma &:= \{x \in \{1, 2, \ldots, n+1\} \big| \sigma(\{1, 2, \ldots, x-1\}) = \{1, 2, \ldots, x-1\}\} \\
&= \{x \in \{1, 2, \ldots, n+1\} \big| \sigma(\{x, x+1, \ldots, n\}) = \{x, x+1, \ldots, n\}\} \\
&= \{x \in \{1, 2, \ldots, n+1\} \big| \alpha(\{x, x+1, \ldots, n\}) = \beta(\{x, x+1, \ldots, n\})\}
\end{aligned}$$

and

$$\sum_{j=a}^{n} \mathbf{e}_{\alpha(j)} = \sum_{j=b}^{n} \mathbf{e}_{\beta(j)},$$

if and only if $a = b$ and $\alpha(\{a, a+1, \ldots, n\}) = \beta(\{b, b+1, \ldots, n\})$.
Hence

$$\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k\} = \left\{\sum_{j=a_1}^{n} \mathbf{e}_{\alpha(j)}, \sum_{j=a_2}^{n} \mathbf{e}_{\alpha(j)}, \ldots, \sum_{j=a_k}^{n} \mathbf{e}_{\alpha(j)}\right\},$$

where $1 = a_1 < a_2 < \ldots < a_k = n+1$ are the elements of $\mathcal{A}_\sigma$ in an increasing order.
Let $\mathbf{x}$ be an arbitrary element in $S_\alpha \cap S_\beta$. Then there are $\mu_i, \lambda_i \in [0, 1]$, for $i = 1, 2, \ldots, n+1$, such that

$$\sum_{i=1}^{n+1} \mu_i = \sum_{i=1}^{n+1} \lambda_i = 1$$

and

$$\mathbf{x} = \sum_{i=1}^{n+1} \mu_i \sum_{j=i}^{n} \mathbf{e}_{\alpha(j)} = \sum_{i=1}^{n+1} \lambda_i \sum_{j=i}^{n} \mathbf{e}_{\beta(j)}.$$

We prove (4.1) by showing by mathematical induction, that $\mu_i = \lambda_i$ for all $i \in \{1, 2, \ldots, n+1\}$ and that $\mu_i = \lambda_i = 0$ for all $i \in \{1, 2, \ldots, n+1\} \setminus \mathcal{A}_\sigma$.
Lemma 4.5 implies

$$\mathbf{x} = \sum_{j=1}^{n} \mathbf{e}_{\alpha(j)} \sum_{i=1}^{j} \mu_i = \sum_{j=1}^{n} \mathbf{e}_{\beta(j)} \sum_{i=1}^{j} \lambda_i.$$

By comparing the components of the vectors on the left-hand and the right-hand side of this equation, we see that for every pair $(r, s) \in \{1, 2, \ldots, n\}^2$, such that $\alpha(r) = \beta(s)$, we must have

$$\sum_{i=1}^{r} \mu_i = \sum_{i=1}^{s} \lambda_i,$$

that is,

$$\sum_{i=1}^{m} \mu_i = \sum_{i=1}^{\sigma(m)} \lambda_i \quad \text{and} \quad \sum_{i=1}^{m} \lambda_i = \sum_{i=1}^{\sigma^{-1}(m)} \mu_i \tag{4.2}$$

for all $m = 1, 2, \ldots, n$.

Denote by $P(r)$ the following proposition, in which the number $r \in \mathbb{N}_{>0}$ is a variable:
"$\mu_i = \lambda_i$ for all $i \in \{1, 2, \ldots, a_r - 1\}$ and $\mu_i = \lambda_i = 0$ for all $i \in \{1, 2, \ldots, a_r - 1\} \setminus \{a_1, a_2, \ldots, a_{r-1}\}$"

From the definition of $\mathcal{A}_\sigma$ it follows that $a_1 = 1$, so the proposition $P(1)$ is obviously true. There is indeed nothing to prove. We now show that for $r < k$, $P(r)$ implies $P(r + 1)$.

Assume that $P(r)$ is true for some $1 \leq r < k$. Then $\mu_i = \lambda_i$ for all $i = 1, 2, \ldots, a_r - 1$ and $\sigma(a_r)$ and $\sigma^{-1}(a_r)$ must be greater than or equal to a_r. If $\sigma(a_r) = \sigma^{-1}(a_r) = a_r$, then trivially $a_r + 1 \in \mathcal{A}_\sigma$, that is, $a_{r+1} = a_r + 1$, and it follows from (4.2) that $\mu_{a_r} = \lambda_{a_r}$, which implies that $P(r + 1)$ is true.

Suppose $\sigma(a_r)$ and $\sigma^{-1}(a_r)$ are greater that a_r. Then it follows from (4.2), that

$$\sum_{i=1}^{a_r} \mu_i = \sum_{i=1}^{\sigma(a_r)} \lambda_i = \sum_{i=1}^{a_r-1} \mu_i + \lambda_{a_r} + \sum_{i=a_r+1}^{\sigma(a_r)} \lambda_i,$$

that is,

$$\mu_{a_r} = \lambda_{a_r} + \sum_{i=a_r+1}^{\sigma(a_r)} \lambda_i,$$

and similarly

$$\lambda_{a_r} = \mu_{a_r} + \sum_{i=a_r+1}^{\sigma^{-1}(a_r)} \mu_i.$$

By adding the two last equation, we see that

$$\sum_{i=a_r+1}^{\sigma(a_r)} \lambda_i + \sum_{i=a_r+1}^{\sigma^{-1}(a_r)} \mu_i = 0$$

and because $\mu_i, \lambda_i \in [0, 1]$ for all $i = 1, 2, \ldots, n$, this implies that

$$\mu_{a_r+1} = \mu_{a_r+2} = \ldots = \mu_{\sigma^{-1}(a_r)} = 0, \quad \lambda_{a_r+1} = \lambda_{a_r+2} = \ldots = \lambda_{\sigma(a_r)} = 0, \quad \text{and} \quad \mu_{a_r} = \lambda_{a_r}.$$

Define the integers a and b by

$$a := \max\{s < a_{r+1} \big| \mu_{a_r+1} = \mu_{a_r+2} = \ldots = \mu_s = 0\}$$

and

$$b := \max\{s < a_{r+1} \big| \lambda_{a_r+1} = \lambda_{a_r+2} = \ldots = \lambda_s = 0\}.$$

For all $m \in \{a_r + 1, a_r + 2, \ldots, a\}$ we have

$$\sum_{i=1}^{\sigma(m)} \lambda_i = \sum_{i=1}^{m} \mu_i = \sum_{i=1}^{a_r} \mu_i$$

and because $\mu_i = \lambda_i$ for all $i = 1, 2, \ldots, a_r$, this implies that

$$\sum_{i=a_r+1}^{\sigma(m)} \lambda_i = 0$$

and then

$$\lambda_{a_r+1} = \lambda_{a_r+2} = \ldots = \lambda_{\sigma(m)} = 0.$$

Therefore and because $\sigma(m) < a_{r+1}$ for all $m = 1, 2, \ldots, a_{r+1} - 1$ we have

$$b \geq \max\{\sigma(m)\big|m = a_r + 1, a_r + 2, \ldots, a\} = \max\{\sigma(m)\big|m = 1, 2, \ldots, a\},$$

where the equality on the right-hand side is a consequence of $\sigma(\{1, 2, \ldots, a_r - 1\}) = \{1, 2, \ldots, a_r - 1\}$ and $b \geq \sigma(a_r)$.

The set $\{\sigma(m)\big|m = 1, 2, \ldots, a\}$ is a subset of $\{1, 2, \ldots, n\}$ with a distinct elements, so

$$\max\{\sigma(m)\big|m = 1, 2, \ldots, a\} \geq a,$$

that is $b \geq a$. With similar reasoning, we can show that $a \geq b$. Hence $a = b$.

We have shown that $\mu_{a_r} = \lambda_{a_r}$, that there is a constant a, with $a_r < a < a_{r+1}$, such that

$$\mu_{a_r+1} = \mu_{a_r+2} = \ldots = \mu_a = \lambda_{a_r+1} = \lambda_{a_r+2} = \ldots = \lambda_a = 0,$$

and that $\sigma(\{1, 2, \ldots, a\}) = \{1, 2, \ldots, a\}$. This implies $a + 1 = a_{r+1} \in \mathcal{A}_\sigma$ and that the proposition $P(r + 1)$ is true, which completes the proof.

■

We now apply the last theorem to prove the general case. That the family of simplices

$$(\mathbf{z} + S_\sigma)_{\mathbf{z} \in \mathbb{Z}^n_{\geq 0}, \sigma \in \text{Perm}[\{1,2,\ldots,n\}]}$$

partitions $\mathbb{R}^n_{\geq 0}$ such that for every pairs $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n_{\geq 0}$ and $\alpha, \beta \in \text{Perm}[\{1, 2, \ldots, n\}]$ the intersection of the sets $\mathbf{a} + S_\alpha$ and $\mathbf{b} + S_\beta$ is a simplex whose vertices are exactly those vertices that are common to $\mathbf{a} + S_\alpha$ and $\mathbf{b} + S_\beta$.

Theorem 4.8 *Let $\alpha, \beta \in \text{Perm}[\{1, 2, \ldots, n\}]$ and $\mathbf{s}_\alpha, \mathbf{s}_\beta \in \mathbb{Z}^n$. Let $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k\}$ be the set of the vertices that are common to the simplices $\mathbf{s}_\alpha + S_\alpha$ and $\mathbf{s}_\beta + S_\beta$. Then $(\mathbf{s}_\alpha + S_\alpha) \cap (\mathbf{s}_\beta + S_\beta) = \emptyset$ if $\mathcal{C} = \emptyset$ and*

$$(\mathbf{s}_\alpha + S_\alpha) \cap (\mathbf{s}_\beta + S_\beta) = \text{con}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k\}$$

if $\mathcal{C} \neq \emptyset$.

PROOF:
Obviously $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k\} \neq \emptyset$ implies

$$(\mathbf{s}_\alpha + S_\alpha) \cap (\mathbf{s}_\beta + S_\beta) \supset \mathrm{con}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k\},$$

and $(\mathbf{s}_\alpha + S_\alpha) \cap (\mathbf{s}_\beta + S_\beta) = \emptyset$ implies $\mathcal{C} = \emptyset$, so we only have to prove that if $(\mathbf{s}_\alpha + S_\alpha) \cap (\mathbf{s}_\beta + S_\beta) \neq \emptyset$, then $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k\} \neq \emptyset$ and

$$(\mathbf{s}_\alpha + S_\alpha) \cap (\mathbf{s}_\beta + S_\beta) \subset \mathrm{con}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k\}.$$

Assume that $(\mathbf{s}_\alpha + S_\alpha) \cap (\mathbf{s}_\beta + S_\beta) \neq \emptyset$. Because

$$(\mathbf{s}_\alpha + S_\alpha) \cap (\mathbf{s}_\beta + S_\beta) = \mathbf{s}_\alpha + S_\alpha \cap (\mathbf{z} + S_\beta),$$

with $\mathbf{z} := \mathbf{s}_\beta - \mathbf{s}_\alpha$ we consider an arbitrary $\mathbf{x} \in S_\alpha \cap (\mathbf{z} + S_\beta)$. Then there are $\mu_i, \lambda_i \in [0,1]$ for $i = 1, 2, \ldots, n+1$, such that

$$\sum_{i=1}^{n+1} \mu_i = \sum_{i=1}^{n+1} \lambda_i = 1$$

and

$$\mathbf{x} = \sum_{i=1}^{n+1} \mu_i \sum_{j=i}^{n} \mathbf{e}_{\alpha(j)} = \sum_{i=1}^{n+1} \lambda_i \sum_{j=i}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z} = \sum_{i=1}^{n+1} \lambda_i \left(\sum_{j=i}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z} \right). \tag{4.3}$$

Because $S_\alpha, S_\beta \subset [0,1]^n$ the components of $\mathbf{z}$ must all be equal to -1, 0, or 1.

If $z_i = -1$, then the i-th component of the vectors

$$\sum_{j=1}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}, \quad \sum_{j=2}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}, \quad \ldots \quad , \quad \sum_{j=\beta^{-1}(i)}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}$$

is equal to 0 and the i-th component of the vectors

$$\sum_{j=\beta^{-1}(i)+1}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}, \quad \sum_{j=\beta^{-1}(i)+2}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}, \quad \ldots \quad , \quad \sum_{j=n+1}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}$$

is equal to -1. Because $x_i \geq 0$, this implies that

$$\lambda_{\beta^{-1}(i)+1} = \lambda_{\beta^{-1}(i)+2} = \ \ldots \ = \lambda_{n+1} = 0.$$

If $z_i = 1$, then the i-th component of the vectors

$$\sum_{j=1}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}, \quad \sum_{j=2}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}, \quad \ldots \quad , \quad \sum_{j=\beta^{-1}(i)}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}$$

is equal to 2 and the i-th component of the vectors

$$\sum_{j=\beta^{-1}(i)+1}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}, \quad \sum_{j=\beta^{-1}(i)+2}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}, \quad \ldots \quad , \quad \sum_{j=n+1}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}$$

is equal to 1. Because $x_i \leq 1$, this implies that

$$\lambda_1 = \lambda_2 = \ \dots \ = \lambda_{\beta^{-1}(i)} = 0.$$

It follows that there are integers $1 \leq r \leq s \leq n+1$ such that $\lambda_1 = \lambda_2 = \dots = \lambda_{r-1} = 0$, $\lambda_{s+1} = \lambda_{s+2} = \dots = \lambda_{n+1} = 0$, and the components of the vectors

$$\sum_{j=r}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}, \quad \sum_{j=r+1}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}, \quad \dots \quad , \sum_{j=s}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}$$

are all equal to 0 or 1. Let $a_1, a_2, \dots, a_{t-1}$ be the indices of those components of

$$\sum_{j=r}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}$$

that are equal to 0. By defining the permutation $\gamma \in \text{Perm}[\{1, 2, \dots, n\}]$ as follows,

$$\begin{aligned} &\gamma(i) := a_i, \text{ for all } i = 1, 2, \dots, t-1, \\ &\gamma(i) := \beta(r+i-t), \text{ for all } i = t, t+1, \dots, t+s-r, \end{aligned}$$

and with $\{b_{t+s-r+1}, b_{t+s-r+2}, \dots, b_n\} := \{1, 2, \dots, n\} \setminus \gamma(\{1, 2, \dots, t+s-r\})$,

$$\gamma(i) := b_i, \text{ for all } i = t+s-r+1, t+s-r+2, \dots, n,$$

we obtain,

$$\sum_{i=t+j}^{n} \mathbf{e}_{\gamma(i)} = \sum_{i=r+j}^{n} \mathbf{e}_{\beta(i)} + \mathbf{z}. \tag{4.4}$$

for all $j = 0, 1, \dots, s-r$. To see this, note that

$$\sum_{i=t}^{n} \mathbf{e}_{\gamma(i)}$$

has all components equal to 1, except for those with an index from the set $\gamma(\{1, 2, \dots, t-1\})$, which are equal to 0. But from $\gamma(\{1, 2, \dots, t-1\}) = \{a_1, a_2, \dots, a_{t-1}\}$ and the definition of the indices a_i, it follows that

$$\sum_{i=t}^{n} \mathbf{e}_{\gamma(i)} = \sum_{i=r}^{n} \mathbf{e}_{\beta(i)} + \mathbf{z}.$$

The equivalence for all $j = 0, 1, \dots, s-r$ then follows from

$$\begin{aligned} \sum_{i=t+j}^{n} \mathbf{e}_{\gamma(i)} &= \sum_{i=t}^{n} \mathbf{e}_{\gamma(i)} - \sum_{i=t}^{t+j-1} \mathbf{e}_{\gamma(i)} = \sum_{i=r}^{n} \mathbf{e}_{\beta(i)} + \mathbf{z} - \sum_{i=t}^{t+j-1} \mathbf{e}_{\beta(r+i-t)} \\ &= \sum_{i=r}^{n} \mathbf{e}_{\beta(i)} + \mathbf{z} - \sum_{i=r}^{r+j-1} \mathbf{e}_{\beta(i)} = \sum_{i=r+j}^{n} \mathbf{e}_{\beta(i)} + \mathbf{z}. \end{aligned}$$

Because of

$$\mathbf{x} \in S_\alpha \cap (\mathbf{z} + S_\beta)$$

and

$$\mathbf{x} = \sum_{i=r}^{s} \lambda_i \left(\sum_{j=i}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z} \right) = \sum_{i=t}^{t+s-r} \lambda_{i+r-t} \sum_{j=i}^{n} \mathbf{e}_{\gamma(j)}$$

we have

$$\mathbf{x} \in S_\alpha \cap S_\gamma.$$

By Theorem 4.7 and equation (4.3), it follows that

$$\mathbf{x} = \sum_{i=1}^{l} \mu_{u_i} \sum_{j=u_i}^{n} \mathbf{e}_{\alpha(j)} = \sum_{i=1}^{l} \mu_{u_i} \sum_{j=u_i}^{n} \mathbf{e}_{\gamma(j)},$$

where

$$\{u_1, u_2, \dots, u_l\} := \{y \in \{1, 2, \dots, n+1\} \big| \alpha(\{1, 2, \dots, y-1\}) = \gamma(\{1, 2, \dots, y-1\})\}.$$

Because the representation of an element of a simplex as a convex sum of its vertices is unique, some of the $u_1, u_2, \dots, u_l$ must be larger than or equal to t and less than or equal to $t+s-r$, say $u_1, u_2, \dots, u_k$, and we have

$$\sum_{j=u_i}^{n} \mathbf{e}_{\alpha(j)} = \sum_{j=u_i+r-t}^{n} \mathbf{e}_{\beta(j)} + \mathbf{z}$$

for all $i = 1, 2, \dots, k$, that is

$$\mathcal{C} := \{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_k\} = \{\mathbf{s}_\alpha + \sum_{j=u_1}^{n} \mathbf{e}_{\alpha(j)},\ \mathbf{s}_\alpha + \sum_{j=u_2}^{n} \mathbf{e}_{\alpha(j)},\ \dots\ ,\ \mathbf{s}_\alpha + \sum_{j=u_k}^{n} \mathbf{e}_{\alpha(j)}\}$$

and

$$(\mathbf{s}_\alpha + S_\alpha) \cap (\mathbf{s}_\beta + S_\beta) = \mathbf{s}_\alpha + S_\alpha \cap (\mathbf{z} + S_\beta) \subset \operatorname{con}\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_k\}.$$

■

We will use a simplicial partition of $\mathbb{R}^n$, invariable with respect to reflections through the hyperplanes $\mathbf{e}_i \cdot \mathbf{x} = 0$, $i = 1, 2, \dots, n$, as a domain for the function space of continuous piecewise affine functions. We will construct such a partition by first partitioning $\mathbb{R}^n_{\geq 0}$ into the family $(\mathbf{z}+S_\sigma)_{\mathbf{z}\in\mathbb{Z}^n_{\geq 0},\ \sigma\in\mathrm{Perm}[\{1,2,\dots,n\}]}$ and then we will extend this partition on $\mathbb{R}^n$ by use of the reflection functions $\mathbf{R}^{\mathcal{J}}$, where $\mathcal{J} \in \mathfrak{P}(\{1, 2, \dots, n\})$.

Definition 4.9 (Reflection functions $\mathbf{R}^{\mathcal{J}}$) *For every $\mathcal{J} \in \mathfrak{P}(\{1, 2, \dots, n\})$, we define the reflection function $\mathbf{R}^{\mathcal{J}} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$,*

$$\mathbf{R}^{\mathcal{J}}(\mathbf{x}) := \sum_{i=1}^{n} (-1)^{\chi_{\mathcal{J}}(i)} x_i \mathbf{e}_i$$

for all $\mathbf{x} \in \mathbb{R}^n$, where $\chi_{\mathcal{J}} : \{1, 2, \dots, n\} \longrightarrow \{0, 1\}$ is the characteristic function of the set $\mathcal{J}$.

□

Clearly $\mathbf{R}^{\mathcal{J}}$, where $\mathcal{J} := \{j_1, j_2, \dots, j_k\}$, represents reflections through the hyperplanes $\mathbf{e}_{j_1} \cdot \mathbf{x} = 0$, $\mathbf{e}_{j_2} \cdot \mathbf{x} = 0, \dots, \mathbf{e}_{j_k} \cdot \mathbf{x} = 0$ in succession.

We now finally have derived a simplicial partition of $\mathbb{R}^n$ that qualifies as a definition domain for continuous piecewise affine functions.

Theorem 4.10 *Let $(\mathbf{q}_{\mathbf{z}})_{\mathbf{z}\in\mathbb{Z}^n}$ be a collection of vectors in $\mathbb{R}^m$. Then there is exactly one continuous function $\mathbf{p} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ with the following properties:*

i) $\mathbf{p}(\mathbf{z}) = \mathbf{q}_{\mathbf{z}}$ *for every* $\mathbf{z} \in \mathbb{Z}^n$.

ii) For every $\mathcal{J} \in \mathfrak{P}(\{1, 2, \dots, n\})$*, every* $\sigma \in \text{Perm}[\{1, 2, \dots, n\}]$*, and every* $\mathbf{z} \in \mathbb{Z}^n_{\geq 0}$*, the restriction of the function* $\mathbf{p}$ *to the simplex*

$$\mathbf{R}^{\mathcal{J}}(\mathbf{z} + S_\sigma) = \text{con}\left\{\mathbf{R}^{\mathcal{J}}(\mathbf{z} + \sum_{j=1}^{n} \mathbf{e}_{\sigma(j)}),\ \mathbf{R}^{\mathcal{J}}(\mathbf{z} + \sum_{j=2}^{n} \mathbf{e}_{\sigma(j)}),\ \dots\ ,\ \mathbf{R}^{\mathcal{J}}(\mathbf{z} + \sum_{j=n+1}^{n} \mathbf{e}_{\sigma(j)})\right\}$$

is affine.

PROOF:
By Corollary 4.3 there is only one candidate for the function $\mathbf{p}$, namely, if

$$\mathbf{x} = \sum_{i=1}^{n+1} \lambda_i \mathbf{R}^{\mathcal{J}}(\mathbf{z} + \sum_{j=i}^{n} \mathbf{e}_{\sigma(j)})$$

for some $\mathcal{J} \in \mathfrak{P}(\{1, 2, \dots, n\})$, some $\sigma \in \text{Perm}[\{1, 2, \dots, n\}]$, and some $\mathbf{z} \in \mathbb{Z}^n_{\geq 0}$, we must have

$$\mathbf{p}(\mathbf{x}) = \sum_{i=1}^{n+1} \lambda_i \mathbf{p}(\mathbf{R}^{\mathcal{J}}(\mathbf{z} + \sum_{j=i}^{n} \mathbf{e}_{\sigma(j)})) = \sum_{i=1}^{n+1} \lambda_i \mathbf{q}_{\mathbf{R}^{\mathcal{J}}(\mathbf{z}+\sum_{j=i}^{n} \mathbf{e}_{\sigma(j)})}.$$

That the function $\mathbf{p}$ is properly defined and continuous by this formula is a direct consequence of Theorem 4.8, Lemma 4.2, Corollary 4.3, and the fact that if $\mathbf{x} \in \mathbf{R}^{\mathcal{I}}(\mathbf{z} + S_\alpha) \cap \mathbf{R}^{\mathcal{J}}(\mathbf{z} + S_\alpha)$ for some $\mathcal{I}, \mathcal{J} \in \mathfrak{P}(\{1, 2, \dots, n\})$, some $\sigma \in \text{Perm}[\{1, 2, \dots, n\}]$, and some $\mathbf{z} \in \mathbb{Z}^n_{\geq 0}$, then $i \in \mathcal{I}\Delta\mathcal{J}$ implies $x_i = 0$, where $\mathcal{I}\Delta\mathcal{J}$ denotes the symmetric difference ($:= [\mathcal{I} \cup \mathcal{J}] \setminus [\mathcal{I} \cap \mathcal{J}]$) of the sets $\mathcal{I}$ and $\mathcal{J}$. ■

After this preparation we have everything we need to define the function spaces CPWA. This will be done in the next section.

4.2 The function spaces CPWA

In this section we will introduce the function spaces CPWA and derive some results regarding the functions in these spaces, that will be useful when we prove that a feasible solution to the linear programming problem we specify in the next chapter can be used to define a CPWA Lyapunov function.

A CPWA space is a set of continuous affine functions from a subset of $\mathbb{R}^n$ into $\mathbb{R}$ with a given boundary configuration. If the subset is compact, then the boundary configuration makes it possible to parameterize the functions in the respective CPWA space with a finite number of real-valued parameters. Further, the CPWA spaces are vector spaces over $\mathbb{R}$ in the canonical way. They are thus well suited as a foundation, in the search of a Lyapunov function with a linear programming problem.

We first define the function spaces CPWA for subsets of $\mathbb{R}^n$ that are the unions of n-dimensional cubes.

Definition 4.11 (CPWA **function on a simple grid)** *Let $\mathcal{Z} \subset \mathbb{Z}^n$, $\mathcal{Z} \neq \emptyset$, be such that the interior of the set*

$$\mathcal{N} := \bigcup_{\mathbf{z} \in \mathcal{Z}} (\mathbf{z} + [0,1]^n),$$

is connected. The function space CPWA$[\mathcal{N}]$ *is then defined as follows.*

A function $p : \mathcal{N} \longrightarrow \mathbb{R}$ is in CPWA$[\mathcal{N}]$*, if and only if:*

i) p is continuous.

ii) For every simplex $\mathbf{R}^{\mathcal{J}}(\mathbf{z} + S_\sigma) \subset \mathcal{N}$, where $\mathbf{z} \in \mathbb{Z}^n_{\geq 0}$, $\mathcal{J} \in \mathfrak{P}(\{1, 2, \ldots, n\})$, and $\sigma \in$ Perm$[\{1, 2, \ldots\}]$, the restriction $p|_{\mathbf{R}^{\mathcal{J}}(\mathbf{z}+S_\sigma)}$ is affine.

□

It follows by Theorem 4.10 that the set CPWA$[\mathcal{N}]$ is not empty and that its elements are uniquely determined by their values on the set $\mathcal{N} \cap \mathbb{Z}^n$.

We will need continuous piecewise affine functions, defined by their values on grids with smaller grid steps than one, and we want to use grids with variable grid steps. We achieve this by using images of $\mathbb{Z}^n$ under mappings $\mathbb{R}^n \longrightarrow \mathbb{R}^n$, of which the components are continuous and strictly increasing functions $\mathbb{R} \longrightarrow \mathbb{R}$, affine on the intervals $[m, m+1]$ for all integers m, and map the origin on itself. We call such $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ mappings *piecewise scaling functions*.

Definition 4.12 (Piecewise scaling function) *A function* $\mathbf{PS} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *is called a piecewise scaling function, if and only if* $\mathbf{PS}(\mathbf{0}) = \mathbf{0}$ *and*

$$\mathbf{PS}(\mathbf{x}) := (\mathrm{PS}_1(\mathbf{x}), \mathrm{PS}_2(\mathbf{x}), \ldots, \mathrm{PS}_n(\mathbf{x})) = (\widetilde{\mathrm{PS}}_1(x_1), \widetilde{\mathrm{PS}}_2(x_2), \ldots, \widetilde{\mathrm{PS}}_n(x_n))$$

for all $\mathbf{x} \in \mathbb{R}^n$, where $\widetilde{\mathrm{PS}}_i \in$ CPWA$[\mathbb{R}]$ and is strictly increasing on $\mathbb{R}$ for all $i = 1, 2, \ldots, n$. Because the i-th component of $\mathbf{PS}$ *only depends on the i-th component of the argument, we will often write $\mathrm{PS}_i(x_i)$ instead of $\mathrm{PS}_i(\mathbf{x})$.*

□

Note that if $y_{i,j}$, $i = 1, 2, \ldots, n$ and $j \in \mathbb{Z}$, are real numbers such that $y_{i,j} < y_{i,j+1}$ and $y_{i,0} = 0$ for all $i = 1, 2, \ldots, n$ and all $j \in \mathbb{Z}$, then we can define a piecewise scaling function $\mathbf{PS} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by $\widetilde{\mathrm{PS}}_{\mathrm{i}}(j) := y_{i,j}$ for all $i = 1, 2, \ldots, n$ and all $j \in \mathbb{Z}$. Moreover, the piecewise scaling functions $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ are exactly the functions, that can be constructed in this way.

In the next definition we use piecewise scaling functions to define general CPWA spaces.

Definition 4.13 (CPWA **function, general)** *Let* $\mathbf{PS} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *be a piecewise scaling function and let $\mathcal{Z} \subset \mathbb{Z}^n$, $\mathcal{Z} \neq \emptyset$, be such that the interior of the set*

$$\mathcal{N} := \bigcup_{\mathbf{z} \in \mathcal{Z}} (\mathbf{z} + [0,1]^n)$$

is connected.

The function space CPWA$[\mathbf{PS}, \mathcal{N}]$ *is defined as*

$$\mathrm{CPWA}[\mathbf{PS}, \mathcal{N}] := \{p \circ \mathbf{PS}^{-1} \,\big|\, p \in \mathrm{CPWA}[\mathcal{N}]\}$$

and we denote by $\mathfrak{S}[\mathbf{PS}, \mathcal{N}]$ the set of the simplices in the family

$$(\mathbf{PS}(\mathbf{R}^{\mathcal{J}}(\mathbf{z} + S_\sigma)))_{\mathbf{z} \in \mathbb{Z}^n_{\geq 0},\ \mathcal{J} \in \mathfrak{P}(\{1,2,\ldots,n\}),\ \sigma \in \mathrm{Perm}[\{1,2,\ldots,n\}]}$$

that are contained in the image $\mathbf{PS}(\mathcal{N})$ of $\mathcal{N}$ under $\mathbf{PS}$.

□

Clearly

$$\{\mathbf{x} \in \mathbb{R}^n \big| \ \mathbf{x} \text{ is a vertex of a simplex in } \mathfrak{S}[\mathbf{PS}, \mathcal{N}]\} = \mathbf{PS}(\mathcal{N} \cap \mathbb{Z}^n)$$

and every function in CPWA$[\mathbf{PS}, \mathcal{N}]$ is continuous and is uniquely determined by its values on the grid $\mathbf{PS}(\mathcal{N} \cap \mathbb{Z}^n)$.

We will use functions from CPWA$[\mathbf{PS}, \mathcal{N}]$ to approximate functions in $\mathcal{C}^2(\mathbf{PS}(\mathcal{N}))$, that have bounded second order derivatives. The next lemma gives an upper bound of the approximation error of such a linearization.

Lemma 4.14 *Let $\sigma \in$ Perm$[\{1, 2, \dots, n\}]$, let $\mathcal{J} \in \mathfrak{P}(\{1, 2, \dots, n\})$, let $\mathbf{z} \in \mathbb{Z}^n_{\geq 0}$, let $\mathbf{R}^\mathcal{J}$ be a reflection function, and let* **PS** *be a piecewise scaling function. Denote by S the n-simplex that is the convex combination of the vertices*

$$\mathbf{y}_i := \mathbf{PS}(\mathbf{R}^\mathcal{J}(\mathbf{z} + \sum_{j=i}^{n+1} \mathbf{e}_{\sigma(j)})), \quad i = 1, 2, \dots, n+1,$$

and let $f \in \mathcal{C}^2(\mathcal{U})$ be a function defined on a domain $S \subset \mathcal{U} \subset \mathbb{R}^n$. For every $i = 1, 2, \dots, n+1$ and every $k = 1, 2, \dots, n$ define the constant

$$A_{k,i} := |\mathbf{e}_k \cdot (\mathbf{y}_i - \mathbf{y}_{n+1})|$$

and for every $r, s = 1, 2, \dots, n$ let B_{rs} be a constant, such that

$$B_{rs} \geq \max_{\mathbf{x} \in S} \left| \frac{\partial^2 f}{\partial x_r \partial x_s}(\mathbf{x}) \right|.$$

Define for every $i = 1, 2, \dots, n+1$ the constant

$$E_i := \frac{1}{2} \sum_{r,s=1}^{n} B_{rs} A_{r,i}(A_{s,1} + A_{s,i}).$$

Then for every convex combination

$$\mathbf{y} := \sum_{i=1}^{n+1} \lambda_i \mathbf{y}_i, \tag{4.5}$$

of the vertices of the simplex S we have

$$\left| f(\mathbf{y}) - \sum_{i=1}^{n+1} \lambda_i f(\mathbf{y}_i) \right| \leq \sum_{i=1}^{n+1} \lambda_i E_i.$$

PROOF:

Let $\mathbf{y}$ be as in equation (4.5). Then, by Taylor's theorem, there is a vector $\mathbf{y}_\mathbf{x}$ on the line-segment between $\mathbf{y}_{n+1}$ and $\mathbf{y}$, such that

$$f(\mathbf{y}) = f(\mathbf{y}_{n+1}) + \nabla f(\mathbf{y}_{n+1}) \cdot (\mathbf{y} - \mathbf{y}_{n+1}) + \frac{1}{2} \sum_{r,s=1}^{n} [\mathbf{e}_r \cdot (\mathbf{y} - \mathbf{y}_{n+1})][\mathbf{e}_s \cdot (\mathbf{y} - \mathbf{y}_{n+1})] \frac{\partial^2 f}{\partial x_r \partial x_s}(\mathbf{y}_\mathbf{x})$$

$$= \sum_{i=1}^{n+1} \lambda_i \Big(f(\mathbf{y}_{n+1}) + \nabla f(\mathbf{y}_{n+1}) \cdot (\mathbf{y}_i - \mathbf{y}_{n+1})$$

$$+ \frac{1}{2} \sum_{r,s=1}^{n} [\mathbf{e}_r \cdot (\mathbf{y}_i - \mathbf{y}_{n+1})][\mathbf{e}_s \cdot (\mathbf{y} - \mathbf{y}_{n+1})] \frac{\partial^2 f}{\partial x_r \partial x_s}(\mathbf{y}_\mathbf{x}) \Big)$$

and for every $i = 1, 2, \dots, n$ there is a vector $\mathbf{y}_{i,\mathbf{x}}$ on the line-segment between $\mathbf{y}_i$ and $\mathbf{y}_{n+1}$ such that

$$f(\mathbf{y}_i) = f(\mathbf{y}_{n+1}) + \nabla f(\mathbf{y}_{n+1}) \cdot (\mathbf{y}_i - \mathbf{y}_{n+1}) + \frac{1}{2}\sum_{r,s=1}^{n} [\mathbf{e}_r \cdot (\mathbf{y}_i - \mathbf{y}_{n+1})][\mathbf{e}_s \cdot (\mathbf{y}_i - \mathbf{y}_{n+1})] \frac{\partial^2 f}{\partial x_r \partial x_s}(\mathbf{y}_{i,\mathbf{x}}).$$

Further, because a simplex is a convex set, the vectors $\mathbf{y}_\mathbf{x}$ and $\mathbf{y}_{1,\mathbf{x}}, \mathbf{y}_{2,\mathbf{x}}, \dots, \mathbf{y}_{n,\mathbf{x}}$ are all in S. But then

$$\begin{aligned}\left| f(\mathbf{y}) - \sum_{i=1}^{n+1} \lambda_i f(\mathbf{y}_i) \right| &\leq \frac{1}{2}\sum_{i=1}^{n+1} \lambda_i \sum_{r,s=1}^{n} |\mathbf{e}_r \cdot (\mathbf{y}_i - \mathbf{y}_{n+1})| \left(|\mathbf{e}_s \cdot (\mathbf{y} - \mathbf{y}_{n+1})| + |\mathbf{e}_s \cdot (\mathbf{y}_i - \mathbf{y}_{n+1})|\right) B_{rs} \\ &= \frac{1}{2}\sum_{i=1}^{n+1} \lambda_i \sum_{r,s=1}^{n} B_{rs} A_{r,i} \left(|\mathbf{e}_s \cdot (\mathbf{y} - \mathbf{y}_{n+1})| + A_{s,i}\right)\end{aligned}$$

and because

$$|\mathbf{e}_s \cdot (\mathbf{y} - \mathbf{y}_{n+1})| \leq \sum_{i=1}^{n+1} \lambda_i |\mathbf{e}_s \cdot (\mathbf{y}_i - \mathbf{y}_{n+1})| \leq |\mathbf{e}_s \cdot (\mathbf{y}_1 - \mathbf{y}_{n+1})| = A_{s,1}$$

it follows that

$$\left| f(\mathbf{y}) - \sum_{i=1}^{n+1} \lambda_i f(\mathbf{y}_i) \right| \leq \frac{1}{2}\sum_{i=1}^{n+1} \lambda_i \sum_{r,s=1}^{n} B_{rs} A_{r,i}(A_{s,1} + A_{s,i}) = \sum_{i=1}^{n+1} \lambda_i E_i.$$

■

An affine function p, defined on a simplex $S \subset \mathbb{R}^n$ and with values in $\mathbb{R}$, has the algebraic form $p(\mathbf{x}) = \mathbf{w} \cdot \mathbf{x} + q$, where $\mathbf{w}$ is a constant vector in $\mathbb{R}^n$ and q is constant in $\mathbb{R}$. Another characterization of p is given by specifying its values at the vertices as stated by Lemma 4.2. The next lemma gives a formula for the components of the vector $\mathbf{w}$ when the values of p at the vertices of S are known and S is a simplex in $\mathfrak{S}[\mathbf{PS}, \mathcal{N}]$.

Lemma 4.15 *Let $\mathbf{PS} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a piecewise scaling function, let $\mathbf{z} \in \mathbb{Z}^n_{\geq 0}$, let $\mathcal{J} \in \mathfrak{P}(\{1, 2, \dots, n\})$, let $\sigma \in \text{Perm}[\{1, 2, \dots, n\}]$, and let $p(\mathbf{x}) := \mathbf{w} \cdot \mathbf{x} + q$ be an affine function defined on the n-simplex with the vertices*

$$\mathbf{y}_i := \mathbf{PS}(\mathbf{R}^\mathcal{J}(\mathbf{z} + \sum_{j=i}^{n} \mathbf{e}_{\sigma(j)})), \quad i = 1, 2, \dots, n.$$

Then

$$\mathbf{w} = \sum_{i=1}^{n} \frac{p(\mathbf{y}_i) - p(\mathbf{y}_{i+1})}{\mathbf{e}_{\sigma(i)} \cdot (\mathbf{y}_i - \mathbf{y}_{i+1})} \mathbf{e}_{\sigma(i)}.$$

PROOF:
For any $i \in \{1, 2, \dots, n\}$ we have

$$p(\mathbf{y}_i) - p(\mathbf{y}_{i+1}) = \mathbf{w} \cdot (\mathbf{y}_i - \mathbf{y}_{i+1}) = \sum_{k=1}^{n} w_{\sigma(k)} [\mathbf{e}_{\sigma(k)} \cdot (\mathbf{y}_i - \mathbf{y}_{i+1})] = w_{\sigma(i)} [\mathbf{e}_{\sigma(i)} \cdot (\mathbf{y}_i - \mathbf{y}_{i+1})]$$

because the components of the vectors $\mathbf{y}_i$ and $\mathbf{y}_{i+1}$ are all equal with except of the $\sigma(i)$-th one. But then

$$w_{\sigma(i)} = \frac{p(\mathbf{y}_i) - p(\mathbf{y}_{i+1})}{\mathbf{e}_{\sigma(i)} \cdot (\mathbf{y}_i - \mathbf{y}_{i+1})}$$

and we have finished the proof.

■

In this chapter we have defined the function spaces CPWA and we have proved some important properties of the functions in these spaces. In the next chapter we will state our linear programming problem, of which every feasible solution parameterizes a CPWA Lyapunov function for the Switched System 1.9 used in the derivation of its linear constraints.

Chapter 5

A linear programming problem to construct Lyapunov functions for arbitrary switched systems

In this chapter we define a linear programming problem, of which every feasible solution parameterizes a Lyapunov function for the Switched System 1.9. The Lyapunov function is of class CPWA. In the first section we define the linear programming problem in Definition 5.1. In the definition the linear constraints are grouped into four classes, **LC1**, **LC2**, **LC3**, and **LC4**, for linear constraints 1, 2, 3, and 4 respectively. In the sections thereafter we show how the variables of the linear programming problem that fulfill these constraints can be used to parameterize functions that meet the conditions **(L1)** and **(L2)** of Definition 2.17, the definition of a Lyapunov function. Then we state and discuss the results in Section 5.7 and in Section 5.8 we consider a simpler linear programming, defined in Definition 5.5, for autonomous systems and we show that it is equivalent to the linear programming problem in Definition 5.1 with additional constraints that force the parameterized CPWA Lyapunov function to be time-invariant.

5.1 The definition of the linear programming problem

The next definition plays a central role in this work. It is generalization of the linear programming problems presented in [36], [35], [11], and [10] to serve the nonautonomous Switched System 1.9.

Definition 5.1 (Linear programming problem $\mathbf{LP}(\{\mathbf{f}_p|p \in \mathcal{P}\}, \mathcal{N}, \mathbf{PS}, \mathbf{t}, \mathcal{D}, \|\cdot\|)$) *Consider the Switched System 1.9 and assume that there are real-valued constants $0 \leq T' < T''$ such that the functions $\mathbf{f}_p$, $p \in \mathcal{P}$, are all in $[\mathcal{C}^2([T', T''] \times \mathcal{U})]^n$. Let $\mathbf{PS} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a piecewise scaling function and $\mathcal{N} \subset \mathcal{U}$ be such that the interior of the set*

$$\mathcal{M} := \bigcup_{\substack{\mathbf{z} \in \mathbb{Z}^n \\ \mathbf{PS}(\mathbf{z}+[0,1]^n) \subset \mathcal{N}}} \mathbf{PS}(\mathbf{z} + [0,1]^n)$$

is a connected set that contains the origin. Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$ and let

$$\mathcal{D} := \mathbf{PS}(\,]d_1^-, d_1^+[\, \times \,]d_2^-, d_2^+[\, \times \ldots \times \,]d_n^-, d_n^+[\,)$$

be a set, of which the closure is contained in the interior of $\mathcal{M}$*, and either* $\mathcal{D} = \emptyset$ *or* d_i^- *and* d_i^+ *are integers such that* $d_i^- \leq -1$ *and* $1 \leq d_i^+$ *for every* $i = 1, 2, \ldots, n$*. Finally, let* $\mathbf{t} := (t_0, t_1, \ldots, t_M) \in \mathbb{R}^{M+1}$*,* $M \in \mathbb{N}_{>0}$*, be a vector such that* $T' =: t_0 < t_1 < \ldots < t_M := T''$*.*

Before we go on, it is very practical to introduce an alternate notation for the vectors $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n$*, because it considerably shortens the formulae in the linear programming problem. We identify the time* t *with the zeroth component* $\tilde{x}_0$ *of the vector*

$$\tilde{\mathbf{x}} := (\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_n)$$

and $\mathbf{x}$ *with the components 1 to* n*, that is* $t := \tilde{x}_0$ *and* $\tilde{x}_i := x_i$ *for all* $i = 1, 2, \ldots, n$*. Then, the systems*

$$\dot{\mathbf{x}} = \mathbf{f}_p(t, \mathbf{x}), \qquad p \in \mathcal{P},$$

can be written in the equivalent form

$$\frac{d}{d\tilde{x}_0}\tilde{\mathbf{x}} = \tilde{\mathbf{f}}_p(\tilde{\mathbf{x}}), \qquad p \in \mathcal{P},$$

where

$$\begin{aligned}\tilde{\mathbf{f}}_p(\tilde{\mathbf{x}}) &:= \left[\tilde{f}_{p,0}(\tilde{\mathbf{x}}), \tilde{f}_{p,1}(\tilde{\mathbf{x}}), \tilde{f}_{p,2}(\tilde{\mathbf{x}}), \ldots, \tilde{f}_{p,n}(\tilde{\mathbf{x}})\right] \\ &:= \left[1, f_{p,1}(t, \mathbf{x}), f_{p,2}(t, \mathbf{x}), \ldots, f_{p,n}(t, \mathbf{x})\right],\end{aligned}$$

(*Recall that* $f_{p,i}$ *denotes the* i*-th component of the function* $\mathbf{f}_p$*.*)

that is, $\tilde{f}_{p,0} := 1$ *and* $\tilde{f}_{p,i}(\tilde{\mathbf{x}}) := f_{p,i}(t, \mathbf{x})$*, where* $\tilde{\mathbf{x}} = (t, \mathbf{x})$*, for all* $p \in \mathcal{P}$ *and all* $i = 1, 2, \ldots, n$*.*

Further, let $\mathrm{PS}_0 : \mathbb{R} \longrightarrow \mathbb{R}$ *be a piecewise scaling function such that* $\mathrm{PS}_0(i) := t_i$ *for all* $i = 0, 1, \ldots, M$ *and define the piecewise scaling function*

$$\widetilde{\mathbf{PS}} : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R} \times \mathbb{R}^n$$

through

$$\widetilde{\mathbf{PS}}(\tilde{\mathbf{x}}) := \left[\mathrm{PS}_0(\tilde{x}_0), \mathrm{PS}_1(\tilde{x}_1), \ldots, \mathrm{PS}_n(\tilde{x}_n))\right],$$

that is,

$$\widetilde{\mathbf{PS}}(\tilde{\mathbf{x}}) = \left[\mathrm{PS}_0(t), \mathbf{PS}(\mathbf{x})\right],$$

where $\tilde{\mathbf{x}} = (t, \mathbf{x})$*.*

We will use the standard orthonormal basis in $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$*, but start the indexing at zero (use* $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_n$*), that is,*

$$\tilde{\mathbf{x}} := \sum_{i=0}^{n} \tilde{x}_i \mathbf{e}_i = t\mathbf{e}_0 + \sum_{i=1}^{n} x_i \mathbf{e}_i.$$

Because we do not have to consider negative time-values $t = \tilde{x}_0 < 0$*, it is more convenient to use reflection functions that do always leave the zeroth-component of* $\tilde{\mathbf{x}} = (t, \mathbf{x})$ *unchanged. Therefore, we define for every reflection function* $\mathbf{R}^{\mathcal{J}} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$*, where* $\mathcal{J} \subset \{1, 2, \ldots, n\}$*, the function* $\widetilde{\mathbf{R}}^{\mathcal{J}} : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R} \times \mathbb{R}^n$ *through*

$$\widetilde{\mathbf{R}}^{\mathcal{J}}(\tilde{\mathbf{x}}) := \left[\tilde{x}_0, \mathbf{R}^{\mathcal{J}}(\mathbf{x})\right] := t\mathbf{e}_0 + \sum_{i=1}^{n} (-1)^{\chi_{\mathcal{J}}(i)} x_i \mathbf{e}_i.$$

We define the mapping $\|\cdot\|_*: \mathbb{R}\times\mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ *through*

$$\|(\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_n)\|_* := \|(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)\|.$$

Then, obviously, $\|\tilde{\mathbf{x}}\|_* = \|\mathbf{x}\|$ *for all* $\tilde{\mathbf{x}} = (t, \mathbf{x}) \in \mathbb{R}\times\mathbb{R}^n$.

The linear programming problem $\mathbf{LP}(\{\mathbf{f}_p \big| p \in \mathcal{P}\}, \mathcal{N}, \mathbf{PS}, \mathbf{t}, \mathcal{D}, \|\cdot\|)$ *is now constructed in the following way:*

i) Define the sets

$$\mathcal{G} := \{\tilde{\mathbf{x}} \in \mathbb{R}\times\mathbb{R}^n \big| \tilde{\mathbf{x}} \in \widetilde{\mathbf{PS}}(\mathbb{Z}\times\mathbb{Z}^n) \cap \big([T', T''] \times (\mathcal{M}\setminus\mathcal{D})\big)\}$$

and

$$\mathcal{X}^{\|\cdot\|} := \{\|\mathbf{x}\| \big| \ \mathbf{x} \in \mathbf{PS}(\mathbb{Z}^n) \cap \mathcal{M}\}.$$

The set $\mathcal{G}$ *is the grid, on which we will derive constraints on the values of the* CPWA *Lyapunov function, and* $\mathcal{X}^{\|\cdot\|}$ *is the set of distances of all relevant points in the state-space to the origin with respect to the norm* $\|\cdot\|$.

ii) Define for every $\sigma \in \operatorname{Perm}[\{0, 1, \ldots, n\}]$ *and every* $i = 0, 1, \ldots, n+1$ *the vector*

$$\mathbf{x}_i^\sigma := \sum_{j=i}^{n} \mathbf{e}_{\sigma(j)},$$

where, of course, the empty sum is interpreted as $\mathbf{0} \in \mathbb{R}\times\mathbb{R}^n$.

iii) Define the set $\mathcal{Z}$ *through:*
The tuple $(\mathbf{z}, \mathcal{J})$*, where* $\mathbf{z} := (z_0, z_1, \ldots, z_n) \in \mathbb{Z}_{\geq 0}\times\mathbb{Z}^n_{\geq 0}$ *and* $\mathcal{J} \in \mathfrak{P}(\{1, 2, \ldots, n\}$*, is an element of* $\mathcal{Z}$*, if and only if*

$$\widetilde{\mathbf{PS}}(\widetilde{\mathbf{R}}^{\mathcal{J}}(\mathbf{z} + [0,1]^{n+1})) \subset [T', T''] \times \big(\mathcal{M}\setminus\mathcal{D}\big).$$

Note that this definition implies that

$$\bigcup_{(\mathbf{z},\mathcal{J})\in\mathcal{Z}} \widetilde{\mathbf{PS}}(\widetilde{\mathbf{R}}^{\mathcal{J}}(\mathbf{z} + [0,1]^{n+1})) = [T', T''] \times \big(\mathcal{M}\setminus\mathcal{D}\big).$$

iv) For every $(\mathbf{z}, \mathcal{J}) \in \mathcal{Z}$*, every* $\sigma \in \operatorname{Perm}[\{0, 1, \ldots, n\}]$*, and every* $i = 0, 1, \ldots, n+1$ *we set*

$$\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})} := \widetilde{\mathbf{PS}}(\widetilde{\mathbf{R}}^{\mathcal{J}}(\mathbf{z} + \mathbf{x}_i^\sigma)).$$

The vectors $\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}$ *are the vertices of the simplices in our simplicial partition of the set* $[T', T''] \times \big(\mathcal{M}\setminus\mathcal{D}\big)$*. The position of the simplex is given by* $(\mathbf{z}, \mathcal{J})$*, where* z_0 *specifies the position in time and* $(z_1, z_2, \ldots, z_n)$ *specifies the position in the state-space. Further,* σ *specifies the simplex and* i *specifies the vertex of the simplex.*

v) Define the set

$$\mathcal{Y} := \left\{\{\mathbf{y}_{\sigma,k}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,k+1}^{(\mathbf{z},\mathcal{J})}\} \Big| \sigma \in \operatorname{Perm}[\{0, 1, \ldots, n\}], (\mathbf{z}, \mathcal{J}) \in \mathcal{Z}, \text{ and } k \in \{0, 1, \ldots, n\}\right\}.$$

The set $\mathcal{Y}$ *is the set of all pairs of neighboring grid points in the grid* $\mathcal{G}$.

vi) *For every* $p \in \mathcal{P}$*, every* $(\mathbf{z},\mathcal{J}) \in \mathcal{Z}$*, and every* $r,s = 0,1,\ldots,n$ *let* $B_{p,rs}^{(\mathbf{z},\mathcal{J})}$ *be a real-valued constant, such that*

$$B_{p,rs}^{(\mathbf{z},\mathcal{J})} \geq \max_{i=1,2,\ldots,n} \sup_{\tilde{\mathbf{x}} \in \widetilde{\mathbf{PS}}(\tilde{\mathbf{R}}^{\mathcal{J}}(\mathbf{z}+[0,1]^{n+1}))} \left| \frac{\partial^2 \tilde{f}_{p,i}}{\partial \tilde{x}_r \partial \tilde{x}_s}(\tilde{\mathbf{x}}) \right|.$$

The constants $B_{p,rs}^{(\mathbf{z},\mathcal{J})}$ *are local bounds on the second-order derivatives of the functions* $\tilde{\mathbf{f}}_p$, $p \in \mathcal{P}$*, with regard to the infinity norm* $\|\cdot\|_\infty$*, similar to the constants* B_{rs} *in Lemma 4.14. Note, that because* $\tilde{f}_{p,0} := 1$*, the zeroth-components can be left out in the definition of the* $B_{p,rs}^{(\mathbf{z},\mathcal{J})}$ *because they are identically zero anyways. Further, for every* $r,s = 0,1,\ldots,n$ *and every* $\tilde{\mathbf{x}} = (t,\mathbf{x})$*,*

$$\frac{\partial^2 \tilde{f}_{p,i}}{\partial \tilde{x}_r \partial \tilde{x}_s}(\tilde{\mathbf{x}}) = \frac{\partial^2 f_{p,i}}{\partial x_r \partial x_s}(t,\mathbf{x})$$

if we read ∂x_0 *as* ∂t *on the right-hand side of the equation. Finally, note that if* B *is real-valued constant such that*

$$B \geq \sup_{\tilde{\mathbf{x}} \in [T',T''] \times (\mathcal{M} \setminus \mathcal{D})} \| \frac{\partial^2 \mathbf{f}_p}{\partial \tilde{x}_r \partial \tilde{x}_s}(\tilde{\mathbf{x}}) \|_\infty, \quad \text{for all } p \in \mathcal{P} \text{ and all } r,s = 0,1,\ldots,n,$$

then, of course, we can set $B_{p,rs}^{(\mathbf{z},\mathcal{J})} = B$ *for all* $p \in \mathcal{P}$*, all* $(\mathbf{z},\mathcal{J}) \in \mathcal{Z}$*, and all* $r,s = 0,1,\ldots,n$*. Tighter bounds, however, might save a lot of computational efforts in a search for a feasible solution to the linear programming problem.*

vii) *For every* $(\mathbf{z},\mathcal{J}) \in \mathcal{Z}$*, every* $i,k = 0,1,\ldots,n$*, and every* $\sigma \in \mathrm{Perm}[\{0,1,\ldots,n\}]$*, define*

$$A_{\sigma,k,i}^{(\mathbf{z},\mathcal{J})} := \left| \mathbf{e}_k \cdot (\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,n+1}^{(\mathbf{z},\mathcal{J})}) \right|.$$

The $A_{\sigma,k,i}^{(\mathbf{z},\mathcal{J})}$ *are constants similar to the constants* $A_{k,i}$ *in Lemma 4.14.*

viii) *Define the constant*

$$x_{\min,\partial\mathcal{M}} := \min\{\|\mathbf{x}\| \,\big|\, \mathbf{x} \in \mathbf{PS}(\mathbb{Z}^n) \cap \partial\mathcal{M}\},$$

where $\partial\mathcal{M}$ *is the boundary of the set* $\mathcal{M}$*.*

ix) *For every* $p \in \mathcal{P}$*, every* $(\mathbf{z},\mathcal{J}) \in \mathcal{Z}$*, every* $\sigma \in \mathrm{Perm}[\{0,1,\ldots,n\}]$*, and every* $i = 0,1,\ldots,n+1$ *set*

$$E_{p,\sigma,i}^{(\mathbf{z},\mathcal{J})} := \frac{1}{2} \sum_{r,s=0}^{n} B_{p,rs}^{(\mathbf{z},\mathcal{J})} A_{\sigma,r,i}^{(\mathbf{z},\mathcal{J})} (A_{\sigma,s,i}^{(\mathbf{z},\mathcal{J})} + A_{\sigma,s,0}^{(\mathbf{z},\mathcal{J})}). \tag{5.1}$$

ix) *Let* $\varepsilon > 0$ *and* $\delta > 0$ *be arbitrary constants.*

The variables of the linear programming problem are:

$$\begin{aligned}
&\Upsilon, \\
&\Psi[y], \quad \text{for all } y \in \mathcal{X}^{\|\cdot\|}, \\
&\Gamma[y], \quad \text{for all } y \in \mathcal{X}^{\|\cdot\|}, \\
&V[\tilde{\mathbf{x}}], \quad \text{for all } \tilde{\mathbf{x}} \in \mathcal{G}, \\
&C[\{\tilde{\mathbf{x}},\tilde{\mathbf{y}}\}], \quad \text{for all } \{\tilde{\mathbf{x}},\tilde{\mathbf{y}}\} \in \mathcal{Y}.
\end{aligned}$$

Considering Definition 2.17, the definition of a Lyapunov function, the variables $\Psi[y]$ correspond to the function α_1, the variables $\Gamma[y]$ to the function ψ, and the variables $V[\tilde{\mathbf{x}}]$ to the Lyapunov function V, the $\tilde{x}_0$ component representing the time t. The variables $C[\{\tilde{\mathbf{x}},\tilde{\mathbf{y}}\}]$ are local bounds on the gradient $\nabla_{\tilde{\mathbf{x}}}V^{Lya}$ of the Lyapunov function V^{Lya} to be constructed and Υ is a corresponding global bound.

The linear constraints of the linear programming problem are:

LC1) *Let $y_0, y_1, \dots, y_K$ be the elements of $\mathcal{X}^{\|\cdot\|}$ in an increasing order. Then*

$$\Psi[y_0] = \Gamma[y_0] = 0,$$
$$\varepsilon y_1 \le \Psi[y_1],$$
$$\varepsilon y_1 \le \Gamma[y_1],$$

and for every $i = 1, 2, \dots, K-1$:

$$\frac{\Psi[y_i]-\Psi[y_{i-1}]}{y_i - y_{i-1}} \le \frac{\Psi[y_{i+1}]-\Psi[y_i]}{y_{i+1}-y_i}$$

and

$$\frac{\Gamma[y_i]-\Gamma[y_{i-1}]}{y_i - y_{i-1}} \le \frac{\Gamma[y_{i+1}]-\Gamma[y_i]}{y_{i+1}-y_i}.$$

LC2) *For every $\tilde{\mathbf{x}} \in \mathcal{G}$:*

$$\Psi[\|\tilde{\mathbf{x}}\|_*] \le V[\tilde{\mathbf{x}}].$$

If $\mathcal{D} = \emptyset$, then, whenever $\|\tilde{\mathbf{x}}\|_* = 0$:

$$V[\tilde{\mathbf{x}}] = 0.$$

If $\mathcal{D} \neq \emptyset$, then, whenever $(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_2) \in \mathbf{PS}(\mathbb{Z}^n) \cap \partial\mathcal{D}$:

$$V[\tilde{\mathbf{x}}] \le \Psi[x_{\min,\partial\mathcal{M}}] - \delta.$$

Further, if $\mathcal{D} \neq \emptyset$, then for every $i = 1, 2, \dots, n$ and every $j = 0, 1, \dots, M$:

$$V[\mathrm{PS}_0(j)\mathbf{e}_0 + \mathrm{PS}_i(d_i^-)\mathbf{e}_i] \le -\Upsilon \cdot \mathrm{PS}_i(d_i^-) \quad \textit{and} \quad V[\mathrm{PS}_0(j)\mathbf{e}_0 + \mathrm{PS}_i(d_i^+)\mathbf{e}_i] \le \Upsilon \cdot \mathrm{PS}_i(d_i^+).$$

LC3) *For every $\{\tilde{\mathbf{x}},\tilde{\mathbf{y}}\} \in \mathcal{Y}$:*

$$-C[\{\tilde{\mathbf{x}},\tilde{\mathbf{y}}\}] \cdot \|\tilde{\mathbf{x}}-\tilde{\mathbf{y}}\|_\infty \le V[\tilde{\mathbf{x}}] - V[\tilde{\mathbf{y}}] \le C[\{\tilde{\mathbf{x}},\tilde{\mathbf{y}}\}] \cdot \|\tilde{\mathbf{x}}-\tilde{\mathbf{y}}\|_\infty \le \Upsilon \cdot \|\tilde{\mathbf{x}}-\tilde{\mathbf{y}}\|_\infty.$$

LC4) *For every $p \in \mathcal{P}$, every $(\mathbf{z},\mathcal{J}) \in \mathcal{Z}$, every $\sigma \in \mathrm{Perm}[\{0,1,\dots,n\}]$, and every $i = 0, 1, \dots, n+1$:*

$$-\Gamma\left[\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_*\right] \ge \sum_{j=0}^{n}\left(\frac{V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - V[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})}\tilde{f}_{p,\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) + E_{p,\sigma,i}^{(\mathbf{z},\mathcal{J})} C[\{\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}\}]\right).$$

As the objective of the linear programming problem is not needed to parameterize a CPWA *Lyapunov function we do not define it here.*

□

Note that the values of the constants $\varepsilon > 0$ and $\delta > 0$ do not affect whether there is a feasible solution to the linear program or not. If there is a feasible solution for $\varepsilon := \varepsilon' > 0$ and $\delta := \delta' > 0$, then there is a feasible solution for all $\varepsilon := \varepsilon^* > 0$ and $\delta := \delta^* > 0$. Just multiply the numerical values of all variables of the feasible solution with

$$\max\{\frac{\varepsilon^*}{\varepsilon'}, \frac{\delta^*}{\delta'}\}.$$

Further note that if $\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_* = 0$, then $\tilde{f}_{p,\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) = 0$ for all $j \in \{0, 1, \ldots, n\}$ such that $\sigma(j) \neq 0$ and if $\sigma(j) = 0$, then $V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - V[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}] = 0$. Thus, the constraints **LC4** reduce to

$$0 \geq \sum_{j=0}^{n} E_{p,\sigma,i}^{(\mathbf{z},\mathcal{J})} C[\{\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}\}],$$

which looks contradictory at first glance. However, if $\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_* = 0$ then necessarily $i = n+1$ and then

$$E_{p,\sigma,n+1}^{(\mathbf{z},\mathcal{J})} := \frac{1}{2}\sum_{r,s=0}^{n} B_{p,rs}^{(\mathbf{z},\mathcal{J})} A_{\sigma,r,n+1}^{(\mathbf{z},\mathcal{J})}(A_{\sigma,s,n+1}^{(\mathbf{z},\mathcal{J})} + A_{\sigma,s,0}^{(\mathbf{z},\mathcal{J})}) = 0$$

because $A_{\sigma,r,n+1}^{(\mathbf{z},\mathcal{J})} = 0$ for all $r = 0, 1, \ldots, n$.

Finally, if the Switched System 1.9 is autonomous, then we know by Theorem 3.10 that there exists a time-invariant Lyapunov function for the system. To reflect this fact one is tempted to additionally include the constraints $V[\mathbf{x}] = V[\mathbf{y}]$ for every $\mathbf{x}, \mathbf{y} \in \mathcal{G}$ such that $\|\mathbf{x} - \mathbf{y}\|_* = 0$ in the linear programming problem to limit the search to time-invariant Lyapunov functions. However, as we will show in Theorem 5.7, this is equivalent to a more simple linear programming problem if the Switched System 1.9 is autonomous, namely, the linear programming problem defined in Definition 5.5.

In the next sections we prove that a feasible solution to the linear programming problem defined in Definition 5.1 parameterizes a CPWA Lyapunov function for the Switched System 1.9 used for its construction. For this proof the variable Υ is not needed. However, it will be needed for the analysis in Section 5.7.

5.2 Definition of the functions $\psi, \gamma,$ and V^{Lya}

Let $y_0, y_1, \ldots, y_K$ be the elements of $\mathcal{X}^{\|\cdot\|}$ in an increasing order. We define the piecewise affine functions $\psi, \gamma : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$,

$$\psi(y) := \Psi[y_i] + \frac{\Psi[y_{i+1}] - \Psi[y_i]}{y_{i+1} - y_i}(y - y_i)$$

and

$$\gamma(y) := \Gamma[y_i] + \frac{\Gamma[y_{i+1}] - \Gamma[y_i]}{y_{i+1} - y_i}(y - y_i),$$

for all $y \in [y_i, y_{i+1}]$ and all $i = 0, 1, \ldots, K-1$. The values of ψ and γ on $]y_K, +\infty[$ do not really matter, but to have everything properly defined, we set

$$\psi(y) := \Psi[y_{K-1}] + \frac{\Psi[y_K] - \Psi[y_{K-1}]}{y_K - y_{K-1}}(y - y_{K-1})$$

and

$$\gamma(y) := \Gamma[y_{K-1}] + \frac{\Gamma[y_K] - \Gamma[y_{K-1}]}{y_K - y_{K-1}}(y - y_{K-1})$$

for all $y > y_K$. Clearly the functions ψ and γ are continuous.

The function $V^{Lya} \in \text{CPWA}[\widetilde{\mathbf{PS}}, \widetilde{\mathbf{PS}}^{-1}\big([T', T''] \times \big(\mathcal{M} \setminus \mathcal{D}\big)\big)]$ is defined by assigning

$$V^{Lya}(\tilde{\mathbf{x}}) := V[\tilde{\mathbf{x}}]$$

for all $\tilde{\mathbf{x}} \in \mathcal{G}$. We will sometimes write $V^{Lya}(t, \mathbf{x})$ for $V^{Lya}(\tilde{\mathbf{x}})$ and $V[t, \mathbf{x}]$ for $V[\tilde{\mathbf{x}}]$. It is then to be understood that $t := \tilde{x}_0$ and $\mathbf{x} := (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$.

In the next four sections we will successively derive the implications the linear constraints **LC1**, **LC2**, **LC3**, and **LC4** have on the functions ψ, γ, and V^{Lya}.

5.3 Implications of the constraints LC1

Let $y_0, y_1, \dots, y_K$ be the elements of $\mathcal{X}^{\|\cdot\|}$ in an increasing order. We are going to show that the constraints **LC1** imply, that the functions ψ and γ are convex and strictly increasing on $[0, +\infty[$. Because $y_0 = 0$, $\psi(y_0) = \Psi[y_0] = 0$, and $\gamma(y_0) = \Gamma[y_0] = 0$, this means that they are convex $\mathcal{K}$ functions. The constraints are the same for Ψ and Γ, so it suffices to show this for the function ψ.

From the definition of ψ, it is clear that it is continuous and that

$$\frac{\psi(x) - \psi(y)}{x - y} = \frac{\Psi[y_{i+1}] - \Psi[y_i]}{y_{i+1} - y_i} \tag{5.2}$$

for all $x, y \in [y_i, y_{i+1}]$ and all $i = 0, 1, \dots, K-1$. From $y_0 = 0$, $\Psi[y_0] = 0$, and $\varepsilon y_1 \leq \Psi[y_1]$ we get

$$\varepsilon \leq \frac{\Psi[y_1] - \Psi[y_0]}{y_1 - y_0} \leq \frac{\Psi[y_2] - \Psi[y_1]}{y_2 - y_1} \leq \dots \leq \frac{\Psi[y_K] - \Psi[y_{K-1}]}{y_K - y_{K-1}}.$$

But then $D^+\psi$ is a positive and increasing function on $\mathbb{R}_{\geq 0}$ and it follows from Corollary 2.10, that ψ is a strictly increasing function.

The function ψ is *convex*, if and only if for every $y \in \mathbb{R}_{>0}$ there are constants $a_y, b_y \in \mathbb{R}$, such that

$$a_y y + b_y = \psi(y) \quad \text{and} \quad a_y x + b_y \leq \psi(x)$$

for all $x \in \mathbb{R}_{\geq 0}$ (see, for example, Section 17 in Chapter 11 in [58]). Let $y \in \mathbb{R}_{>0}$. Because the function $D^+\psi$ is increasing, it follows by Theorem 2.9, that for every $x \in \mathbb{R}_{\geq 0}$, there is a $c_{x,y} \in \mathbb{R}$, such that

$$\psi(x) = \psi(y) + c_{x,y}(x - y)$$

and $c_{x,y} \leq D^+\psi(y)$ if $x < y$ and $c_{x,y} \geq D^+\psi(y)$ if $x > y$. This means that

$$\psi(x) = \psi(y) + c_{x,y}(x - y) \geq D^+\psi(y)x + \psi(y) - D^+\psi(y)y$$

for all $x \in \mathbb{R}_{\geq 0}$. Because y was arbitrary, the function ψ is convex.

5.4 Implications of the constraints LC2

Define the constant

$$V^{Lya}_{\partial\mathcal{M},\min} := \min_{\substack{\mathbf{x}\in\partial\mathcal{M}\\ t\in[T',T'']}} V^{Lya}(t,\mathbf{x})$$

and if $\mathcal{D}\neq\emptyset$ the constant

$$V^{Lya}_{\partial\mathcal{D},\max} := \max_{\substack{\mathbf{x}\in\partial\mathcal{D}\\ t\in[T',T'']}} V^{Lya}(t,\mathbf{x}).$$

We are going to show that the constraints **LC2** imply, that

$$\psi(\|\mathbf{x}\|) \leq V^{Lya}(t,\mathbf{x}) \tag{5.3}$$

for all $t\in[T',T'']$ and all $\mathbf{x}\in\mathcal{M}\setminus\mathcal{D}$ and that

$$V^{Lya}_{\partial\mathcal{D},\max} \leq V^{Lya}_{\partial\mathcal{M},\min} - \delta$$

if $\mathcal{D}\neq\emptyset$.

We first show that they imply, that

$$\psi(\|\tilde{\mathbf{x}}\|_*) \leq V^{Lya}(\tilde{\mathbf{x}})$$

for all $\tilde{\mathbf{x}}\in\mathcal{G}$, which obviously implies (5.3). Let $\tilde{\mathbf{x}}\in\mathcal{G}$. Then there is a $(\mathbf{z},\mathcal{J})\in\mathcal{Z}$, a $\sigma\in\text{Perm}[\{0,1,\dots,n\}]$, and constants $\lambda_0,\lambda_1,\dots,\lambda_{n+1}\in[0,1]$, such that

$$\tilde{\mathbf{x}} = \sum_{i=0}^{n+1}\lambda_i\mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,i} \quad\text{and}\quad \sum_{i=0}^{n+1}\lambda_i = 1.$$

Then

$$\begin{aligned}
\psi(\|\tilde{\mathbf{x}}\|_*) &= \psi(\|\sum_{i=0}^{n+1}\lambda_i\mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,i}\|_*) \leq \psi(\sum_{i=0}^{n+1}\lambda_i\|\mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,i}\|_*)\\
&\leq \sum_{i=0}^{n+1}\lambda_i\psi(\|\mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,i}\|_*) = \sum_{i=0}^{n+1}\lambda_i\Psi[\|\mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,i}\|_*] \leq \sum_{i=0}^{n+1}\lambda_i V[\mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,i}]\\
&= \sum_{i=0}^{n+1}\lambda_i V^{Lya}(\mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,i}) = V^{Lya}(\sum_{i=0}^{n+1}\lambda_i\mathbf{y}^{(\mathbf{z},\mathcal{J})}_{\sigma,i}) = V^{Lya}(\tilde{\mathbf{x}}).
\end{aligned}$$

Now consider the case $\mathcal{D}\neq\emptyset$. From the definition of V^{Lya} and the constants $V^{Lya}_{\partial\mathcal{D},\max}$ and $V^{Lya}_{\partial\mathcal{M},\min}$ it is clear, that

$$V^{Lya}_{\partial\mathcal{D},\max} = \max_{\substack{\mathbf{x}\in\partial\mathcal{D}\cap\mathbf{PS}(\mathbb{Z}^n)\\ u=0,1,\dots,M}} V[t_u,\mathbf{x}]$$

and

$$V^{Lya}_{\partial\mathcal{M},\min} = \min_{\substack{\mathbf{x}\in\partial\mathcal{M}\cap\mathbf{PS}(\mathbb{Z}^n)\\ u=0,1,\dots,M}} V[t_u,\mathbf{x}].$$

Let $\mathbf{x}\in\partial\mathcal{M}\cap\mathbf{PS}(\mathbb{Z}^n)$ and $u\in\{0,1,\dots,M\}$ be such that $V[t_u,\mathbf{x}] = V^{Lya}_{\partial\mathcal{M},\min}$, then

$$\begin{aligned}
V^{Lya}_{\partial\mathcal{D},\max} &\leq \Psi[x_{\min,\partial\mathcal{M}}] - \delta = \psi(x_{\min,\partial\mathcal{M}}) - \delta\\
&\leq \psi(\|\mathbf{x}\|) - \delta \leq V[t_u,\mathbf{x}] - \delta\\
&= V^{Lya}_{\partial\mathcal{M},\min} - \delta.
\end{aligned}$$

5.5 Implications of the constraints LC3

The constraints **LC3** imply that

$$\left|\frac{V[\tilde{\mathbf{x}}] - V[\tilde{\mathbf{y}}]}{\|\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\|_\infty}\right| \leq C[\{\tilde{\mathbf{x}}, \tilde{\mathbf{y}}\}] \leq \Upsilon$$

for every $\{\tilde{\mathbf{x}}, \tilde{\mathbf{y}}\} \in \mathcal{Y}$ and these local bounds $C[\{\tilde{\mathbf{x}}, \tilde{\mathbf{y}}\}]$ on the gradient $\nabla_{\tilde{\mathbf{x}}} V^{Lya}$ will be used in the next section.

5.6 Implications of the constraints LC4

We are going to show that the constraints **LC4** and **LC3** together imply that

$$-\gamma(\|\boldsymbol{\phi}_\varsigma(t, t', \boldsymbol{\xi})\|) \geq \limsup_{h \to 0+} \frac{V^{Lya}(t+h, \boldsymbol{\phi}_\varsigma(t+h, t', \boldsymbol{\xi})) - V^{Lya}(t, \boldsymbol{\phi}_\varsigma(t, t', \boldsymbol{\xi}))}{h} \tag{5.4}$$

for all $\varsigma \in \mathcal{S}_\mathcal{P}$ and all $(t, \boldsymbol{\phi}_\varsigma(t, t', \boldsymbol{\xi}))$ in the interior of $[T', T''] \times (\mathcal{M} \setminus \mathcal{D})$.

Let $\varsigma \in \mathcal{S}_\mathcal{P}$ and $\tilde{\mathbf{x}} := (t, \boldsymbol{\phi}_\varsigma(t, t', \boldsymbol{\xi}))$ in the interior of $[T', T''] \times (\mathcal{M} \setminus \mathcal{D})$ be arbitrary, but fixed throughout this section, and set $\mathbf{x} := \boldsymbol{\phi}_\varsigma(t, t', \boldsymbol{\xi})$ and $p := \varsigma(t)$.

We claim that there is a $(\mathbf{z}, \mathcal{J}) \in \mathcal{Z}$, a $\sigma \in \text{Perm}[\{0, 1, \ldots, n\}]$, and constants $\lambda_0, \lambda_1, \ldots, \lambda_{n+1} \in [0, 1]$, such that

$$\tilde{\mathbf{x}} = \sum_{i=0}^{n+1} \lambda_i \mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}, \quad \sum_{i=0}^{n+1} \lambda_i = 1, \quad \text{and} \quad \tilde{\mathbf{x}} + h\tilde{\mathbf{f}}_p(\tilde{\mathbf{x}}) \in \text{con}\{\mathbf{y}_{\sigma,0}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,1}^{(\mathbf{z},\mathcal{J})}, \ldots, \mathbf{y}_{\sigma,n+1}^{(\mathbf{z},\mathcal{J})}\} \tag{5.5}$$

for all $h \in [0, a]$, where $a > 0$ is some constant.

We prove this claim by a contradiction. Assume that it does not hold true. The vector $\tilde{\mathbf{x}}$ is contained in some of the simplices in the simplicial partition of $[T', T''] \times \big(\mathcal{M} \setminus \mathcal{D}\big)$, say $S_1, S_2, \ldots, S_k$. Simplices are convex sets so we necessarily have

$$\left\{\tilde{\mathbf{x}} + \frac{1}{j}\tilde{\mathbf{f}}_p(\tilde{\mathbf{x}}) \big| j \in \mathbb{N}_{>0}\right\} \cap S_i = \emptyset$$

for every $i = 1, 2, \ldots, k$. But then there must be a simplex S in the simplicial partition, different to the simplices $S_1, S_2, \ldots, S_k$, such that the intersection

$$\left\{\tilde{\mathbf{x}} + \frac{1}{j}\tilde{\mathbf{f}}_p(\tilde{\mathbf{x}}) \big| \, j \in \mathbb{N}_{>0}\right\} \cap S$$

contains an infinite number of elements. This implies that there is a sequence in S that converges to $\tilde{\mathbf{x}}$, which is a contradiction, because S is a closed set and $\tilde{\mathbf{x}} \notin S$. Therefore (5.5) holds true.

Because γ is a convex function, we have

$$-\gamma(\|\tilde{\mathbf{x}}\|_*) \geq -\sum_{i=0}^{n+1} \lambda_i \Gamma\big[\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_*\big] \tag{5.6}$$

as was shown in Section 5.4. From the definition of V^{Lya} it follows, that there is a vector $\mathbf{w} \in \mathbb{R} \times \mathbb{R}^n$, such that

$$V^{Lya}(\tilde{\mathbf{y}}) = \mathbf{w} \cdot (\tilde{\mathbf{y}} - \mathbf{y}_{\sigma,n+1}^{(\mathbf{z},\mathcal{J})}) + V^{Lya}(\mathbf{y}_{\sigma,n+1}^{(\mathbf{z},\mathcal{J})}) \tag{5.7}$$

for all $\tilde{\mathbf{y}} \in \text{con}\{\mathbf{y}_{\sigma,0}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,1}^{(\mathbf{z},\mathcal{J})}, \ldots, \mathbf{y}_{\sigma,n+1}^{(\mathbf{z},\mathcal{J})}\}$.

It follows by Hölder's inequality, that

$$\begin{aligned} \mathbf{w} \cdot \tilde{\mathbf{f}}_p(\tilde{\mathbf{x}}) &= \mathbf{w} \cdot \sum_{i=0}^{n+1} \lambda_i \tilde{\mathbf{f}}_p(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) + \mathbf{w} \cdot \left(\tilde{\mathbf{f}}_p(\tilde{\mathbf{x}}) - \sum_{i=0}^{n+1} \lambda_i \tilde{\mathbf{f}}_p(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) \right) \\ &\leq \sum_{i=0}^{n+1} \lambda_i \mathbf{w} \cdot \tilde{\mathbf{f}}_p(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) + \|\mathbf{w}\|_1 \|\tilde{\mathbf{f}}_p(\tilde{\mathbf{x}}) - \sum_{i=0}^{n+1} \lambda_i \tilde{\mathbf{f}}_p(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})\|_\infty \end{aligned} \tag{5.8}$$

and by Lemma 4.14 and the assignment in (5.1),

$$\begin{aligned} \|\tilde{\mathbf{f}}_p(\tilde{\mathbf{x}}) - \sum_{i=0}^{n+1} \lambda_i \tilde{\mathbf{f}}_p(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})\|_\infty &= \max_{j=0,1,\ldots,n} \left| \tilde{f}_{p,j}(\tilde{\mathbf{x}}) - \sum_{i=0}^{n+1} \lambda_i \tilde{f}_{p,j}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) \right| \\ &\leq \frac{1}{2} \sum_{i=0}^{n+1} \lambda_i \sum_{r,s=0}^{n} B_{p,rs}^{(\mathbf{z},\mathcal{J})} A_{\sigma,r,i}^{(\mathbf{z},\mathcal{J})} (A_{\sigma,s,0}^{(\mathbf{z},\mathcal{J})} + A_{\sigma,s,i}^{(\mathbf{z},\mathcal{J})}) \\ &\leq \sum_{i=0}^{n+1} \lambda_i E_{p,\sigma,i}^{(\mathbf{z},\mathcal{J})}, \end{aligned}$$

which implies that we have derived the inequality

$$\mathbf{w} \cdot \tilde{\mathbf{f}}_p(\tilde{\mathbf{x}}) \leq \sum_{i=0}^{n+1} \lambda_i \left(\mathbf{w} \cdot \tilde{\mathbf{f}}_p(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) + \|\mathbf{w}\|_1 E_{p,\sigma,i}^{(\mathbf{z},\mathcal{J})} \right). \tag{5.9}$$

We come to the vector $\mathbf{w}$. By Lemma 4.15, the constraints **LC3**, and because

$$\left| \mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}) \right| = \|\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}\|_\infty$$

for all $j = 0, 1, \ldots, n$, we obtain the inequality

$$\|\mathbf{w}\|_1 = \sum_{j=0}^{n} \left| \frac{V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - V[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\|\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}\|_\infty} \right| \leq \sum_{j=0}^{n} C[\{\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}\}].$$

This inequality combined with (5.9) gives

$$\mathbf{w} \cdot \tilde{\mathbf{f}}_p(\tilde{\mathbf{x}}) \leq \sum_{i=0}^{n+1} \lambda_i \sum_{j=0}^{n} \left(\frac{V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - V[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})} \tilde{f}_{p,\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) + E_{p,\sigma,i}^{(\mathbf{z},\mathcal{J})} C[\{\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}\}] \right). \tag{5.10}$$

We are going to show that inequality (5.10) together with the constraints **LC4** imply that inequality (5.4) holds true. First, note that because V^{Lya} is Lipschitz with a Lipschitz constant, say $L_V > 0$

with respect to the norm $\|\cdot\|$, we have

$$\begin{aligned}\limsup_{h\to 0+}&\left|\frac{V^{Lya}(t+h,\boldsymbol{\phi}_\varsigma(t+h,t',\boldsymbol{\xi}))-V^{Lya}(t+h,\mathbf{x}+h\mathbf{f}_p(t,\mathbf{x}))}{h}\right|\\&\leq \limsup_{h\to 0+} L_V\left\|\frac{\boldsymbol{\phi}_\varsigma(t+h,t',\boldsymbol{\xi})-\mathbf{x}}{h}-\mathbf{f}_p(t,\mathbf{x})\right\|\\&= L_V\,\|\mathbf{f}_p(t,\mathbf{x})-\mathbf{f}_p(t,\mathbf{x})\|\\&=0.\end{aligned}$$

Hence, by Lemma 2.8 and the representation (5.7) of V^{Lya},

$$\begin{aligned}\limsup_{h\to 0+}&\frac{V^{Lya}(t+h,\boldsymbol{\phi}_\varsigma(t+h,t',\boldsymbol{\xi}))-V^{Lya}(t,\boldsymbol{\phi}_\varsigma(t,t',\boldsymbol{\xi}))}{h}\\&=\limsup_{h\to 0+}\frac{V^{Lya}(\tilde{\mathbf{x}}+h\tilde{\mathbf{f}}_p(\tilde{\mathbf{x}}))-V^{Lya}(\tilde{\mathbf{x}})}{h}\\&=\limsup_{h\to 0+}\frac{h\mathbf{w}\cdot\tilde{\mathbf{f}}_p(\tilde{\mathbf{x}})}{h}\\&=\mathbf{w}\cdot\tilde{\mathbf{f}}_p(\tilde{\mathbf{x}}),\end{aligned}$$

and we obtain by (5.10), **LC4**, and (5.6) that

$$\begin{aligned}\limsup_{h\to 0+}&\frac{V^{Lya}(t+h,\boldsymbol{\phi}_\varsigma(t+h,t',\boldsymbol{\xi}))-V^{Lya}(t,\boldsymbol{\phi}_\varsigma(t,t',\boldsymbol{\xi}))}{h}\\&=\mathbf{w}\cdot\tilde{\mathbf{f}}_p(\tilde{\mathbf{x}})\\&\leq\sum_{i=0}^{n+1}\lambda_i\sum_{j=0}^{n}\left(\frac{V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}]-V[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)}\cdot(\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}-\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})}\tilde{f}_{p,\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})+E_{p,\sigma,i}^{(\mathbf{z},\mathcal{J})}C[\{\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})},\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}\}]\right)\\&\leq-\sum_{i=1}^{n+1}\lambda_i\Gamma\left[\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|\right]\\&\leq-\gamma(\|\boldsymbol{\phi}_\varsigma(t,t',\boldsymbol{\xi})\|).\end{aligned}$$

Hence, inequality (5.4) holds true for all $\varsigma\in\mathcal{S}_\mathcal{P}$ and all $(t,\boldsymbol{\phi}_\varsigma(t,t',\boldsymbol{\xi}))$ in the interior of $[T',T'']\times(\mathcal{M}\setminus\mathcal{D})$.

5.7 Summary of the results and their consequences

We start by summing up the results in the previous sections in this chapter in a theorem.

Theorem 5.2 (Parametrization of a CPWA **Lyapunov function by linear programming)** *Consider the linear programming problem* $\mathbf{LP}(\{\mathbf{f}_p|p\in\mathcal{P}\},\mathcal{N},\mathbf{PS},\mathbf{t},\mathcal{D},\|\cdot\|)$ *in Definition 5.1 and assume that it possesses a feasible solution. Let the functions* $\psi,\gamma,$ *and* V^{Lya} *be defined as in Section 5.2 from the numerical values of the variables* $\Psi[x]$, $\Gamma[x]$, *and* $V[\tilde{\mathbf{x}}]$ *from a feasible solution. Then the inequality*

$$\psi(\|\mathbf{x}\|)\leq V^{Lya}(t,\mathbf{x})$$

holds true for all $\mathbf{x} \in \mathcal{M} \setminus \mathcal{D}$ *and all* $t \in [T', T'']$*. If* $\mathcal{D} = \emptyset$ *we have* $\psi(0) = V^{Lya}(t, \mathbf{0}) = 0$ *for all* $t \in [T', T'']$*. If* $\mathcal{D} \neq \emptyset$ *we have, with*

$$V^{Lya}_{\partial\mathcal{M},\min} := \min_{\substack{\mathbf{x}\in\partial\mathcal{M}\\ t\in[T',T'']}} V^{Lya}(t,\mathbf{x})$$

and

$$V^{Lya}_{\partial\mathcal{D},\max} := \max_{\substack{\mathbf{x}\in\partial\mathcal{D}\\ t\in[T',T'']}} V^{Lya}(t,\mathbf{x}),$$

that

$$V^{Lya}_{\partial\mathcal{D},\max} \leq V^{Lya}_{\partial\mathcal{M},\min} - \delta.$$

Further, with $\boldsymbol{\phi}$ *as the solution to the Switched System 1.9 that we used in the construction of the linear programming problem, the inequality*

$$-\gamma(\|\boldsymbol{\phi}_\varsigma(t,t',\boldsymbol{\xi})\|) \geq \limsup_{h\to 0+} \frac{V^{Lya}(t+h, \boldsymbol{\phi}_\varsigma(t+h,t',\boldsymbol{\xi})) - V^{Lya}(t, \boldsymbol{\phi}_\varsigma(t,t',\boldsymbol{\xi}))}{h} \tag{5.11}$$

hold true for all $\varsigma \in \mathcal{S}_\mathcal{P}$ *and all* $(t, \boldsymbol{\phi}_\varsigma(t,t',\boldsymbol{\xi}))$ *in the interior of* $[T', T''] \times (\mathcal{M} \setminus \mathcal{D})$*.* ■

We now come to the important question:

> *Which information on the stability behavior of the Switched System 1.9 can we extract from the Lyapunov-like function* V^{Lya} *defined in Section 5.2 ?*

Before we answer this question we discuss the implications secured by a continuously differentiable Lyapunov function on the stability behavior of a non-switched system to get an idea what we can expect. To do this consider the system

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}),$$

where $\mathbf{f} \in [\mathcal{C}^1(\mathbb{R}_{\geq 0} \times \mathcal{V})]^n$ and $\mathcal{V}$ is a bounded domain in $\mathbb{R}^n$ containing the origin, and assume that there is a function $W \in \mathcal{C}^1(\mathbb{R}_{\geq 0} \times \mathcal{V})$ and functions $a, b, c \in \mathcal{K}$, such that

$$a(\|\boldsymbol{\xi}\|) \leq W(t, \boldsymbol{\xi}) \leq b(\|\boldsymbol{\xi}\|)$$

for all $\boldsymbol{\xi} \in \mathcal{V}$ and all $t \geq 0$ and

$$\begin{aligned}\frac{d}{dt}W(t, \boldsymbol{\phi}(t,t',\boldsymbol{\xi})) &= [\nabla_\mathbf{x} W](t, \boldsymbol{\phi}(t,t',\boldsymbol{\xi})) \cdot \mathbf{f}(t, \boldsymbol{\phi}(t,t',\boldsymbol{\xi})) + \frac{\partial W}{\partial t}(t, \boldsymbol{\phi}(t,t',\boldsymbol{\xi})) \\ &\leq -c(\|\boldsymbol{\phi}(t,t',\boldsymbol{\xi})\|)\end{aligned}$$

for all $(t, \boldsymbol{\phi}(t,t',\boldsymbol{\xi})) \in \mathbb{R}_{\geq 0} \times \mathcal{V}$, where $\boldsymbol{\phi}$ is the solution to the differential equation $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$.

For our analysis we let $(t', \boldsymbol{\xi}) \in \mathbb{R}_{\geq 0} \times \mathcal{V}$ be arbitrary but constant and set $y(t) := W(t, \boldsymbol{\phi}(t,t',\boldsymbol{\xi}))$. Then $y(t') = W(t', \boldsymbol{\xi})$ and y satisfies the differential inequality

$$\dot{y}(t) \leq -c(b^{-1}(y(t)))$$

for all t such that $\boldsymbol{\phi}(t,t',\boldsymbol{\xi}) \in \mathcal{V}$. Now, assume that there are constants $b^* > 0$ and $c^* > 0$, such that $b(\|\mathbf{x}\|) \leq b^*\|\mathbf{x}\|$ and $c^*\|\boldsymbol{\xi}\| \leq c(\|\mathbf{x}\|)$ for all $\mathbf{x} \in \mathcal{V}$. In this simple case it is quite simple to derive the inequality

$$y(t) \leq y(t') \exp\left(-\frac{c^*}{b^*}(t-t')\right),$$

which is valid for all $t \geq t'$ if

$$W(t', \boldsymbol{\xi}) < \inf_{\substack{s \geq t' \\ \mathbf{y} \in \partial \mathcal{V}}} W(s, \mathbf{y}).$$

We are going to show that a very similar analysis can be done for a switched system and the corresponding Lyapunov-like function V^{Lya} if the arbitrary norm $\|\cdot\|$ used in Definition 5.1 of the linear programming problem is a p-norm $\|\cdot\|_p$, $1 \leq p \leq +\infty$, but first we prove a technical lemma that will be used in the proof of the theorem.

Lemma 5.3 *Let $[a, b[$ be an interval in $\mathbb{R}$, $-\infty < a < b \leq +\infty$, and let $y, z : [a, b[\longrightarrow \mathbb{R}$ be functions such that $y(a) \leq z(a)$, y is continuous, and z is differentiable. Assume that there is a locally Lipschitz function $s : \mathbb{R} \longrightarrow \mathbb{R}$ such that*

$$D^+ y(t) \leq -s(y(t)) \quad \text{and} \quad \dot{z}(t) = -s(z(t))$$

for all $t \in [a, b[$. Then $y(t) \leq z(t)$ for all $t \in [a, b[$.

Proof:
Assume that the proposition of the lemma does not hold true. Then there is a $t_0 \in [a, b[$ such that $y(t) \leq z(t)$ for all $t \in [a, t_0]$ and an $\epsilon > 0$ such that $y(t) > z(t)$ for all $t \in]t_0, t_0 + \epsilon]$. Let $L > 0$ be a local Lipschitz constant for s on the interval $[y(t_0), y(t_0 + \epsilon)]$. Then, by Lemma 2.8,

$$D^+(y - z)(t) = D^+ y(t) - \dot{z}(t) \leq -s(y(t)) + s(z(t)) \leq L(y(t) - z(t))$$

for every $t \in [t_0, t_0 + \epsilon]$. But then, with $w(t) := y(t) - z(t)$ for all $t \in [a, b[$, we have

$$\begin{aligned}
&\limsup_{h \to 0+} \frac{w(t+h)e^{-L(t+h)} - w(t)e^{-Lt}}{h} \\
&\qquad \leq e^{-Lt} \limsup_{h \to 0+} \frac{w(t+h)(e^{-Lh} - 1)}{h} + e^{-Lt} \limsup_{h \to 0+} \frac{w(t+h) - w(t)}{h} \\
&\qquad = -Le^{-Lt} w(t) + e^{-Lt} D^+ w(t) \\
&\qquad \leq -Le^{-Lt} w(t) + Le^{-Lt} w(t) \\
&\qquad = 0,
\end{aligned}$$

for all $t \in [t_0, t_0 + \epsilon]$, which implies, by Corollary 2.10, that the function $t \mapsto e^{-Lt} w(t)$ is monotonically decreasing on the same interval. Because $w(t_0) = 0$ this is contradictory to $y(t) > z(t)$ for all $t \in]t_0, t_0 + \epsilon]$ and therefore the proposition of the lemma must hold true. ■

We come to the promised theorem, where the implications of the function V^{Lya} on the stability behavior of the Switched System 1.9 are specified.

Theorem 5.4 (Implications of the Lyapunov function V^{Lya}) *Make the same assumptions and definitions as in Theorem 5.2 and assume additionally that the arbitrary norm $\|\cdot\|$ in the linear programming problem $\mathbf{LP}(\{\mathbf{f}_p | p \in \mathcal{P}\}, \mathcal{N}, \mathbf{PS}, \mathbf{t}, \mathcal{D}, \|\cdot\|)$ is a k-norm [1], $1 \leq k \leq +\infty$. Define the set $\mathcal{T}$ through $\mathcal{T} := \{\mathbf{0}\}$ if $\mathcal{D} = \emptyset$ and*

$$\mathcal{T} := \mathcal{D} \cup \left\{ \mathbf{x} \in \mathcal{M} \setminus \mathcal{D} \;\middle|\; \max_{t \in [T', T'']} V^{Lya}(t, \mathbf{x}) \leq V^{Lya}_{\partial \mathcal{D}, \max} \right\}, \qquad \text{if } \mathcal{D} \neq \emptyset,$$

[1] With k-norm we mean the norm $\|\mathbf{x}\|_k := \left(\sum_{i=1}^n |x_i|^k\right)^{1/k}$ if $1 \leq k < +\infty$ and $\|\mathbf{x}\|_\infty := \max_{i=1,2,\ldots,n} |x_i|$. Unfortunately, these norms are usually called p-norms, which is inappropriate in this context because the alphabet p is used to index the functions $\mathbf{f}_p$, $p \in \mathcal{P}$.

and the set $\mathcal{A}$ through

$$\mathcal{A} := \left\{ \mathbf{x} \in \mathcal{M} \setminus \mathcal{D} \;\Big|\; \max_{t \in [T',T'']} V^{Lya}(t,\mathbf{x}) < V^{Lya}_{\partial\mathcal{M},\min} \right\}.$$

Set $q := k \cdot (k-1)^{-1}$ if $1 < k < +\infty$, $q := 1$ if $k = +\infty$, and $q := +\infty$ if $k = 1$, and define the constant

$$E_q := \| \sum_{i=1}^{n} \mathbf{e}_i \|_q.$$

Then the following propositions hold true:

i) If $\boldsymbol{\phi}_\varsigma(t,t',\boldsymbol{\xi}) \in \mathcal{T}$ for some particular $\varsigma \in \mathcal{S}_\mathcal{P}$, $T'' \geq t \geq T'$, $t' \geq 0$, and $\boldsymbol{\xi} \in \mathcal{U}$, then $\boldsymbol{\phi}_\varsigma(s,t',\boldsymbol{\xi}) \in \mathcal{T}$ for all $s \in [t,T'']$.

ii) If $\boldsymbol{\phi}_\varsigma(t,t',\boldsymbol{\xi}) \in \mathcal{M} \setminus \mathcal{D}$ for some particular $\varsigma \in \mathcal{S}_\mathcal{P}$, $T'' \geq t \geq T'$, $t' \geq 0$, and $\boldsymbol{\xi} \in \mathcal{U}$, then the inequality

$$V^{Lya}(s,\boldsymbol{\phi}_\varsigma(s,t',\boldsymbol{\xi})) \leq V^{Lya}(t,\boldsymbol{\phi}_\varsigma(t,t',\boldsymbol{\xi})) \exp\left(-\frac{\Upsilon}{\varepsilon E_q}(s-t)\right) \tag{5.12}$$

holds true for all s such that $\boldsymbol{\phi}_\varsigma(s',t',\boldsymbol{\xi}) \in \mathcal{M} \setminus \mathcal{D}$ for all $t \leq s' \leq s \leq T''$.

iii) If $\boldsymbol{\phi}_\varsigma(t,t',\boldsymbol{\xi}) \in \mathcal{A}$ for some particular $\varsigma \in \mathcal{S}_\mathcal{P}$, $T'' \geq t \geq T'$, $t' \geq 0$, and $\boldsymbol{\xi} \in \mathcal{U}$, then the solution $\boldsymbol{\phi}_\varsigma$ either fulfills inequality (5.12) for all $t \leq s \leq T''$, or there is a $T^* \in]t,T'']$, such that the solution $\boldsymbol{\phi}_\varsigma$ fulfills inequality (5.12) for all $t \leq s \leq T^*$, $\boldsymbol{\phi}_\varsigma(T^*,t',\boldsymbol{\xi}) \in \partial\mathcal{D}$, and $\boldsymbol{\phi}_\varsigma(s,t',\boldsymbol{\xi}) \in \mathcal{T}$ for all $T^* \leq s \leq T''$.

Proof:
Proposition *i)* is trivial if $\mathcal{D} = \emptyset$. To prove proposition *i)* when $\mathcal{D} \neq \emptyset$ define for every $\kappa > 0$ the set

$$\mathcal{T}_\kappa := \{\mathbf{x} \in \mathbb{R}^n \big| \|\mathbf{x}-\mathbf{y}\|_2 < \kappa \text{ for some } \mathbf{y} \in \mathcal{T}\}.$$

Because $V^{Lya}_{\partial\mathcal{D},\max} \leq V^{Lya}_{\partial\mathcal{M},\min} - \delta$ by Theorem 5.2, it follows that $\mathcal{T}_\kappa \subset \mathcal{M}$ for all small enough $\kappa > 0$. For every such small $\kappa > 0$ notice, that inequality (5.11) and Corollary 2.10 together imply, that if $\boldsymbol{\phi}_\varsigma(t,t',\boldsymbol{\xi}) \in \mathcal{T}_\kappa$ for some particular $\varsigma \in \mathcal{S}_\mathcal{P}$, $T'' \geq t \geq T'$, $t' \geq 0$, and $\boldsymbol{\xi} \in \mathcal{U}$, then $\boldsymbol{\phi}_\varsigma(s,t',\boldsymbol{\xi}) \in \mathcal{T}_\kappa$ for all $s \in [t,T'']$. But then the proposition *i)* follows, because if $\boldsymbol{\phi}_\varsigma(t,t',\boldsymbol{\xi}) \in \mathcal{T}$, then $\boldsymbol{\phi}_\varsigma(t,t',\boldsymbol{\xi}) \in \bigcap_{\kappa>0}\mathcal{T}_\kappa$, and therefore $\boldsymbol{\phi}_\varsigma(s,t',\boldsymbol{\xi}) \in \bigcap_{\kappa>0}\mathcal{T}_\kappa = \mathcal{T}$ for all $s \in [t,T'']$.

To prove proposition *ii)* first note that the linear constraints **LC2** and **LC3** imply that $V^{Lya}(t,\mathbf{x}) \leq \Upsilon\|\mathbf{x}\|_1$ for all $t \in [T',T'']$ and all $\mathbf{x} \in \mathcal{M}$. To see this just notice that at least for one $i \in \{1,2,\ldots,n\}$ we must have either

$$x_i \geq \mathrm{PS}_i(d_i^+) \quad \text{or} \quad x_i \leq \mathrm{PS}_i(d_i^-)$$

because $\mathbf{x} \notin \mathcal{D}$. But then either

$$V^{Lya}(t,x_i\mathbf{e}_i) \leq \Upsilon \cdot \mathrm{PS}_i(d_i^+) + \Upsilon \cdot |x_1 - \mathrm{PS}_i(d_i^+)| = \Upsilon|x_i|$$

or

$$V^{Lya}(t,x_i\mathbf{e}_i) \leq -\Upsilon \cdot \mathrm{PS}_i(d_i^-) + \Upsilon \cdot |x_1 - \mathrm{PS}_i(d_i^-)| = \Upsilon|x_i|,$$

so

$$V^{Lya}(t,x_i\mathbf{e}_i) \leq \Upsilon|x_i|,$$

which in turn implies, for any $j \in \{1, 2, \dots, n\}$, $j \neq i$, that

$$V^{Lya}(t, x_i\mathbf{e}_i + x_j\mathbf{e}_j) \le V^{Lya}(t, x_i\mathbf{e}_i) + \Upsilon|x_j| \le \Upsilon(|x_i| + |x_j|)$$

and by mathematical induction $V^{Lya}(t, \mathbf{x}) \le \Upsilon\|\mathbf{x}\|_1$.

But then, by Hölder's inequality,

$$V^{Lya}(t, \mathbf{x}) \le \Upsilon\|\mathbf{x}\|_1 = \Upsilon\left(\sum_{i=1}^{n} \mathbf{e}_i\right) \cdot \left(\sum_{i=1}^{n} |x_i|\mathbf{e}_i\right) \le \Upsilon E_q\|\mathbf{x}\|_k,$$

so by the linear constraints **LC1** and inequality (5.11), we have for every $\varsigma \in \mathcal{S}_\mathcal{P}$, $T'' \ge t \ge T'$, $t' \ge 0$, and $\boldsymbol{\xi} \in \mathcal{U}$, such that $\boldsymbol{\phi}_\varsigma(t, t', \boldsymbol{\xi}) \in \mathcal{M} \setminus \mathcal{D}$, that

$$\begin{aligned}
-\frac{\varepsilon}{\Upsilon E_q} V^{Lya}(t, \boldsymbol{\phi}_\varsigma(t, t', \boldsymbol{\xi})) &\ge -\varepsilon\|\boldsymbol{\phi}_\varsigma(t, t', \boldsymbol{\xi})\|_k \\
&\ge -\gamma(\|\boldsymbol{\phi}_\varsigma(t, t', \boldsymbol{\xi})\|_k) \\
&\ge \limsup_{h \to 0+} \frac{V^{Lya}(t+h, \boldsymbol{\phi}_\varsigma(t+h, t', \boldsymbol{\xi})) - V^{Lya}(t, \boldsymbol{\phi}_\varsigma(t, t', \boldsymbol{\xi}))}{h}.
\end{aligned}$$

The solution to the differential equation $\dot{y}(s) = -\Upsilon E_q/\varepsilon \cdot y(s)$ is $y(s) = \exp[-\Upsilon E_q(s - t')/\varepsilon]\, y(t')$. Hence, by Lemma 5.3,

$$V(s, \boldsymbol{\phi}_\varsigma(s, t', \boldsymbol{\xi})) \le V(t, \boldsymbol{\phi}_\varsigma(t, t', \boldsymbol{\xi})) \exp\left(-\frac{\Upsilon}{\varepsilon E_q}(s - t)\right)$$

and proposition *ii)* holds true.

Proposition *iii)* is a direct consequence of the propositions *i)* and *ii)* and the definition of the set $\mathcal{A}$. It merely states that if it is impossible for a solution to exit the set $\mathcal{M} \setminus \mathcal{D}$ at the boundary $\partial\mathcal{M}$, then it either exits at the boundary $\partial\mathcal{D}$ or it does not exit at all.

■

5.8 The autonomous case

As was discussed after Definition 5.1, one is tempted to try to parameterize a time-invariant Lyapunov function for the Switched System 1.9 if it is autonomous. The reason for this is that we proved in Theorem 3.10 that if it is autonomous, then there exists a time-invariant Lyapunov function. In the next definition we present a linear programming problem that does exactly this. It is a generalization of the linear programming problem presented in [36], [35], [11], and [10] to serve the Switched System 1.9 in the particular case that it is autonomous.

Definition 5.5 (Linear programming problem LP($\{\mathbf{f}_p | p \in \mathcal{P}\}, \mathcal{N}, \mathbf{PS}, \mathcal{D}, \|\cdot\|$)) *Consider the Switched System 1.9, assume that it is autonomous, and assume that the functions $\mathbf{f}_p$, $p \in \mathcal{P}$, are all in $[\mathcal{C}^2(\mathcal{U})]^n$ (because the $\mathbf{f}_p$ do not depend on the time t we consider them to be functions $\mathcal{U} \longrightarrow \mathbb{R}^n$). Let $\mathbf{PS} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a piecewise scaling function and $\mathcal{N} \subset \mathcal{U}$ be such that the interior of the set*

$$\mathcal{M} := \bigcup_{\substack{\mathbf{z} \in \mathbb{Z}^n \\ \mathbf{PS}(\mathbf{z} + [0,1]^n) \subset \mathcal{N}}} \mathbf{PS}(\mathbf{z} + [0, 1]^n)$$

is a connected set that contains the origin. Let $\|\cdot\|$ *be an arbitrary norm on* $\mathbb{R}^n$ *and let*

$$\mathcal{D} := \mathbf{PS}(\,]d_1^-, d_1^+[\, \times \,]d_2^-, d_2^+[\, \times \, \ldots \, \times \,]d_n^-, d_n^+[\,)$$

be a set, of which the closure is contained in the interior of $\mathcal{M}$*, and either* $\mathcal{D} = \emptyset$ *or* d_i^- *and* d_i^+ *are integers such that* $d_i^- \leq -1$ *and* $1 \leq d_i^+$ *for all* $i = 1, 2, \ldots, n$.

The linear programming problem $\mathbf{LP}(\{\mathbf{f}_p | p \in \mathcal{P}\}, \mathcal{N}, \mathbf{PS}, \mathcal{D}, \|\cdot\|)$ *is now constructed in the following way:*

i) Define the sets

$$\mathcal{G}_a := \{\mathbf{x} \in \mathbb{R}^n \big| \mathbf{x} \in \mathbf{PS}(\mathbb{Z}^n) \cap \big(\mathcal{M} \setminus \mathcal{D}\big)\}$$

and

$$\mathcal{X}^{\|\cdot\|} := \{\|\mathbf{x}\| \big| \ \mathbf{x} \in \mathbf{PS}(\mathbb{Z}^n) \cap \mathcal{M}\}.$$

ii) Define for every $\sigma \in \mathrm{Perm}[\{1, 2, \ldots, n\}]$ *and every* $i = 1, \ldots, n+1$ *the vector*

$$\mathbf{x}_i^\sigma := \sum_{j=i}^{n} \mathbf{e}_{\sigma(j)}.$$

iii) Define the set $\mathcal{Z}_a$ *through:*

$$\mathcal{Z}_a := \big\{(\mathbf{z}, \mathcal{J}) \in \mathbb{Z}^n_{\geq 0} \times \mathfrak{P}(\{1, 2, \ldots, n\}) \big| \mathbf{PS}(\mathbf{R}^{\mathcal{J}}(\mathbf{z} + [0,1]^n)) \subset \mathcal{M} \setminus \mathcal{D}\big\}.$$

iv) For every $(\mathbf{z}, \mathcal{J}) \in \mathcal{Z}_a$*, every* $\sigma \in \mathrm{Perm}[\{1, 2, \ldots, n\}]$*, and every* $i = 1, 2, \ldots, n+1$ *we set*

$$\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})} := \mathbf{PS}(\mathbf{R}^{\mathcal{J}}(\mathbf{z} + \mathbf{x}_i^\sigma)).$$

v) Define the set

$$\mathcal{Y}_a := \left\{\{\mathbf{y}_{\sigma,k}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,k+1}^{(\mathbf{z},\mathcal{J})}\} \middle| \sigma \in \mathrm{Perm}[\{1, 2, \ldots, n\}], (\mathbf{z}, \mathcal{J}) \in \mathcal{Z}_a, \text{ and } k \in \{1, 2, \ldots, n\}\right\}.$$

vi) For every $p \in \mathcal{P}$*, every* $(\mathbf{z}, \mathcal{J}) \in \mathcal{Z}_a$*, and every* $r, s = 1, 2, \ldots, n$ *let* $B_{p,rs}^{(\mathbf{z},\mathcal{J})}$ *be a real-valued constant, such that*

$$B_{p,rs}^{(\mathbf{z},\mathcal{J})} \geq \max_{i=1,2,\ldots,n} \sup_{\mathbf{x} \in \mathbf{PS}(\mathbf{R}^{\mathcal{J}}(\mathbf{z}+[0,1]^n))} \left| \frac{\partial^2 f_{p,i}}{\partial x_r \partial x_s}(\mathbf{x}) \right|.$$

vii) For every $(\mathbf{z}, \mathcal{J}) \in \mathcal{Z}_a$*, every* $i, k = 1, 2, \ldots, n$*, and every* $\sigma \in \mathrm{Perm}[\{1, 2, \ldots, n\}]$*, define*

$$A_{\sigma,k,i}^{(\mathbf{z},\mathcal{J})} := \left| \mathbf{e}_k \cdot (\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,n+1}^{(\mathbf{z},\mathcal{J})}) \right|.$$

viii) Define the constant

$$x_{\min,\partial\mathcal{M}} := \min\{\|\mathbf{x}\| \big| \mathbf{x} \in \mathbf{PS}(\mathbb{Z}^n) \cap \partial\mathcal{M}\},$$

where $\partial\mathcal{M}$ *is the boundary of the set* $\mathcal{M}$.

ix) For every $p \in \mathcal{P}$*, every* $(\mathbf{z}, \mathcal{J}) \in \mathcal{Z}_a$*, every* $\sigma \in \mathrm{Perm}[\{1, 2, \ldots, n\}]$*, and every* $i = 1, 2, \ldots, n+1$ *set*

$$E_{p,\sigma,i}^{(\mathbf{z},\mathcal{J})} := \frac{1}{2} \sum_{r,s=1}^{n} B_{p,rs}^{(\mathbf{z},\mathcal{J})} A_{\sigma,r,i}^{(\mathbf{z},\mathcal{J})} (A_{\sigma,s,i}^{(\mathbf{z},\mathcal{J})} + A_{\sigma,s,1}^{(\mathbf{z},\mathcal{J})}). \tag{5.13}$$

ix) Let $\varepsilon > 0$ and $\delta > 0$ be arbitrary constants.

The variables of the linear programming problem are:

$$\begin{aligned}&\Upsilon_a,\\ &\Psi_a[y], \quad \textit{for all } y \in \mathcal{X}^{\|\cdot\|},\\ &\Gamma_a[y], \quad \textit{for all } y \in \mathcal{X}^{\|\cdot\|},\\ &V_a[\mathbf{x}], \quad \textit{for all } \mathbf{x} \in \mathcal{G}_a,\\ &C_a[\{\mathbf{x},\mathbf{y}\}], \quad \textit{for all } \{\mathbf{x},\mathbf{y}\} \in \mathcal{Y}_a.\end{aligned}$$

The linear constraints of the linear programming problem are:

LC1a) *Let $y_0, y_1, \dots, y_K$ be the elements of $\mathcal{X}^{\|\cdot\|}$ in an increasing order. Then*

$$\begin{aligned}&\Psi_a[y_0] = \Gamma_a[y_0] = 0,\\ &\varepsilon y_1 \leq \Psi_a[y_1],\\ &\varepsilon y_1 \leq \Gamma_a[y_1],\end{aligned}$$

and for every $i = 1, 2, \dots, K-1$:

$$\frac{\Psi_a[y_i] - \Psi_a[y_{i-1}]}{y_i - y_{i-1}} \leq \frac{\Psi_a[y_{i+1}] - \Psi_a[y_i]}{y_{i+1} - y_i}$$

and

$$\frac{\Gamma_a[y_i] - \Gamma_a[y_{i-1}]}{y_i - y_{i-1}} \leq \frac{\Gamma_a[y_{i+1}] - \Gamma_a[y_i]}{y_{i+1} - y_i}.$$

LC2a) *For every $\mathbf{x} \in \mathcal{G}_a$:*

$$\Psi_a[\|\mathbf{x}\|] \leq V_a[\mathbf{x}].$$

If $\mathcal{D} = \emptyset$, then:

$$V_a[\mathbf{0}] = 0.$$

If $\mathcal{D} \neq \emptyset$, then, for every $\mathbf{x} \in \mathbf{PS}(\mathbb{Z}^n) \cap \partial\mathcal{D}$:

$$V_a[\mathbf{x}] \leq \Psi_a[x_{\min,\partial\mathcal{M}}] - \delta.$$

Further, if $\mathcal{D} \neq \emptyset$, then for every $i = 1, 2, \dots, n$:

$$V_a[\mathrm{PS}_i(d_i^-)\mathbf{e}_i] \leq -\Upsilon_a \cdot \mathrm{PS}_i(d_i^-) \quad \textit{and} \quad V_a[\mathrm{PS}_i(d_i^+)\mathbf{e}_i] \leq \Upsilon_a \cdot \mathrm{PS}_i(d_i^+).$$

LC3a) *For every $\{\mathbf{x},\mathbf{y}\} \in \mathcal{Y}_a$:*

$$-C_a[\{\mathbf{x},\mathbf{y}\}] \cdot \|\mathbf{x}-\mathbf{y}\|_\infty \leq V_a[\mathbf{x}] - V_a[\mathbf{y}] \leq C_a[\{\mathbf{x},\mathbf{y}\}] \cdot \|\mathbf{x}-\mathbf{y}\|_\infty \leq \Upsilon_a \cdot \|\mathbf{x}-\mathbf{y}\|_\infty.$$

LC4a) *For every $p \in \mathcal{P}$, every $(\mathbf{z},\mathcal{J}) \in \mathcal{Z}_a$, every $\sigma \in \mathrm{Perm}[\{1,2,\dots,n\}]$, and every $i = 1, 2, \dots, n+1$:*

$$-\Gamma_a\big[\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|\big] \geq \sum_{j=1}^{n} \left(\frac{V_a[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - V_a[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})} f_{p,\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) + E_{p,\sigma,i}^{(\mathbf{z},\mathcal{J})} C_a[\{\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}\}] \right).$$

As the objective of the linear programming problem is not needed to parameterize a CPWA *Lyapunov function we do not define it here.*

□

Obviously, the two first comments after Definition 5.1 apply equally to the linear programming problem from this definition. Further, if the functions $\mathbf{f}_p$, $p \in \mathcal{P}$, in Definition 5.5 are linear, then obviously we can set $B_{p,rs}^{(\mathbf{z},\mathcal{J})} := 0$ for all $p \in \mathcal{P}$, all $(\mathbf{z},\mathcal{J}) \in \mathcal{Z}_a$, and all $r, s = 1, 2, \dots, n$, and then the "error terms" $E_{p,\sigma,i}^{(\mathbf{z},\mathcal{J})}$ are all identically zero. Linear problems are thus the most easy to solve with the linear programming problem because we can drop the variables $C[\{\mathbf{x},\mathbf{y}\}]$ and the constraints **LC3** out of the linear programming problem altogether .

If the linear programming problem $\mathbf{LP}(\{\mathbf{f}_p \big| p \in \mathcal{P}\}, \mathcal{N}, \mathbf{PS}, \mathcal{D}, \|\cdot\|)$ from Definition 5.5 possesses a feasible solution, then we can use this solution to parameterize a time-invariant CPWA Lyapunov function for the autonomous Switched System 1.9 used in the construction of the linear programming problem. The definition of the parameterized CPWA Lyapunov function in the autonomous case is in essence identical to the definition in the nonautonomous case.

Definition 5.6 *Assume that*

$$\begin{aligned}
&\Upsilon_a, \\
&\Psi_a[y], \quad \textit{for all } y \in \mathcal{X}^{\|\cdot\|}, \\
&\Gamma_a[y], \quad \textit{for all } y \in \mathcal{X}^{\|\cdot\|}, \\
&V_a[\mathbf{x}], \quad \textit{for all } \mathbf{x} \in \mathcal{G}_a, \\
&C_a[\{\mathbf{x},\mathbf{y}\}], \quad \textit{for all } \{\mathbf{x},\mathbf{y}\} \in \mathcal{Y}_a.
\end{aligned}$$

is a feasible solution to the linear programming problem $\mathbf{LP}(\{\mathbf{f}_p \big| p \in \mathcal{P}\}, \mathcal{N}, \mathbf{PS}, \mathcal{D}, \|\cdot\|)$ *from Definition 5.5. Then we define the function* V_a^{Lya} *trough* $V_a^{Lya} \in \mathrm{CPWA}[\mathbf{PS}, \mathbf{PS}^{-1}(\mathcal{M} \setminus \mathcal{D})]$ *and*

$$V_a^{Lya}(\mathbf{x}) := V_a[\mathbf{x}] \quad \textit{for all } \mathbf{x} \in \mathcal{G}_a.$$

Further, we define the function ψ_a *from the numerical values of the variables* $\Psi_a[y]$ *and* γ_a *from the numerical values of the variables* $\Gamma_a[y]$*, just as the functions* ψ *and* γ *were defined in Section 5.2 from the numerical values of the variables* $\Psi[y]$ *and* $\Gamma[y]$ *respectively.* □

That V_a^{Lya} in Definition 5.6 is a Lyapunov function for the autonomous Switched System 1.9, that is equivalent to a time-invariant Lyapunov function parameterized by the linear programming problem $\mathbf{LP}(\{\mathbf{f}_p \big| p \in \mathcal{P}\}, \mathcal{N}, \mathbf{PS}, \mathbf{t}, \mathcal{D}, \|\cdot\|)$ from Definition 5.1, is proved in the next theorem.

Theorem 5.7 *Consider the Switched System 1.9 and assume that there are real-valued constants* $0 \le T' < T''$ *such that the functions* $\mathbf{f}_p$*,* $p \in \mathcal{P}$*, are all in* $[\mathcal{C}^2([T', T''] \times \mathcal{U})]^n$*. Let* $\mathbf{PS} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *be a piecewise scaling function and* $\mathcal{N} \subset \mathcal{U}$ *be such that the interior of the set*

$$\mathcal{M} := \bigcup_{\substack{\mathbf{z} \in \mathbb{Z}^n \\ \mathbf{PS}(\mathbf{z}+[0,1]^n) \subset \mathcal{N}}} \mathbf{PS}(\mathbf{z} + [0,1]^n)$$

is a connected set that contains the origin. Let $\|\cdot\|$ *be an arbitrary norm on* $\mathbb{R}^n$ *and let*

$$\mathcal{D} := \mathbf{PS}(\,]d_1^-, d_1^+[\, \times \,]d_2^-, d_2^+[\, \times \, \dots \, \times \,]d_n^-, d_n^+[\,)$$

be a set, of which the closure is contained in the interior of $\mathcal{M}$*, and either* $\mathcal{D} = \emptyset$ *or* d_i^- *and* d_i^+ *are integers such that* $d_i^- \le -1$ *and* $1 \le d_i^+$ *for all* $i = 1, 2, \dots, n$*. Finally, let* $\mathbf{t} := (t_0, t_1, \dots, t_M) \in \mathbb{R}^{M+1}$*,* $M \in \mathbb{N}_{>0}$ *be a vector such that* $T' =: t_0 < t_1 < \dots < t_M := T''$*.*

Assume the the Switched System 1.9 is autonomous. Then, the linear programming problem $\mathbf{LP}(\{\mathbf{f}_p \big| p \in \mathcal{P}\}, \mathcal{N}, \mathbf{PS}, \mathbf{t}, \mathcal{D}, \|\cdot\|)$ *from Definition 5.1 with the additional linear constraints:*

LC-A) *For every* $\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \mathcal{G}$ *such that* $\|\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\|_* = 0$:

$$V[\tilde{\mathbf{x}}] = V[\tilde{\mathbf{y}}].$$

For every $\{\tilde{\mathbf{x}}, \tilde{\mathbf{y}}\} \in \mathcal{Y}$ *such that* $\|\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\|_* = 0$:

$$C[\{\tilde{\mathbf{x}}, \tilde{\mathbf{y}}\}] = 0.$$

Is equivalent to the linear programming problem $\mathbf{LP}(\{\mathbf{f}_p \big| p \in \mathcal{P}\}, \mathcal{N}, \mathbf{PS}, \mathcal{D}, \|\cdot\|)$ *from Definition 5.5, in the following sense:*

i) If V^{Lya} *is a Lyapunov function, defined as in Section 5.2 from a feasible solution to the linear programming problem* $\mathbf{LP}(\{\mathbf{f}_p \big| p \in \mathcal{P}\}, \mathcal{N}, \mathbf{PS}, \mathbf{t}, \mathcal{D}, \|\cdot\|)$ *from Definition 5.1 that additionally satisfies the constraints* **LC-A**, *then* V^{Lya} *does not depend on* t *and we can parameterize a Lyapunov function* W^{Lya} *with the linear programming problem* $\mathbf{LP}(\{\mathbf{f}_p \big| p \in \mathcal{P}\}, \mathcal{N}, \mathbf{PS}, \mathcal{D}, \|\cdot\|)$ *from Definition 5.5, such that*

$$W^{Lya}(\mathbf{x}) = V^{Lya}(T', \mathbf{x}) \quad \textit{for all } \mathbf{x} \in \mathcal{M} \setminus \mathcal{D}.$$

ii) If W^{Lya} *is a Lyapunov function, defined as the function* V_a^{Lya} *in Definition 5.6, from a feasible solution to the linear programming problem* $\mathbf{LP}(\{\mathbf{f}_p \big| p \in \mathcal{P}\}, \mathcal{N}, \mathbf{PS}, \mathcal{D}, \|\cdot\|)$ *from Definition 5.5, then we can parameterize a Lyapunov function* V^{Lya} *by use of the linear programming problem* $\mathbf{LP}(\{\mathbf{f}_p \big| p \in \mathcal{P}\}, \mathcal{N}, \mathbf{PS}, \mathbf{t}, \mathcal{D}, \|\cdot\|)$ *from Definition 5.1 with* **LC-A** *as additional constraints, such that*

$$V^{Lya}(t, \mathbf{x}) = W^{Lya}(\mathbf{x}) \quad \textit{for all } t \in [T', T''] \textit{ and all } \mathbf{x} \in \mathcal{M} \setminus \mathcal{D}.$$

In both cases one should use the same numerical values for the bounds $B_{p,rs}^{(\mathbf{z},\mathcal{J})}$ *on the second-order derivatives of the functions* $\mathbf{f}_p$, $p \in \mathcal{P}$, *and for the constants* ε *and* δ.

Proof:
We start by proving proposition *i)*:

Assume that

$$\begin{aligned}
&\Upsilon_a, \\
&\Psi_a[y], \quad \text{for all } y \in \mathcal{X}^{\|\cdot\|}, \\
&\Gamma_a[y], \quad \text{for all } y \in \mathcal{X}^{\|\cdot\|}, \\
&V_a[\mathbf{x}], \quad \text{for all } \mathbf{x} \in \mathcal{G}_a, \\
&C_a[\{\mathbf{x}, \mathbf{y}\}], \quad \text{for all } \{\mathbf{x}, \mathbf{y}\} \in \mathcal{Y}_a.
\end{aligned}$$

is a feasible solution to the linear programming problem $\mathbf{LP}(\{\mathbf{f}_p \big| p \in \mathcal{P}\}, \mathcal{N}, \mathbf{PS}, \mathcal{D}, \|\cdot\|)$ from Definition 5.5, define $C_a[\{\mathbf{x}, \mathbf{x}\}] := 0$ for all $\mathbf{x} \in \mathcal{G}_a$, and set

$$\begin{aligned}
&\Upsilon := \Upsilon_a, \\
&\Psi[y] := \Psi_a[y], \quad \text{for all } y \in \mathcal{X}^{\|\cdot\|}, \\
&\Gamma[y] := \Gamma_a[y], \quad \text{for all } y \in \mathcal{X}^{\|\cdot\|}, \\
&V[\tilde{\mathbf{x}}] := V_a[(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)], \quad \text{for all } \tilde{\mathbf{x}} \in \mathcal{G}, \\
&C[\{\tilde{\mathbf{x}}, \tilde{\mathbf{y}}\}] := C_a[\{(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n), (\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_n)\}], \quad \text{for all } \{\tilde{\mathbf{x}}, \tilde{\mathbf{y}}\} \in \mathcal{Y}.
\end{aligned}$$

We claim that Υ, $\Psi[y]$, $\Gamma[y]$, $V[\tilde{\mathbf{x}}]$, and $C[\{\tilde{\mathbf{x}},\tilde{\mathbf{y}}\}]$ is a feasible solution to the linear programming problem $\mathbf{LP}(\{\mathbf{f}_p|p\in\mathcal{P}\},\mathcal{N},\mathbf{PS},\mathbf{t},\mathcal{D},\|\cdot\|)$ from Definition 5.1 that additionally satisfies the constraints **LC-A**. If this is the case, then clearly $V^{Lya}\in \text{CPWA}[\widetilde{\mathbf{PS}},\widetilde{\mathbf{PS}}^{-1}\big([T',T'']\times\big(\mathcal{M}\setminus\mathcal{D}\big)\big)]$, defined through $V^{Lya}(\tilde{\mathbf{x}}):=V[\tilde{\mathbf{x}}]$ for all $\tilde{\mathbf{x}}\in\mathcal{G}$, is the promised Lyapunov function.

It is a simple task so confirm that they satisfy the constraints **LC1**, **LC2**, **LC3**, and **LC-A**, so we only prove that they fulfill the constraints **LC4**, which is not as obvious.

Let $p\in\mathcal{P}$, $(\mathbf{z},\mathcal{J})\in\mathcal{Z}$, $\sigma\in\text{Perm}[\{0,1,\ldots,n\}]$, and $i\in\{0,1,\ldots,n+1\}$ be arbitrary, but fixed throughout this part of the proof. We have to show that

$$-\Gamma\big[\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_*\big]\geq\sum_{j=0}^{n}\left(\frac{V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}]-V[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)}\cdot(\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}-\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})}\tilde{f}_{p,\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})+E_{p,\sigma,i}^{(\mathbf{z},\mathcal{J})}C[\{\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})},\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}\}]\right). \tag{5.14}$$

We define the mapping $l_\sigma:\{0,1,\ldots,n+1\}\longrightarrow\{1,\ldots,n+1\}$ through

$$l_\sigma(k):=\begin{cases}k+1, & \text{if } 0\leq k\leq\sigma^{-1}(0),\\ k, & \text{otherwise,}\end{cases}$$

and $\varsigma\in\text{Perm}[\{1,2,\ldots,n\}]$ through

$$\varsigma(k):=\begin{cases}\sigma(k-1), & \text{if } 1\leq k\leq\sigma^{-1}(0),\\ \sigma(k), & \text{if } \sigma^{-1}(0)<k\leq n.\end{cases}$$

Further, we set

$$\mathbf{z}':=(z_1,z_2,\ldots,z_n),\quad\text{where}\quad\mathbf{z}=(z_0,z_1,\ldots,z_n).$$

Note that by these definitions and the definitions of $\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}$ and $\mathbf{y}_{\varsigma,j}^{(\mathbf{z}',\mathcal{J})}$ we have for all $j=0,1,\ldots,n+1$ and all $r=1,2,\ldots,n$, that

$$\begin{aligned}\mathbf{e}_r\cdot\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}&=\mathbf{e}_r\cdot\widetilde{\mathbf{PS}}(\widetilde{\mathbf{R}}^{\mathcal{J}}(\mathbf{z}+\sum_{k=j}^{n}\mathbf{e}_{\sigma(k)}))\\&=\mathbf{e}_r\cdot\mathbf{PS}(\mathbf{R}^{\mathcal{J}}(\mathbf{z}'+\sum_{\substack{k=j\\ \sigma(k)\neq0}}^{n}\mathbf{e}_{\sigma(k)}))\\&=\mathbf{e}_r\cdot\mathbf{PS}(\mathbf{R}^{\mathcal{J}}(\mathbf{z}'+\sum_{k=l_\sigma(j)}^{n}\mathbf{e}_{\varsigma(k)}))\\&=\mathbf{e}_r\cdot\mathbf{y}_{\varsigma,l_\sigma(j)}^{(\mathbf{z}',\mathcal{J})}.\end{aligned}\tag{5.15}$$

Especially, because $l_\sigma(0)=1$, we have

$$\mathbf{e}_r\cdot\mathbf{y}_{\sigma,0}^{(\mathbf{z},\mathcal{J})}=\mathbf{e}_r\cdot\mathbf{y}_{\varsigma,1}^{(\mathbf{z}',\mathcal{J})}\quad\text{for all } r=1,2,\ldots,n.$$

But then $\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_*=\|\mathbf{y}_{\varsigma,l_\sigma(i)}^{(\mathbf{z}',\mathcal{J})}\|$ and for every $r=1,2,\ldots,n$ and every $j=0,1,\ldots,n$ we have

$$A_{\sigma,r,j}^{(\mathbf{z},\mathcal{J})}=\left|\mathbf{e}_r\cdot(\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}-\mathbf{y}_{\sigma,n+1}^{(\mathbf{z},\mathcal{J})})\right|=\left|\mathbf{e}_r\cdot(\mathbf{y}_{\varsigma,l_\sigma(j)}^{(\mathbf{z}',\mathcal{J})}-\mathbf{y}_{\varsigma,n+1}^{(\mathbf{z}',\mathcal{J})})\right|=A_{\varsigma,r,l_\sigma(j)}^{(\mathbf{z}',\mathcal{J})}. \tag{5.16}$$

Further, because $\tilde{\mathbf{f}}_p$ does not depend on the first argument,

$$\tilde{f}_{p,\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) = \begin{cases} 1, & \text{if } \sigma(j) = 0, \\ f_{p,\varsigma(l_\sigma(j))}(\mathbf{y}_{\varsigma,l_\sigma(i)}^{(\mathbf{z}',\mathcal{J})}), & \text{if } j \in \{1,2,\ldots,n\} \setminus \{\sigma^{-1}(0)\}, \end{cases} \tag{5.17}$$

and similarly

$$V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] = V_a[\mathbf{y}_{\varsigma,l_\sigma(j)}^{(\mathbf{z}',\mathcal{J})}] \quad \text{for all } j = 0,1,\ldots,n+1, \tag{5.18}$$

and

$$C[\{\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}\}] = C_a[\{\mathbf{y}_{\varsigma,l_\sigma(j)}^{(\mathbf{z}',\mathcal{J})}, \mathbf{y}_{\varsigma,l_\sigma(j+1)}^{(\mathbf{z}',\mathcal{J})}] \quad \text{for all } j = 0,1,\ldots,n. \tag{5.19}$$

Especially,

$$V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - V[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}] = C[\{\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}\}] = 0 \quad \text{if } \sigma(j) = 0. \tag{5.20}$$

For the bounds $B_{p,rs}^{(\mathbf{z},\mathcal{J})}$ on the second-order derivatives of $\mathbf{f}_p$ we demand,

$$B_{p,rs}^{(\mathbf{z},\mathcal{J})} \geq \max_{i=1,2,\ldots,n} \sup_{\tilde{\mathbf{x}} \in \widetilde{\mathbf{PS}}(\tilde{\mathbf{R}}^{\mathcal{J}}(\mathbf{z}+[0,1]^{n+1}))} \left| \frac{\partial^2 \tilde{f}_{p,i}}{\partial \tilde{x}_r \partial \tilde{x}_s}(\tilde{\mathbf{x}}) \right|,$$

which is compatible with

$$B_{p,rs}^{(\mathbf{z},\mathcal{J})} := 0 \quad \text{if } r = 0 \text{ or } s = 0$$

and

$$B_{p,rs}^{(\mathbf{z},\mathcal{J})} := B_{p,rs}^{(\mathbf{z}',\mathcal{J})} \quad \text{for } r,s = 1,2,\ldots,n.$$

This together with (5.16) implies that

$$\begin{aligned} E_{p,\sigma,i}^{(\mathbf{z},\mathcal{J})} &:= \frac{1}{2}\sum_{r,s=0}^{n} B_{p,rs}^{(\mathbf{z},\mathcal{J})} A_{\sigma,r,i}^{(\mathbf{z},\mathcal{J})}(A_{\sigma,s,i}^{(\mathbf{z},\mathcal{J})} + A_{\sigma,s,0}^{(\mathbf{z},\mathcal{J})}) \\ &= \frac{1}{2}\sum_{r,s=1}^{n} B_{p,rs}^{(\mathbf{z}',\mathcal{J})} A_{\varsigma,r,l_\sigma(i)}^{(\mathbf{z},\mathcal{J})}(A_{\varsigma,s,l_\sigma(i)}^{(\mathbf{z}',\mathcal{J})} + A_{\varsigma,s,1}^{(\mathbf{z}',\mathcal{J})}) \\ &= E_{p,\varsigma,l_\sigma(i)}^{(\mathbf{z}',\mathcal{J})}. \end{aligned} \tag{5.21}$$

Now, by assumption,

$$-\Gamma_a\big[\|\mathbf{y}_{\varsigma,l_\sigma(i)}^{(\mathbf{z}',\mathcal{J})}\|\big] \geq \sum_{j=1}^{n} \left(\frac{V_a[\mathbf{y}_{\varsigma,j}^{(\mathbf{z}',\mathcal{J})}] - V_a[\mathbf{y}_{\varsigma,j+1}^{(\mathbf{z}',\mathcal{J})}]}{\mathbf{e}_{\varsigma(j)} \cdot (\mathbf{y}_{\varsigma,j}^{(\mathbf{z}',\mathcal{J})} - \mathbf{y}_{\varsigma,j+1}^{(\mathbf{z}',\mathcal{J})})} f_{p,\varsigma(j)}(\mathbf{y}_{\varsigma,l_\sigma(i)}^{(\mathbf{z}',\mathcal{J})}) + E_{p,\varsigma,l_\sigma(i)}^{(\mathbf{z}',\mathcal{J})} C_a[\{\mathbf{y}_{\varsigma,j}^{(\mathbf{z}',\mathcal{J})}, \mathbf{y}_{\varsigma,j+1}^{(\mathbf{z}',\mathcal{J})}\}] \right),$$

so by (5.15), the definition of the function l_σ, (5.21), (5.17), (5.18), (5.19), and (5.20), we have

$$\begin{aligned}
-\Gamma\big[\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_*\big] &= -\Gamma_a\big[\|\mathbf{y}_{\varsigma,l_\sigma(i)}^{(\mathbf{z}',\mathcal{J})}\|\big] \\
&\geq \sum_{j=1}^{n}\left(\frac{V_a[\mathbf{y}_{\varsigma,j}^{(\mathbf{z}',\mathcal{J})}] - V_a[\mathbf{y}_{\varsigma,j+1}^{(\mathbf{z}',\mathcal{J})}]}{\mathbf{e}_{\varsigma(j)}\cdot(\mathbf{y}_{\varsigma,j}^{(\mathbf{z}',\mathcal{J})} - \mathbf{y}_{\varsigma,j+1}^{(\mathbf{z}',\mathcal{J})})} f_{p,\varsigma(j)}(\mathbf{y}_{\varsigma,l_\sigma(i)}^{(\mathbf{z}',\mathcal{J})}) + E_{p,\varsigma,l_\sigma(i)}^{(\mathbf{z}',\mathcal{J})} C_a[\{\mathbf{y}_{\varsigma,j}^{(\mathbf{z}',\mathcal{J})}, \mathbf{y}_{\varsigma,j+1}^{(\mathbf{z}',\mathcal{J})}\}]\right) \\
&= \sum_{\substack{j=0\\ \sigma(j)\neq 0}}^{n}\left(\frac{V_a[\mathbf{y}_{\varsigma,l_\sigma(j)}^{(\mathbf{z}',\mathcal{J})}] - V_a[\mathbf{y}_{\varsigma,l_\sigma(j+1)}^{(\mathbf{z}',\mathcal{J})}]}{\mathbf{e}_{\varsigma(l_\sigma(j))}\cdot(\mathbf{y}_{\varsigma,l_\sigma(j)}^{(\mathbf{z}',\mathcal{J})} - \mathbf{y}_{\varsigma,l_\sigma(j+1)}^{(\mathbf{z}',\mathcal{J})})} f_{p,\varsigma(l_\sigma(j))}(\mathbf{y}_{\varsigma,l_\sigma(i)}^{(\mathbf{z}',\mathcal{J})}) + E_{p,\varsigma,l_\sigma(i)}^{(\mathbf{z}',\mathcal{J})} C_a[\{\mathbf{y}_{\varsigma,l_\sigma(j)}^{(\mathbf{z}',\mathcal{J})}, \mathbf{y}_{\varsigma,l_\sigma(j+1)}^{(\mathbf{z}',\mathcal{J})}\}]\right) \\
&= \sum_{\substack{j=0\\ \sigma(j)\neq 0}}^{n}\left(\frac{V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - V[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)}\cdot(\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})} \tilde{f}_{p,\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) + E_{p,\sigma,i}^{(\mathbf{z},\mathcal{J})} C[\{\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}\}]\right) \\
&= \sum_{j=0}^{n}\left(\frac{V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - V[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)}\cdot(\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})} \tilde{f}_{p,\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) + E_{p,\sigma,i}^{(\mathbf{z},\mathcal{J})} C[\{\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}\}]\right)
\end{aligned}$$

and we have proved (5.14).

We now prove proposition *ii)*:

Assume that

$$\begin{aligned}
&\Upsilon, \\
&\Psi[y], \quad \text{for all } y \in \mathcal{X}^{\|\cdot\|}, \\
&\Gamma[y], \quad \text{for all } y \in \mathcal{X}^{\|\cdot\|}, \\
&V[\tilde{\mathbf{x}}], \quad \text{for all } \tilde{\mathbf{x}} \in \mathcal{G}, \\
&C[\{\tilde{\mathbf{x}}, \tilde{\mathbf{y}}\}], \quad \text{for all } \{\tilde{\mathbf{x}}, \tilde{\mathbf{y}}\} \in \mathcal{Y}.
\end{aligned}$$

is a solution to the linear programming problem $\mathbf{LP}(\{\mathbf{f}_p \big| p \in \mathcal{P}\}, \mathcal{N}, \mathbf{PS}, \mathbf{t}, \mathcal{D}, \|\cdot\|)$ from Definition 5.1 and set

$$\begin{aligned}
&\Upsilon_a := \Upsilon, \\
&\Psi_a[y] := \Psi[y], \quad \text{for all } y \in \mathcal{X}^{\|\cdot\|}, \\
&\Gamma_a[y] := \Gamma[y], \quad \text{for all } y \in \mathcal{X}^{\|\cdot\|}, \\
&V_a[\mathbf{x}] := V[(T', \mathbf{x})], \quad \text{for all } \mathbf{x} \in \mathcal{G}_a, \\
&C_a[\{\mathbf{x}, \mathbf{y}\}] := C[\{(T', \mathbf{x}), (T', \mathbf{y})\}], \quad \text{for all } \{\mathbf{x}, \mathbf{y}\} \in \mathcal{Y}_a.
\end{aligned}$$

We claim that, with these numerical values, the variables Υ_a, $\Psi_a[y]$, $\Gamma_a[y]$, $V_a[\mathbf{x}]$, and $C_a[\{\mathbf{x}, \mathbf{y}\}]$ are a solution to the linear programming problem $\mathbf{LP}(\{\mathbf{f}_p \big| p \in \mathcal{P}\}, \mathcal{N}, \mathbf{PS}, \mathcal{D}, \|\cdot\|)$ from Definition 5.5, that

$$V^{Lya} \in \text{CPWA}[\widetilde{\mathbf{PS}}, \widetilde{\mathbf{PS}}^{-1}\big([T', T''] \times \big(\mathcal{M} \setminus \mathcal{D}\big)\big)], \quad \text{defined through } V^{Lya}(\tilde{\mathbf{x}}) := V[\tilde{\mathbf{x}}] \text{ for all } \tilde{\mathbf{x}} \in \mathcal{G},$$

does not depend on the first argument, and that $W^{Lya} \in \text{CPWA}[\mathbf{PS}, \mathbf{PS}^{-1}, \mathcal{M} \setminus \mathcal{D}]$, defined through $W^{Lya}(\mathbf{x}) := V_a[\mathbf{x}]$ for all $\mathbf{x} \in \mathcal{G}_a$, is the promised Lyapunov function.

First we prove that the function values of V^{Lya} do not depend on the first argument. To do this it is obviously enough to show that this holds true for every simplex in the simplicial partition of $[T', T''] \times \big(\mathcal{M} \setminus \mathcal{D}\big)$. Let $(\mathbf{z}, \mathcal{J}) \in \mathcal{Z}$ and $\sigma \in \text{Perm}[\{0, 1, \ldots, n\}]$ be arbitrary and let

$$\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \text{con}\{\mathbf{y}_{\sigma,0}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,1}^{(\mathbf{z},\mathcal{J})}, \ldots, \mathbf{y}_{\sigma,n+1}^{(\mathbf{z},\mathcal{J})}\}$$

such that

$$\mathbf{e}_r \cdot \tilde{\mathbf{x}} = \mathbf{e}_r \cdot \tilde{\mathbf{y}} \quad \text{for all } r = 1, 2, \dots, n.$$

We are going to show that $V^{Lya}(\tilde{\mathbf{x}}) = V^{Lya}(\tilde{\mathbf{y}})$.

Set $k := \sigma^{-1}(0)$ and let $\lambda_0, \lambda_1, \dots, \lambda_{n+1} \in [0,1]$ and $\mu_0, \mu_1, \dots, \mu_{n+1} \in [0,1]$ be such that

$$\tilde{\mathbf{x}} = \sum_{i=0}^{n+1} \lambda_i \mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}, \quad \tilde{\mathbf{y}} = \sum_{i=0}^{n+1} \mu_i \mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}, \quad \text{and} \quad \sum_{i=0}^{n+1} \lambda_i = \sum_{i=0}^{n+1} \mu_i = 1.$$

From the definition of the $\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}$ it follows that:

$$\mathbf{e}_{\sigma(0)} \cdot \sum_{i=0}^{n+1} \lambda_i \mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})} = \mathbf{e}_{\sigma(0)} \cdot \sum_{i=0}^{n+1} \mu_i \mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})} \quad \text{from which } \lambda_0 = \mu_0 \text{ follows},$$

which implies

$$\mathbf{e}_{\sigma(1)} \cdot \sum_{i=1}^{n+1} \lambda_i \mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})} = \mathbf{e}_{\sigma(1)} \cdot \sum_{i=1}^{n+1} \mu_i \mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})} \quad \text{from which } \lambda_1 = \mu_1 \text{ follows},$$

$$\vdots$$

which implies

$$\mathbf{e}_{\sigma(k-1)} \cdot \sum_{i=k-1}^{n+1} \lambda_i \mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})} = \mathbf{e}_{\sigma(k-1)} \cdot \sum_{i=k-1}^{n+1} \mu_i \mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})} \quad \text{from which } \lambda_{k-1} = \mu_{k-1} \text{ follows},$$

which implies

$$\mathbf{e}_{\sigma(k+1)} \cdot \sum_{i=k}^{n+1} \lambda_i \mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})} = \mathbf{e}_{\sigma(k+1)} \cdot \sum_{i=k}^{n+1} \mu_i \mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})} \quad \text{from which } \lambda_k + \lambda_{k+1} = \mu_k + \mu_{k+1} \text{ follows},$$

which implies

$$\mathbf{e}_{\sigma(k+2)} \cdot \sum_{i=k+2}^{n+1} \lambda_i \mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})} = \mathbf{e}_{\sigma(k+2)} \cdot \sum_{i=k+2}^{n+1} \mu_i \mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})} \quad \text{from which } \lambda_{k+2} = \mu_{k+2} \text{ follows},$$

$$\vdots$$

which implies

$$\mathbf{e}_{\sigma(n)} \cdot \sum_{i=n}^{n+1} \lambda_i \mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})} = \mathbf{e}_{\sigma(n)} \cdot \sum_{i=n}^{n+1} \mu_i \mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})} \quad \text{from which } \lambda_n = \mu_n \text{ follows}.$$

But then $\lambda_{n+1} = \mu_{n+1}$ and because by **LC-A** we have $V[\mathbf{y}_{\sigma,k}^{(\mathbf{z},\mathcal{J})}] = V[\mathbf{y}_{\sigma,k+1}^{(\mathbf{z},\mathcal{J})}]$ we get

$$\begin{aligned} V^{Lya}(\tilde{\mathbf{x}}) = V^{Lya}(\sum_{i=0}^{n+1} \lambda_i \mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) &= \sum_{i=0}^{n+1} \lambda_i V[\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}] \\ &= \sum_{\substack{i=0 \\ i\neq k,k+1}}^{n+1} \lambda_i V[\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}] + (\lambda_k + \lambda_{k+1}) V[\mathbf{y}_{\sigma,k}^{(\mathbf{z},\mathcal{J})}] \\ &= \sum_{\substack{i=0 \\ i\neq k,k+1}}^{n+1} \mu_i V[\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}] + (\mu_k + \mu_{k+1}) V[\mathbf{y}_{\sigma,k}^{(\mathbf{z},\mathcal{J})}] \\ &= \sum_{i=0}^{n+1} \mu_i V[\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}] = V^{Lya}(\sum_{i=0}^{n+1} \mu_i \mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) \\ &= V^{Lya}(\tilde{\mathbf{y}}) \end{aligned}$$

and we have proved that V^{Lya} does not depend on the first argument.

Now, let $p \in \mathcal{P}$, $(\mathbf{z},\mathcal{J}) \in \mathcal{Z}_a$, $\sigma \in \text{Perm}[\{1,2,\dots,n\}]$, and $i \in \{1,2,\dots,n+1\}$ be arbitrary, but fixed throughout the rest of the proof. To finish the proof we have to show that

$$-\Gamma_a\big[\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|\big] \geq \sum_{j=1}^{n} \left(\frac{V_a[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - V_a[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})} f_{p,\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) + E_{p,\sigma,i}^{(\mathbf{z},\mathcal{J})} C_a[\{\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}\}] \right). \tag{5.22}$$

We define $\varsigma \in \text{Perm}[\{0,1,2,\dots,n\}]$ trough

$$\varsigma(k) := \begin{cases} 0, & \text{if } k = 0, \\ \sigma(k), & \text{if } k \in \{1,2,\dots,n\}, \end{cases}$$

and $\mathbf{z}' \in \mathcal{G}$ through

$$\mathbf{z}' := (T', z_1, z_2, \dots, z_n), \quad \text{where} \quad \mathbf{z} = (z_1, z_2, \dots, z_n).$$

Then, for every $r = 1,2,\dots,n$, one easily verifies that

$$\mathbf{e}_r \cdot \mathbf{y}_{\varsigma,0}^{(\mathbf{z}',\mathcal{J})} = \mathbf{e}_r \cdot \mathbf{y}_{\varsigma,1}^{(\mathbf{z}',\mathcal{J})}$$

and, for all $r = 1,2,\dots,n$ and all $j = 1,2,\dots,n+1$, that

$$\mathbf{e}_r \cdot \mathbf{y}_{\varsigma,j}^{(\mathbf{z}',\mathcal{J})} = \mathbf{e}_r \cdot \widetilde{\mathbf{PS}}(\widetilde{\mathbf{R}}^{\mathcal{J}}(\mathbf{z}' + \sum_{k=j}^{n} \mathbf{e}_{\varsigma(k)})) = \mathbf{e}_r \cdot \mathbf{PS}(\mathbf{R}^{\mathcal{J}}(\mathbf{z} + \sum_{k=j}^{n} \mathbf{e}_{\sigma(k)})) = \mathbf{e}_r \cdot \mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}. \tag{5.23}$$

But then $\|\mathbf{y}_{\varsigma,i}^{(\mathbf{z}',\mathcal{J})}\|_* = \|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|$ and for every $r = 1,2,\dots,n$

$$A_{\varsigma,r,0}^{(\mathbf{z}',\mathcal{J})} = \left|\mathbf{e}_r \cdot (\mathbf{y}_{\varsigma,0}^{(\mathbf{z}',\mathcal{J})} - \mathbf{y}_{\varsigma,n+1}^{(\mathbf{z}',\mathcal{J})})\right| = \left|\mathbf{e}_r \cdot (\mathbf{y}_{\sigma,1}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,n+1}^{(\mathbf{z},\mathcal{J})})\right| = A_{\sigma,r,1}^{(\mathbf{z},\mathcal{J})}. \tag{5.24}$$

and for every every $j, r = 1,2,\dots,n$ we have

$$A_{\varsigma,r,j}^{(\mathbf{z}',\mathcal{J})} = \left|\mathbf{e}_r \cdot (\mathbf{y}_{\varsigma,j}^{(\mathbf{z}',\mathcal{J})} - \mathbf{y}_{\varsigma,n+1}^{(\mathbf{z}',\mathcal{J})})\right| = \left|\mathbf{e}_r \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,n+1}^{(\mathbf{z},\mathcal{J})})\right| = A_{\sigma,r,j}^{(\mathbf{z},\mathcal{J})}. \tag{5.25}$$

For every $k = 0, 1, \ldots, M$ define

$$\mathbf{z}_k := (t_k, z_1, z_2, \ldots, z_n), \quad \text{where} \quad \mathbf{z} := (z_1, z_2, \ldots, z_n),$$

and define for every $r, s = 1, 2, \ldots, n$

$$B_{p,rs}^{(\mathbf{z},\mathcal{J})} := \min_{k=0,1,\ldots,n} B_{p,rs}^{(\mathbf{z}_k,\mathcal{J})}.$$

Now set

$$B_{p,00}^{(\mathbf{z}_k,\mathcal{J})} := 0 \quad \text{for all } k = 0, 1, \ldots, M$$

and

$$B_{p,rs}^{(\mathbf{z}_k,\mathcal{J})} := B_{p,rs}^{(\mathbf{z},\mathcal{J})} \quad \text{for all } k = 0, 1, \ldots, M \text{ and all } r, s = 1, 2, \ldots, n,$$

and consider, that with these possibly tighter bounds $B_{p,rs}^{(\mathbf{z}_k,\mathcal{J})}$ in the linear programming problem $\mathbf{LP}(\{\mathbf{f}_p\big|p \in \mathcal{P}\}, \mathcal{N}, \mathbf{PS}, \mathbf{t}, \mathcal{D}, \|\cdot\|)$, the Υ, $\Psi[y]$, $\Gamma[y]$, $V[\tilde{\mathbf{x}}]$, and $C[\{\tilde{\mathbf{x}}, \tilde{\mathbf{y}}\}]$ are, of course, still a solution. Therefore we can just as well assume that these were the bounds ab initio.

It follows by (5.25) that

$$\begin{aligned}
E_{p,\varsigma,i}^{(\mathbf{z}',\mathcal{J})} &= \frac{1}{2}\sum_{r,s=0}^{n} B_{p,rs}^{(\mathbf{z}',\mathcal{J})} A_{\varsigma,r,i}^{(\mathbf{z}',\mathcal{J})}(A_{\varsigma,s,i}^{(\mathbf{z}',\mathcal{J})} + A_{\varsigma,s,0}^{(\mathbf{z},\mathcal{J})}) \\
&= \frac{1}{2}\sum_{r,s=1}^{n} B_{p,rs}^{(\mathbf{z},\mathcal{J})} A_{\sigma,r,i}^{(\mathbf{z},\mathcal{J})}(A_{\sigma,s,i}^{(\mathbf{z},\mathcal{J})} + A_{\sigma,s,1}^{(\mathbf{z},\mathcal{J})}) \\
&= E_{p,\sigma,i}^{(\mathbf{z},\mathcal{J})}.
\end{aligned} \tag{5.26}$$

By assumption

$$-\Gamma\big[\|\mathbf{y}_{\varsigma,i}^{(\mathbf{z}',\mathcal{J})}\|_*\big] \geq \sum_{j=0}^{n}\left(\frac{V[\mathbf{y}_{\varsigma,j}^{(\mathbf{z}',\mathcal{J})}] - V[\mathbf{y}_{\varsigma,j+1}^{(\mathbf{z}',\mathcal{J})}]}{\mathbf{e}_{\varsigma(j)}\cdot(\mathbf{y}_{\varsigma,j}^{(\mathbf{z}',\mathcal{J})} - \mathbf{y}_{\varsigma,j+1}^{(\mathbf{z}',\mathcal{J})})}\tilde{f}_{p,\varsigma(j)}(\mathbf{y}_{\varsigma,i}^{(\mathbf{z}',\mathcal{J})}) + E_{p,\varsigma,i}^{(\mathbf{z}',\mathcal{J})}C[\{\mathbf{y}_{\varsigma,j}^{(\mathbf{z}',\mathcal{J})}, \mathbf{y}_{\varsigma,j+1}^{(\mathbf{z}',\mathcal{J})}\}]\right),$$

so, by (5.23), **LC-A**, because $\mathbf{f}_p$ does not depend on the first argument, and (5.26), we get

$$\begin{aligned}
-\Gamma\big[\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|\big] &= -\Gamma\big[\|\mathbf{y}_{\varsigma,i}^{(\mathbf{z}',\mathcal{J})}\|_*\big] \\
&\geq \sum_{j=0}^{n}\left(\frac{V[\mathbf{y}_{\varsigma,j}^{(\mathbf{z}',\mathcal{J})}] - V[\mathbf{y}_{\varsigma,j+1}^{(\mathbf{z}',\mathcal{J})}]}{\mathbf{e}_{\varsigma(j)}\cdot(\mathbf{y}_{\varsigma,j}^{(\mathbf{z}',\mathcal{J})} - \mathbf{y}_{\varsigma,j+1}^{(\mathbf{z}',\mathcal{J})})}\tilde{f}_{p,\varsigma(j)}(\mathbf{y}_{\varsigma,i}^{(\mathbf{z}',\mathcal{J})}) + E_{p,\varsigma,i}^{(\mathbf{z}',\mathcal{J})}C[\{\mathbf{y}_{\varsigma,j}^{(\mathbf{z}',\mathcal{J})}, \mathbf{y}_{\varsigma,j+1}^{(\mathbf{z}',\mathcal{J})}\}]\right) \\
&= \sum_{j=1}^{n}\left(\frac{V_a[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - V_a[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)}\cdot(\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})}f_{p,\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) + E_{p,\sigma,i}^{(\mathbf{z},\mathcal{J})}C_a[\{\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}\}]\right)
\end{aligned}$$

and we have proved (5.22) and finished the proof.

■

An immediate consequence of Theorem 5.7 is a theorem, similar to Theorem 5.2, but for autonomous systems with a time-invariant CPWA Lyapunov function.

Theorem 5.8 (Parametrization of a CPWA **Lyapunov function by linear programming)** *Consider the linear programming problem* $\mathbf{LP}(\{\mathbf{f}_p \big| p \in \mathcal{P}\}, \mathcal{N}, \mathbf{PS}, \mathcal{D}, \|\cdot\|)$ *from Definition 5.5 and assume that it possesses a feasible solution. Let the functions* $\psi_a, \gamma_a,$ *and* V_a^{Lya} *be defined as in Definition 5.6 from the numerical values of the variables* $\Psi_a[x]$, $\Gamma_a[x]$, *and* $V_a[\tilde{\mathbf{x}}]$ *from a feasible solution. Then the inequality*

$$\psi_a(\|\mathbf{x}\|) \leq V_a^{Lya}(\mathbf{x})$$

holds true for all $\mathbf{x} \in \mathcal{M} \setminus \mathcal{D}$. *If* $\mathcal{D} = \emptyset$ *we have* $\psi_a(0) = V_a^{Lya}(t, \mathbf{0}) = 0$. *If* $\mathcal{D} \neq \emptyset$ *we have, with*

$$V_{\partial\mathcal{M},\min}^{Lya} := \min_{\mathbf{x}\in\partial\mathcal{M}} V_a^{Lya}(\mathbf{x})$$

and

$$V_{\partial\mathcal{D},\max}^{Lya} := \max_{\mathbf{x}\in\partial\mathcal{D}} V_a^{Lya}(\mathbf{x}),$$

that

$$V_{\partial\mathcal{D},\max}^{Lya} \leq V_{\partial\mathcal{M},\min}^{Lya} - \delta.$$

Further, with $\boldsymbol{\phi}$ *as the solution to the Switched System 1.9 that we used in the construction of the linear programming problem, the inequality*

$$-\gamma_a(\|\boldsymbol{\phi}_\varsigma(t,\boldsymbol{\xi})\|) \geq \limsup_{h\to 0+} \frac{V^{Lya}(\boldsymbol{\phi}_\varsigma(t+h,\boldsymbol{\xi})) - V^{Lya}(\boldsymbol{\phi}_\varsigma(t,\boldsymbol{\xi}))}{h}$$

hold true for all $\varsigma \in \mathcal{S}_\mathcal{P}$ *and all* $\boldsymbol{\xi}$ *in the interior of* $\mathcal{M} \setminus \mathcal{D}$.

PROOF:
Follows directly by Theorem 5.2 and Theorem 5.7. ■

We end this discussion with a theorem, that is the equivalent of Theorem 5.4 for autonomous systems with a time-invariant CPWA Lyapunov function, parameterized by the linear programming problem $\mathbf{LP}(\{\mathbf{f}_p \big| p \in \mathcal{P}\}, \mathcal{N}, \mathbf{PS}, \mathcal{D}, \|\cdot\|)$ from Definition 5.5. It states some stability properties the system must have, if it possesses such a Lyapunov function.

Theorem 5.9 (Implications of the Lyapunov function V_a^{Lya}**)** *Make the same assumptions and definitions as in Theorem 5.2 and assume additionally that the arbitrary norm* $\|\cdot\|$ *in the linear programming problem* $\mathbf{LP}(\{\mathbf{f}_p \big| p \in \mathcal{P}\}, \mathcal{N}, \mathbf{PS}, \mathcal{D}, \|\cdot\|)$ *is a k-norm* [2], $1 \leq k \leq +\infty$. *Define the set* $\mathcal{T}$ *through* $\mathcal{T} := \{\mathbf{0}\}$ *if* $\mathcal{D} = \emptyset$ *and*

$$\mathcal{T} := \mathcal{D} \cup \left\{\mathbf{x} \in \mathcal{M} \setminus \mathcal{D} \;\middle|\; V_a^{Lya}(\mathbf{x}) \leq V_{\partial\mathcal{D},\max}^{Lya}\right\}, \quad \text{if } \mathcal{D} \neq \emptyset,$$

and the set $\mathcal{A}$ *through*

$$\mathcal{A} := \left\{\mathbf{x} \in \mathcal{M} \setminus \mathcal{D} \;\middle|\; V^{Lya}(\mathbf{x}) < V_{\partial\mathcal{M},\min}^{Lya}\right\}.$$

Set $q := k \cdot (k-1)^{-1}$ *if* $1 < k < +\infty$, $q := 1$ *if* $k = +\infty$, *and* $q := +\infty$ *if* $k = 1$, *and define the constant*

$$E_q := \|\sum_{i=1}^n \mathbf{e}_i\|_q.$$

Then the following propositions hold true:

[2] With k-norm we mean the norm $\|\mathbf{x}\|_k := \left(\sum_{i=1}^n |x_i|^k\right)^{1/k}$ if $1 \leq k < +\infty$ and $\|\mathbf{x}\|_\infty := \max_{i=1,2,\ldots,n} |x_i|$. Unfortunately, these norms are usually called p-norms, which is inappropriate in this context because the alphabet p is used to index the functions $\mathbf{f}_p$, $p \in \mathcal{P}$.

i) If $\boldsymbol{\xi} \in \mathcal{T}$*, then* $\boldsymbol{\phi}_\sigma(t, \boldsymbol{\xi}) \in \mathcal{T}$ *for all* $\sigma \in \mathcal{S}_\mathcal{P}$ *and all* $t \geq 0$*.*

ii) If $\boldsymbol{\xi} \in \mathcal{M} \setminus \mathcal{D}$*, the inequality*

$$V_a^{Lya}(\boldsymbol{\phi}_\sigma(t, \boldsymbol{\xi})) \leq V_a^{Lya}(\boldsymbol{\xi}) \exp\left(-\frac{\Upsilon_a}{\varepsilon E_q} t\right) \tag{5.27}$$

holds true for all t *such that* $\boldsymbol{\phi}_\sigma(t', \boldsymbol{\xi}) \in \mathcal{M} \setminus \mathcal{D}$ *for all* $0 \leq t' \leq t$*.*

iii) If $\boldsymbol{\xi} \in \mathcal{A} \setminus \mathcal{T}$ *and* $\mathcal{D} = \emptyset$*, then inequality (5.27) holds true for all* $t \geq 0$ *and all* $\sigma \in \mathcal{S}_\mathcal{P}$*. If* $\boldsymbol{\xi} \in \mathcal{A} \setminus \mathcal{T}$ *and* $\mathcal{D} \neq \emptyset$*, then, for every* $\sigma \in \mathcal{S}_\mathcal{P}$ *there is a* $t' \geq 0$*, such that inequality (5.27) holds true for all* $0 \leq t \leq t'$*,* $\boldsymbol{\phi}_\sigma(t', \boldsymbol{\xi}) \in \partial\mathcal{T}$*, and* $\boldsymbol{\phi}_\sigma(t, \boldsymbol{\xi}) \in \mathcal{T}$ *for all* $t \geq t'$*.*

PROOF:
Follows directly by Theorem 5.4.

■

Chapter 6

Constructive converse theorems

Consider the Switched System 1.9 and assume that the set $\mathcal{P}$ is finite. In this chapter we will prove that whenever this system possesses a two-times continuously differentiable Lyapunov function, then it is always possible to parameterize a Lyapunov function for the system by use of the linear programming problem in Definition 5.1. Further, we will prove that if the Switched System 1.9 is autonomous, then we can alternatively use the linear programming problem from Definition 5.5 to parameterize a time-invariant Lyapunov function for the system.

We will start by showing how to combine the results from Theorem 3.10, a non-constructive converse Lyapunov theorem, and the linear programming problem from Definition 5.1, to prove a constructive converse Lyapunov theorem. We will do this for the general, not necessarily autonomous, case. Thereafter, we will do the same for autonomous switched systems and we will prove that in this case, we can parameterize a time-invariant Lyapunov function by the use of the linear programming problem from Definition 5.5.

The structure of this chapter is somewhat unconventional because we start with a proof of the yet to be stated Theorem 6.1. In the proof we assign values to the constants and to the variables of the linear programming problem such that a feasible solution results. By these assignments we will use the numerical values of the Lyapunov function from Theorem 3.10. Note, that because Theorem 3.10 is a pure existence theorem, the numerical values of this Lyapunov function are not known. However, our knowledge about these numerical values and their relations is substantial. Indeed, we have enough information to prove that the linear constraints **LC1**, **LC2**, **LC3**, and **LC4**, of the linear programming problem in Definition 5.1 are fulfilled by the numerical values we assign to the variables and to the constants. Because there are well-known algorithms to find a feasible solution to a linear programming problem if the set of feasible solutions is not empty, this implies that we can always parameterize a Lyapunov function by the use of the linear programming problem in Definition 5.1, whenever the underlying system possesses a Lyapunov function at all.

6.1 The assumptions

Consider the Switched System 1.9 and assume that the set $\mathcal{P}$ is finite and that $\mathbf{f}_p$ is a $[\mathcal{C}^2(\mathbb{R}_{\geq 0} \times \mathcal{U})]^n$ function for every $p \in \mathcal{P}$. Further, assume that there is a constant $a > 0$ such that $[-a, a]^n \subset \mathcal{U}$ and $W \in \mathcal{C}^2(\mathbb{R}_{\geq 0} \times [-a, a]^n)$ is a Lyapunov function for the switched system. By Theorem 3.10, for example, this is the case if the functions $\mathbf{f}_p$ are globally Lipschitz in the time argument and locally Lipschitz in the state-space argument, the origin is a uniformly asymptotically stable equilibrium of the Switched System 1.9, and $[-a,a]^n$ is a subset of its region of attraction. By Definition 2.17 there exist, for an arbitrary norm $\|\cdot\|$ on $\mathbb{R}^n$, class $\mathcal{K}$ functions α, β, and ω, such that

$$\alpha(\|\mathbf{x}\|) \leq W(t,\mathbf{x}) \leq \beta(\|\mathbf{x}\|)$$

and

$$[\nabla_{\mathbf{x}} W](t,\mathbf{x}) \cdot \mathbf{f}_p(t,\mathbf{x}) + \frac{\partial W}{\partial t}(t,\mathbf{x}) \leq -\omega(\|\mathbf{x}\|) \tag{6.1}$$

for all $(t,\mathbf{x}) \in \mathbb{R}_{>0} \times]-a,a[^n$ and all $p \in \mathcal{P}$. Further, by Lemma 2.13, we can assume without loss of generality that α and ω are convex functions. Now, let $0 \leq T' < T'' < +\infty$ be arbitrary and let $\mathcal{D}' \subset [-a,a]^n$ be an arbitrary neighborhood of the origin. Especially, the set $\mathcal{D}' \neq \emptyset$ can be taken as small as one wishes. We are going to prove that we can parameterize a CPWA Lyapunov function on the set $[T',T''] \times \big([-a,a]^n \setminus \mathcal{D}'\big)$. We will start by assigning values to the constants and the variables of the linear programming problem $\mathbf{LP}(\{\mathbf{f}_p \big| p \in \mathcal{P}\},]-a,a[^n, \mathbf{PS}, \mathbf{t}, \mathcal{D}, \|\cdot\|)$ in Definition 5.1. This includes that we define the piecewise scaling function $\mathbf{PS}$, the vector $\mathbf{t}$, and the set $\mathcal{D} \subset \mathcal{D}'$. Thereafter, we will prove that the linear constraints of the linear programming problem are all fulfilled by these values.

6.2 The assignments

First, we determine a constant B that is an upper bound on all second-order derivatives of the components of the functions $\mathbf{f}_p$, $p \in \mathcal{P}$. That is, with $\tilde{\mathbf{x}} = (\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_n) := (t,\mathbf{x})$ and

$$\tilde{\mathbf{f}}_p(\tilde{\mathbf{x}}) = (\tilde{f}_{p,0}(\tilde{\mathbf{x}}), \tilde{f}_{p,1}(\tilde{\mathbf{x}}), \ldots, \tilde{f}_{p,n}(\tilde{\mathbf{x}})) = (1, f_{p,1}(t,\mathbf{x}), f_{p,2}(t,\mathbf{x}), \ldots, f_{p,n}(t,\mathbf{x})),$$

we need a constant $B < +\infty$ such that

$$B \geq \max_{\substack{p \in \mathcal{P} \\ i,r,s=0,1,\ldots,n \\ \tilde{\mathbf{x}} \in [T',T''] \times [-a,a]^n}} \left| \frac{\partial^2 \tilde{f}_{p,i}}{\partial \tilde{x}_r \partial \tilde{x}_s}(\tilde{\mathbf{x}}) \right|.$$

We must, at least in principle, be able to assign a numerical value to the constant B. This is in contrast to the rest of the constants and variables, where the mere knowledge of the existence of the appropriate values suffices. However, because B is an arbitrary upper bound (no assumptions are needed about its quality) on the second-order partial derivatives of the components of the functions $\mathbf{f}_p$ on the compact set $[T',T''] \times [-a,a]^n$, this should not cause any difficulties if the algebraic form of the components is known. It might sound strange that the mere existence of the appropriate values to be assigned to the other variables suffices in a constructive theorem. However, as we will prove in this chapter, if they exist then the simplex algorithm, for example, will successfully determine valid values for them.

With

$$x^*_{\min} := \min_{\|\mathbf{x}\|_\infty = a} \|\mathbf{x}\|$$

we set

$$\delta := \frac{\alpha(x^*_{\min})}{2}$$

and let m^* be a strictly positive integer, such that

$$[-\frac{a}{2^{m^*}}, \frac{a}{2^{m^*}}]^n \subset \{\mathbf{x} \in \mathbb{R}^n \big| \beta(\|\mathbf{x}\|) \leq \delta\} \cap \mathcal{D}' \tag{6.2}$$

and set

$$\mathcal{D} :=]-\frac{a}{2^{m^*}}, \frac{a}{2^{m^*}}[^n.$$

Note that we do not know the numerical values of the constants δ and m^* because α and β are unknown. However, their mere existence allows us to properly define δ and m^*. We will keep on introducing constants in this way. Their existence is secured in the sense that *there exists a constant with the following property.*

Set

$$\begin{aligned} x^* &:= 2^{-m^*} x^*_{\min}, \\ \omega^* &:= \frac{1}{2}\omega(x^*), \\ \varepsilon &:= \min\{\omega^*, \alpha(y_1)/y_1\}, \end{aligned}$$

where y_1 is the second smallest element in $\mathcal{X}^{\|\cdot\|}$, and

$$A^* := \sup_{\substack{p\in\mathcal{P} \\ \tilde{\mathbf{x}}\in[T',T'']\times[-a,a]^n}} \|\tilde{\mathbf{f}}_p(\tilde{\mathbf{x}})\|_2.$$

We define $\widetilde{W}(\tilde{\mathbf{x}}) := W(t, \mathbf{x})$, where $\tilde{\mathbf{x}} := (t, \mathbf{x})$, and assign

$$\begin{aligned} C &:= \max_{\substack{r=0,1,\dots,n \\ \tilde{\mathbf{x}}\in[T',T'']\times[-a,a]^n}} \left|\frac{\partial \widetilde{W}}{\partial \tilde{x}_r}(\tilde{\mathbf{x}})\right|, \\ B^* &:= (n+1)^{\frac{3}{2}} \cdot \max_{\substack{r,s=0,1,\dots,n \\ \tilde{\mathbf{x}}\in[T',T'']\times[-a,a]^n}} \left|\frac{\partial^2 \widetilde{W}}{\partial \tilde{x}_r \partial \tilde{x}_s}(\tilde{\mathbf{x}})\right|, \end{aligned}$$

and

$$C^* := (n+1)^3 CB.$$

We set

$$a^* := \max\{T'' - T', a\}$$

and let $m \geq m^*$ be an integer, such that

$$\frac{a^*}{2^m} \leq \frac{\sqrt{(A^*B^*)^2 + 4x^*\omega^*C^*} - A^*B^*}{2C^*},$$

and set

$$d := 2^{m-m^*}.$$

We define the piecewise scaling function $\mathbf{PS} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ through

$$\mathbf{PS}(j_1, j_2, \dots, j_n) := a2^{-m}(j_1, j_2, \dots, j_n)$$

for all $(j_1, j_2, \dots, j_n) \in \mathbb{Z}^n$ and the vector $\mathbf{t}$,

$$\mathbf{t} := (t_0, t_1, \dots, t_{2^m}),$$

where

$$t_j := T' + 2^{-m} j(T'' - T')$$

for all $j = 0, 1, \dots, 2^m$.

We assign the following values to the variables and the remaining constants of the linear programming problem $\mathbf{LP}(\{\mathbf{f}_p \big| p \in \mathcal{P}\}, \mathcal{N}, \mathbf{PS}, \mathbf{t}, \mathcal{D}, \|\cdot\|)$:

$$\begin{aligned}
&B^{(\mathbf{z},\mathcal{J})}_{p,rs} := B, \quad \text{for all } p \in \mathcal{P}, \text{ all } (\mathbf{z},\mathcal{J}) \in \mathcal{Z}, \text{ and all } r,s = 0,1,\dots,n,\\
&\Psi[y] := \alpha(y), \quad \text{for all } y \in \mathcal{X}^{\|\cdot\|},\\
&\Gamma[y] := \omega^* y, \quad \text{for all } y \in \mathcal{X}^{\|\cdot\|},\\
&V[\tilde{\mathbf{x}}] := \widetilde{W}(\tilde{\mathbf{x}}) \quad \text{for all } \tilde{\mathbf{x}} \in \mathcal{G},\\
&C[\{\tilde{\mathbf{x}},\tilde{\mathbf{y}}\}] := C, \quad \text{for all } \{\tilde{\mathbf{x}},\tilde{\mathbf{y}}\} \in \mathcal{Y},\\
&\Upsilon := \max\left\{C,\ a^{-1}2^{m^*} \cdot \max_{i=1,2,\dots,n} \beta(a2^{-m^*}\|\mathbf{e}_i\|)\right\}.
\end{aligned}$$

We now show that the linear constraints **LC1**, **LC2**, **LC3**, and **LC4** of the linear programming problem $\mathbf{LP}(\{\mathbf{f}_p \big| p \in \mathcal{P}\}, \mathcal{N}, \mathbf{PS}, \mathbf{t}, \mathcal{D}, \|\cdot\|)$ are satisfied by these values.

6.3 The constraints LC1 are fulfilled

Let $y_0, y_1, \dots, y_K$ be the elements of $\mathcal{X}^{\|\cdot\|}$ in an increasing order. We have to show that $\Psi[y_0] = \Gamma[y_0] = 0$, $\varepsilon y_1 \le \Psi[y_1]$, $\varepsilon y_1 \le \Gamma[y_1]$, and that for every $i = 1, 2, \dots, K-1$:

$$\frac{\Psi[y_i] - \Psi[y_{i-1}]}{y_i - y_{i-1}} \le \frac{\Psi[y_{i+1}] - \Psi[y_i]}{y_{i+1} - y_i}$$

and

$$\frac{\Gamma[y_i] - \Gamma[y_{i-1}]}{y_i - y_{i-1}} \le \frac{\Gamma[y_{i+1}] - \Gamma[y_i]}{y_{i+1} - y_i}.$$

Proof:
Clearly $\Psi[y_0] = \Gamma[y_0] = 0$ because $y_0 = 0$ and

$$\varepsilon y_1 \le \omega^* y_1 = \Gamma[y_1] \quad \text{and} \quad \varepsilon y_1 \le \frac{\alpha(y_1)}{y_1} y_1 = \Psi[y_1].$$

Because α is convex we have for all $i = 1, 2, \dots, K-1$ that

$$\frac{y_i - y_{i-1}}{y_{i+1} - y_{i-1}}\, \alpha(y_{i+1}) + \frac{y_{i+1} - y_i}{y_{i+1} - y_{i-1}}\, \alpha(y_{i-1}) \ge \alpha(y_i),$$

that is

$$\frac{\alpha(y_i) - \alpha(y_{i-1})}{y_i - y_{i-1}} = \frac{\Psi[y_i] - \Psi[y_{i-1}]}{y_i - y_{i-1}} \le \frac{\Psi[y_{i+1}] - \Psi[y_i]}{y_{i+1} - y_i} = \frac{\alpha(y_{i+1}) - \alpha(y_i)}{y_{i+1} - y_i}.$$

Finally, we clearly have for every $i = 1, 2, \dots, K-1$ that

$$\frac{\omega^*}{2} = \frac{\Gamma[y_i] - \Gamma[y_{i-1}]}{y_i - y_{i-1}} \le \frac{\Gamma[y_{i+1}] - \Gamma[y_i]}{y_{i+1} - y_i} = \frac{\omega^*}{2}.$$

■

6.4 The constraints LC2 are fulfilled

We have to show that for every $\tilde{\mathbf{x}} \in \mathcal{G}$ we have

$$\Psi[\|\tilde{\mathbf{x}}\|_*] \leq V[\tilde{\mathbf{x}}],$$

that for every $\tilde{\mathbf{x}} = (\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_n)$, such that $(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n) \in \mathbf{PS}(\mathbb{Z}^n) \cap \partial\mathcal{D}$ we have

$$V[\tilde{\mathbf{x}}] \leq \Psi[x_{\min,\partial\mathcal{M}}] - \delta,$$

and that for every $i = 1, 2, \ldots, n$ and every $j = 0, 1, \ldots, 2^m$ we have

$$V[\mathrm{PS}_0(j)\mathbf{e}_0 + \mathrm{PS}_i(d_i^-)\mathbf{e}_i] \leq -\Upsilon\mathrm{PS}(d_i^-) \quad \text{and} \quad V[\mathrm{PS}_0(j)\mathbf{e}_0 + \mathrm{PS}_i(d_i^-)\mathbf{e}_i] \leq \Upsilon\mathrm{PS}_i(d_i^+).$$

Proof:

Clearly,

$$\Psi[\|\tilde{\mathbf{x}}\|_*] = \alpha(\|\tilde{\mathbf{x}}\|_*) \leq \widetilde{W}(\tilde{\mathbf{x}}) = V[\tilde{\mathbf{x}}]$$

for all $\tilde{\mathbf{x}} \in \mathcal{G}$.

For every $\tilde{\mathbf{x}} = (\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_n)$, such that $(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n) \in \mathbf{PS}(\mathbb{Z}^n) \cap \partial\mathcal{D}$, we have by (6.2) that

$$V[\tilde{\mathbf{x}}] = \widetilde{W}(\tilde{\mathbf{x}}) \leq \beta(\|\tilde{\mathbf{x}}\|_*) \leq \delta = \alpha(x^*_{\min}) - \delta \leq \alpha(x_{\min,\partial\mathcal{M}}) - \delta = \Psi[x_{\min,\partial\mathcal{M}}] - \delta.$$

Finally, note that $d_i^+ = -d_i^- = d = 2^{m-m^*}$ for all $i = 1, 2, \ldots, n$, which implies that for every $i = 1, 2, \ldots, n$ and $j = 0, 1, \ldots, 2^m$ we have

$$\begin{aligned}
V[\mathrm{PS}_0(j)\mathbf{e}_0 + \mathrm{PS}(d_i^+)\mathbf{e}_i] &= V[\mathrm{PS}_0(j)\mathbf{e}_0 + a2^{-m^*}\mathbf{e}_i] \\
&= W(t_j, a2^{-m^*}\mathbf{e}_i) \\
&\leq \beta(a2^{-m^*}\|\mathbf{e}_i\|) \\
&\leq \Upsilon a2^{-m^*} \\
&= \Upsilon \cdot \mathrm{PS}_i(d_i^+)
\end{aligned}$$

and

$$\begin{aligned}
V[\mathrm{PS}_0(j)\mathbf{e}_0 + \mathrm{PS}(d_i^-)\mathbf{e}_i] &= V[\mathrm{PS}_0(j)\mathbf{e}_0 - a2^{-m^*}\mathbf{e}_i] \\
&= W(t_j, -a2^{-m^*}\mathbf{e}_i) \\
&\leq \beta(a2^{-m^*}\|\mathbf{e}_i\|) \\
&\leq \Upsilon a2^{-m^*} \\
&= -\Upsilon \cdot \mathrm{PS}(d_i^-).
\end{aligned}$$

■

6.5 The constraints LC3 are fulfilled

We must show that for every $\{\tilde{\mathbf{x}}, \tilde{\mathbf{y}}\} \in \mathcal{Y}$:

$$-C[\{\tilde{\mathbf{x}}, \tilde{\mathbf{y}}\}] \cdot \|\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\|_\infty \leq V[\tilde{\mathbf{x}}] - V[\tilde{\mathbf{y}}] \leq C[\{\tilde{\mathbf{x}}, \tilde{\mathbf{y}}\}] \cdot \|\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\|_\infty \leq \Upsilon \cdot \|\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\|_\infty.$$

PROOF:
Let $\{\tilde{\mathbf{x}}, \tilde{\mathbf{y}}\} \in \mathcal{Y}$. Then there is an $i \in \{0, 1, \ldots, n\}$ such that $\tilde{\mathbf{x}} - \tilde{\mathbf{y}} = \pm\mathbf{e}_i\|\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\|_\infty$. By the Mean-value theorem there is a $\vartheta \in]0, 1[$ such that

$$\left|\frac{V[\tilde{\mathbf{x}}] - V[\tilde{\mathbf{y}}]}{\|\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\|_\infty}\right| = \left|\frac{\widetilde{W}(\tilde{\mathbf{x}}) - \widetilde{W}(\tilde{\mathbf{y}})}{\|\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\|_\infty}\right| = \left|\frac{\partial\widetilde{W}}{\partial\tilde{x}_i}(\tilde{\mathbf{y}} + \vartheta(\tilde{\mathbf{x}} - \tilde{\mathbf{y}}))\right|.$$

Hence, by the definition of the constants C and Υ,

$$\left|\frac{V[\tilde{\mathbf{x}}] - V[\tilde{\mathbf{y}}]}{\|\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\|_\infty}\right| \le C \le \Upsilon,$$

which implies that the constraints **LC3** are fulfilled. ■

6.6 The constraints LC4 are fulfilled

Let $p \in \mathcal{P}$, $(\mathbf{z}, \mathcal{J}) \in \mathcal{Z}$, $\sigma \in \mathrm{Perm}[\{0, 1, \ldots, n\}]$, and $i \in \{0, 1, \ldots, n+1\}$ be arbitrary. We have to show that

$$-\Gamma\left[\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_*\right] \ge \sum_{j=0}^{n}\left(\frac{V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - V[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})}\tilde{f}_{p,\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) + E_{p,\sigma,i}^{(\mathbf{z},\mathcal{J})}C[\{\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}\}]\right). \tag{6.3}$$

PROOF:
With the values we have assigned to the variables and the constants of the linear programming problem we have for every $p \in \mathcal{P}$, every $(\mathbf{z}, \mathcal{J}) \in \mathcal{Z}$, every $\sigma \in \mathrm{Perm}[\{0, 1, \ldots, n\}]$, every $i, j = 0, 1, \ldots, n$, and with $h := a^*2^{-m}$, that

$$A_{\sigma,i,j}^{(\mathbf{z},\mathcal{J})} \le h,$$

$$E_{p,\sigma,i}^{(\mathbf{z},\mathcal{J})} := \frac{1}{2}\sum_{r,s=0}^{n} B_{p,rs}^{(\mathbf{z},\mathcal{J})}A_{\sigma,r,i}^{(\mathbf{z},\mathcal{J})}(A_{\sigma,s,i}^{(\mathbf{z},\mathcal{J})} + A_{\sigma,s,1}^{(\mathbf{z},\mathcal{J})}) \le (n+1)^2Bh^2,$$

and

$$\sum_{j=0}^{n} C[\{\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}\}] \le (n+1)C.$$

Hence, inequality (6.3) follows if we can prove that

$$-\Gamma\left[\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_*\right] = -\omega^*\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_* \ge \sum_{j=0}^{n}\frac{\widetilde{W}(\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}) - \widetilde{W}(\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})}\tilde{f}_{p,\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) + C^*h^2. \tag{6.4}$$

Now, by the Cauchy-Schwarz inequality and inequality (6.1),

$$\begin{aligned}
&\sum_{j=0}^{n}\frac{\widetilde{W}(\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}) - \widetilde{W}(\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})}\tilde{f}_{p,\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) \\
&\quad = \sum_{j=0}^{n}\left(\frac{\widetilde{W}(\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}) - \widetilde{W}(\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})} - \frac{\partial\widetilde{W}}{\partial\tilde{x}_{\sigma(j)}}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})\right)\tilde{f}_{p,\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) + \nabla_{\tilde{\mathbf{x}}}\widetilde{W}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) \cdot \tilde{\mathbf{f}}_p(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) \\
&\quad \le -\omega(\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_*) + \left\|\sum_{j=0}^{n}\left(\frac{\widetilde{W}(\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}) - \widetilde{W}(\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})} - \frac{\partial\widetilde{W}}{\partial\tilde{x}_{\sigma(j)}}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})\right)\mathbf{e}_j\right\|_2 \|\tilde{\mathbf{f}}_p(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})\|_2.
\end{aligned}$$

By the Mean-value theorem there is an $\mathbf{y}$ on the line-segment between the vectors $\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}$ and $\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}$, such that

$$\frac{\widetilde{W}(\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}) - \widetilde{W}(\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})} = \frac{\partial \widetilde{W}}{\partial \tilde{x}_{\sigma(j)}}(\mathbf{y})$$

and an $\mathbf{y}^*$ on the line-segment between the vectors $\mathbf{y}$ and $\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}$ such that

$$\frac{\partial \widetilde{W}}{\partial \tilde{x}_{\sigma(j)}}(\mathbf{y}) - \frac{\partial \widetilde{W}}{\partial \tilde{x}_{\sigma(j)}}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) = \left[\nabla_{\tilde{\mathbf{x}}} \frac{\partial \widetilde{W}}{\partial \tilde{x}_{\sigma(j)}}\right](\mathbf{y}^*) \cdot (\mathbf{y} - \mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}).$$

Because $\mathbf{y}$ and $\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}$ are both elements of the simplex $\mathbf{PS}(\mathbf{R}^{\mathcal{J}}(\mathbf{z}+S_\sigma))$, we have

$$\|\mathbf{y} - \mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_2 \leq h\sqrt{n+1}$$

and because

$$\left\| \left[\nabla_{\tilde{\mathbf{x}}} \frac{\partial \widetilde{W}}{\partial \tilde{x}_{\sigma(j)}}\right](\mathbf{y}^*) \right\|_2 \leq \sqrt{n+1} \cdot \max_{\substack{r,s=0,1,\ldots,n \\ \tilde{\mathbf{x}} \in [T',T''] \times [-a,a]^n}} \left| \frac{\partial^2 \widetilde{W}}{\partial \tilde{x}_r \partial \tilde{x}_s}(\tilde{\mathbf{x}}) \right|,$$

we obtain

$$\left\| \sum_{j=0}^{n} \left(\frac{\widetilde{W}(\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}) - \widetilde{W}(\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})} - \frac{\partial \widetilde{W}}{\partial \tilde{x}_{\sigma(j)}}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) \right) \mathbf{e}_j \right\|_2 \leq hB^*.$$

Finally, by the definition of A^*,

$$\|\tilde{\mathbf{f}}_p(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})\|_2 \leq A^*.$$

Putting the pieces together delivers the inequality

$$\sum_{j=0}^{n} \frac{\widetilde{W}(\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}) - \widetilde{W}(\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})} \tilde{f}_{p,\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) \leq hB^*A^* - \omega(\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_*).$$

From this inequality and because $\omega(x) \geq 2\omega^* x$ for all $x \geq x^*$ and because of the fact that $\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_* \geq x^*$, inequality (6.4) holds true if

$$-\omega^*\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_* \geq hA^*B^* - 2\omega^*\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_* + h^2C^*.$$

But, this last inequality follows from

$$h := \frac{a^*}{2^m} \leq \frac{\sqrt{(A^*B^*)^2 + 4x^*\omega^*C^*} - A^*B^*}{2C^*},$$

which implies

$$0 \geq hA^*B^* - \omega^*x^* + h^2C^* \geq hA^*B^* - \omega^*\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|_* + h^2C^*.$$

Because $p \in \mathcal{P}$, $(\mathbf{z}, \mathcal{J}) \in \mathcal{Z}$, $\sigma \in \mathrm{Perm}[\{0,1,\ldots,n\}]$, and $i \in \{0,1,\ldots,n+1\}$ were arbitrary, we have finished the proof.

■

In the last proof we took care of that the second order polynomial

$$P(z) := z^2C^* + zA^*B^* - \omega^*x^*$$

has two distinct real-valued roots, one smaller than zero and one larger then zero. Further, because $h := a^*2^{-m} > 0$ is not larger than the positive root, we have $P(h) \leq 0$, which is exactly what we need in the proof so that everything adds up.

6.7 Summary of the results

In this chapter we have, among other things, delivered a proof of the following theorem:

Theorem 6.1 (Constructive converse theorem for arbitrary switched systems) *Consider the Switched System 1.9 where $\mathcal{P}$ is a finite set, let $a > 0$ be a real-valued constant such that $[-a,a]^n \subset \mathcal{U}$, and assume that at least one of the following two assumptions holds true:*

i) There exists a Lyapunov function $W \in \mathcal{C}^2(\mathbb{R}_{\geq 0} \times [-a,a]^n)$ for the Switched System 1.9.

ii) The functions $\mathbf{f}_p$ are all Lipschitz on the set $\mathbb{R}_{\geq 0} \times [-a,a]^n$, the origin is a uniformly asymptotically stable equilibrium point of the Switched System 1.9, and the set $[-a,a]^n$ is contained in its region of attraction.

Then, for every constants $0 \leq T' < T'' < +\infty$ and every neighborhood $\mathcal{N} \subset [-a,a]^n$ of the origin, no matter how small, it is possible to parameterize a Lyapunov function V^{Lya} of class CPWA,

$$V^{Lya} : [T', T''] \times \big([-a,a]^n \setminus \mathcal{N}\big) \longrightarrow \mathbb{R},$$

for the Switched System 1.9 by using the linear programming problem defined in Definition 5.1.

More concrete: Let m be a positive integer and define the piecewise scaling function $\mathbf{PS} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, the set $\mathcal{D}$, and the vector $\mathbf{t}$ of the linear programming problem through

$$\mathbf{PS}(j_1, j_2, \ldots, j_n) := a2^{-m}(j_1, j_2, \ldots, j_n),$$

$$\mathcal{D} :=]-2^k \frac{a}{2^m}, 2^k \frac{a}{2^m}[^n \subset \mathcal{N},$$

for some integer $1 \leq k < m$, and

$$\mathbf{t} := (t_0, t_1, \ldots, t_M), \quad \textit{where} \;\; t_i := T' + j2^{-m}(T'' - T') \;\; \textit{for all } j = 0, 1, \ldots, 2^m.$$

Then, the linear programming problem $\mathbf{LP}(\{\mathbf{f}_p \big| p \in \mathcal{P}\}, [-a,a]^n, \mathbf{PS}, \mathbf{t}, \mathcal{D}, \|\cdot\|)$ in Definition 5.1 possesses a feasible solution, whenever m is large enough.

Proof:
Note that by Theorem 3.10 assumption *ii)* implies assumption *i)*. But then, by the arguments already delivered in this section, the propositions of the theorem follow.

■

Note that we have, in this chapter, actually proved substantially more than stated in Theorem 6.1. Namely, we did derive formulae for the values of the parameters that are needed to initialize the linear programming problem in Definition 5.1. These formulae do depend on the unknown Lyapunov function W, so we cannot extract the numerical values. However, these formulae are concrete enough to derive the promised algorithm to generate a CPWA Lyapunov function. This will be done in the next chapter.

6.8 The autonomous case

Theorem 6.2 (Constructive converse theorem for autonomous switched systems) *Consider the Switched System 1.9 where $\mathcal{P}$ is a finite set and assume that it is autonomous. Let $a > 0$ be a real-valued constant such that $[-a, a]^n \subset \mathcal{U}$, and assume that at least one of the following two assumptions holds true:*

i) There exists a Lyapunov function $W \in \mathcal{C}^2([-a, a]^n)$ for the Switched System 1.9.

ii) The functions $\mathbf{f}_p$ are all locally Lipschitz, the origin is an asymptotically stable equilibrium point of the Switched System 1.9, and the set $[-a, a]^n$ is contained in its region of attraction.

Then, for every neighborhood $\mathcal{N} \subset [-a, a]^n$ of the origin, no matter how small, it is possible to parameterize a time-invariant Lyapunov function V^{Lya} of class CPWA*,*

$$V^{Lya} : [-a, a]^n \setminus \mathcal{N} \longrightarrow \mathbb{R},$$

for the Switched System 1.9 by using the linear programming problem from Definition 5.5.

More concrete: Let m be a positive integer and define the piecewise scaling function $\mathbf{PS} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and the set $\mathcal{D}$ of the linear programming problem through

$$\mathbf{PS}(j_1, j_2, \ldots, j_n) := a2^{-m}(j_1, j_2, \ldots, j_n)$$

and

$$\mathcal{D} :=]-2^k \frac{a}{2^m}, 2^k \frac{a}{2^m}[^n \subset \mathcal{N},$$

for some integer $1 \le k < m$.

Then, the linear programming problem $\mathbf{LP}(\{\mathbf{f}_p \big| p \in \mathcal{P}\}, [-a, a]^n, \mathbf{PS}, \mathcal{D}, \|\cdot\|)$ in Definition 5.5 possesses a feasible solution, whenever m is large enough.

The proof is essentially a slimmed down version of the proof in the last section, so we will not go very thoroughly into details.

PROOF:

First, note that by Theorem 3.10 assumption *ii)* implies assumption *i)*, so in both cases there are class $\mathcal{K}$ functions α, β, and ω, and a function $W \in \mathcal{C}^2([-a, a]^n) \longrightarrow \mathbb{R}$, such that

$$\alpha(\|\mathbf{x}\|) \le W(\mathbf{x}) \le \beta(\|\mathbf{x}\|)$$

and

$$\nabla W(\mathbf{x}) \cdot \mathbf{f}_p(\mathbf{x}) \le -\omega(\|\mathbf{x}\|)$$

for all $\mathbf{x} \in]-a, a[^n$ and all $p \in \mathcal{P}$. Further, by Lemma 2.13, we can assume without loss of generality that α and ω are convex functions. With

$$x^*_{\min} := \min_{\|\mathbf{x}\|_\infty = a} \|\mathbf{x}\|$$

we set

$$\delta := \frac{\alpha(x^*_{\min})}{2}$$

and let m^* be a strictly positive integer, such that

$$[-\frac{a}{2^{m^*}}, \frac{a}{2^{m^*}}]^n \subset \{\mathbf{x} \in \mathbb{R}^n \big| \beta(\|\mathbf{x}\|) \leq \delta\} \cap \mathcal{N}.$$

Set

$$\begin{aligned}
x^* &:= \min_{\|\mathbf{x}\|_\infty = a2^{-m^*}} \|\mathbf{x}\|, \\
\omega^* &:= \frac{1}{2}\omega(x^*), \\
\varepsilon &:= \min\{\omega^*, \alpha(y_1)/y_1\},
\end{aligned}$$

where y_1 is the second smallest element of $\mathcal{X}^{\|\cdot\|}$,

$$C := \max_{\substack{i=1,2,\ldots,n \\ \mathbf{x}\in[-a,a]^n}} \left| \frac{\partial W}{\partial x_i}(\mathbf{x}) \right|,$$

and determine a constant B such that

$$B \geq \max_{\substack{p\in\mathcal{P} \\ i,r,s=1,2,\ldots,n \\ \mathbf{x}\in[-a,a]^n}} \left| \frac{\partial^2 f_{p,i}}{\partial x_r \partial x_s}(\mathbf{x}) \right|.$$

Assign

$$\begin{aligned}
A^* &:= \sup_{\substack{p\in\mathcal{P} \\ \mathbf{x}\in[-a,a]^n}} \|\mathbf{f}_p(\mathbf{x})\|_2, \\
B^* &:= n^{\frac{3}{2}} \cdot \max_{\substack{r,s=1,2,\ldots,n \\ \mathbf{x}\in[-a,a]^n}} \left| \frac{\partial^2 W}{\partial x_r \partial x_s}(\mathbf{x}) \right|, \\
C^* &:= n^3 BC,
\end{aligned}$$

and let $m \geq m^*$ be an integer such that

$$\frac{a}{2^m} \leq \frac{\sqrt{(A^*B^*)^2 + 4x^*\omega^*C^*} - A^*B^*}{2C^*}$$

and set

$$d := 2^{m-m^*}.$$

With $\mathbf{y} := a2^{-m}(0, 1, \ldots, 2^m)$ we assign the following values to the variables and the remaining constants of the linear programming problem $\mathbf{LP}(\{\mathbf{f}_p \big| p \in \mathcal{P}\}, [-a,a]^n, \mathbf{PS}, \mathcal{D}, \|\cdot\|)$:

$$\begin{aligned}
&B_{rs}^{(\mathbf{z},\mathcal{J})} := B, \quad \text{for all } (\mathbf{z},\mathcal{J}) \in \mathcal{Z}_a \text{ and all } r,s = 1,2,\ldots,n, \\
&\Psi_a[x] := \alpha(x), \quad \text{for all } x \in \mathcal{X}^{\|\cdot\|}, \\
&\Gamma_a[x] := \omega^* x, \quad \text{for all } x \in \mathcal{X}^{\|\cdot\|}, \\
&V_a[\mathbf{x}] := W(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathcal{G}_a, \\
&C_a[\{\mathbf{x},\mathbf{y}\}] := C, \quad \text{for all } \{\mathbf{x},\mathbf{y}\} \in \mathcal{Y}_a.
\end{aligned}$$

Now, that the linear constraints **LC1a**, **LC2a**, and **LC3a** are all satisfied follows very similarly to how the linear constraints **LC1**, **LC2**, and **LC3** follow in the nonautonomous case, so we only show that the constraints **LC4a** are fulfilled.

Let $(\mathbf{z}, \mathcal{J}) \in \mathcal{Z}_a$, $\sigma \in \text{Perm}[\{1, 2, \ldots, n\}]$, and $i \in \{1, 2, \ldots, n+1\}$ be arbitrary, but fixed throughout the rest of the proof. We have to show that

$$-\Gamma_a[\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|] \geq \sum_{j=1}^{n} \frac{V_a[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - V_a[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})} f_{p,\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) + E_{p,\sigma,i}^{(\mathbf{z},\mathcal{J})} \sum_{j=1}^{n} C_a[\{\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}\}]. \tag{6.5}$$

With the values we have assigned to the variables and the constants of the linear programming problem, inequality (6.5) holds true if

$$-\omega^* \|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\| \geq \sum_{j=1}^{n} \frac{W[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - W[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})} f_{\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) + h^2 C^*$$

with $h := a2^{-m}$. Now, by the Mean-value theorem and because $\omega(x) \geq 2\omega^* x$ for all $x \geq x^*$,

$$\begin{aligned}
&\sum_{j=1}^{n} \frac{W[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - W[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})} f_{\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) + h^2 C^* \\
&= \sum_{j=1}^{n} \left(\frac{W[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - W[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})} - \frac{\partial W}{\partial \xi_{\sigma(j)}}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) \right) f_{\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) + \nabla W(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) \cdot \mathbf{f}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) + h^2 C^* \\
&\leq \left\| \sum_{j=1}^{n} \left(\frac{W[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - W[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})} - \frac{\partial W}{\partial \xi_{\sigma(j)}}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) \right) \mathbf{e}_j \right\|_2 \|f_{\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})})\|_2 - \omega(\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|) + h^2 C^* \\
&\leq B^* h A^* - 2\omega^* \|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\| + h^2 C^*.
\end{aligned}$$

Hence, if

$$-\omega^* \|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\| \geq hA^*B^* - 2\omega^* \|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\| + h^2 C^*,$$

inequality (6.5) follows. But, this last inequality follows from

$$h := \frac{a}{2^m} \leq \frac{\sqrt{(A^*B^*)^2 + 4x^*\omega^* C^*} - A^*B^*}{2C^*},$$

which implies

$$0 \geq hA^*B^* - \omega^* x^* + h^2 C^* \geq hA^*B^* - \omega^* \|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\| + h^2 C^*.$$

■

In the next chapter we will use Theorem 6.1 to derive an algorithm to parameterize a CPWA Lyapunov function for the Switched System 1.9 and, if the Switched System 1.9 is autonomous, we will use Theorem 6.2 to derive an algorithm to parameterize a time-invariant CPWA Lyapunov function for the system.

Chapter 7

Algorithms to construct Lyapunov functions

In this chapter we use the results from Theorem 6.1 and Theorem 6.2 to prove that the systematic scan of the parameters of the linear programming problem from Definition 5.1 in Procedure 7.1 is an algorithm to construct Lyapunov functions for the Switched System 1.9, whenever one exists, and that Procedure 7.3 is an algorithm to construct time-invariant Lyapunov functions for the Switched System 1.9 if it is autonomous, again, whenever one exists. However, first we will give a short discussion on *algorithms*, because we intend to prove that our procedure to generate Lyapunov functions is concordant with the concept of an algorithm, whenever the system in question possesses a Lyapunov function.

7.1 What is an algorithm?

Donald Knuth writes in his classic work *The Art of Computer Programming* on algorithms [25]:

> The modern meaning for algorithm is quite similar to that of *recipe*, *process*, *method*, *technique*, *procedure*, *routine*, *rigmarole*, except that the word "algorithm" connotes something just a little different. Besides merely being a finite set of rules that gives a sequence of operations for solving a specific type of problem, an algorithm has five important features:
>
> 1) *Finiteness.* An algorithm must always terminate in a finite number of steps. [...]
>
> 2) *Definiteness.* Each step of an algorithm must be precisely defined; the actions to be carried out must be be rigorously and unambiguously specified for each case. [...]
>
> 3) *Input.* An algorithm has zero or more *inputs*: quantities that are given to it initially before the algorithm begins, or dynamically as the algorithm runs. These inputs are taken from specified sets of objects. [...]
>
> 4) *Output.* An algorithm has one or more *outputs*: quantities that have a specified relation to the inputs. [...]
>
> 5) *Effectiveness.* An algorithm is also generally expected to be *effective*, in the sense that its operations must all be sufficiently basic that they can in principle be done exactly and in a finite length of time by someone using pencil and paper.

The construction scheme for a Lyapunov function we are going to derive here does comply to all of these features whenever the equilibrium at the origin is uniformly asymptotically stable, and is therefore an algorithm to construct Lyapunov functions for arbitrary switched systems possessing a uniformly asymptotically stable equilibrium.

7.2 The algorithm in the nonautonomous case

We begin by defining a procedure to construct Lyapunov functions and then we prove that it is an algorithm to construct Lyapunov functions for arbitrary switched systems possessing a uniformly asymptotically stable equilibrium.

Procedure 7.1 *Consider the Switched System 1.9 where $\mathcal{P}$ is a finite set, let $a > 0$ be a constant such that $[-a,a]^n \subset \mathcal{U}$, and let $\mathcal{N} \subset \mathcal{U}$ be an arbitrary neighborhood of the origin. Further, let T' and T'' be arbitrary real-valued constants such that $0 \leq T' < T''$ and let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$.*

First, we have to determine a constant B such that

$$B \geq \max_{\substack{p\in\mathcal{P} \\ i,r,s=0,1,\dots,n \\ \tilde{\mathbf{x}}\in[T',T'']\times[-a,a]^n}} \left| \frac{\partial^2 \tilde{f}_{p,i}}{\partial \tilde{x}_r \partial \tilde{x}_s}(\tilde{\mathbf{x}}) \right|.$$

The procedure has two integer variables that have to be initialized, namely m and N. They should be initialized as follows: Set $N := 0$ and assign the smallest possible positive integer to m such that

$$]-a2^{-m}, a2^{-m}[^n \subset \mathcal{N}.$$

The procedure consists of the following steps:

i) Define the piecewise scaling function $\mathbf{PS} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and the vector $\mathbf{t} := (t_0, t_1, \dots, t_{2^m})$, through

$$\mathbf{PS}(j_1, j_2, \dots, j_n) := a2^{-m}(j_1, j_2, \dots, j_n), \quad \textit{for all } (j_1, j_2, \dots, j_n) \in \mathbb{Z}^n$$

and

$$t_i := T' + i\frac{T''-T'}{2^m}, \quad \textit{for } i = 0, 1, \dots, 2^m.$$

ii) For every $N^* = 0, 1, \dots, N$ we do the following: Generate the linear programming problem

$$\mathbf{LP}(\{\mathbf{f}_p | p \in \mathcal{P}\}, [-a,a]^n, \mathbf{PS}, \mathbf{t},]-a2^{N^*-m}, a2^{N^*-m}[, \|\cdot\|)$$

as defined in Definition 5.1 and check whether it possesses a feasible solution or not. If one of the linear programming problems possesses a feasible solution, then go to step iii). If none of them possesses a feasible solution, then assign $m := m+1$ and $N := N+1$ and go back to step i).

iii) Use the feasible solution to parameterize a CPWA *Lyapunov function for the Switched System 1.9 as described in Section 5.2.*

□

After all the preparation we have done, the proof that Procedure 7.1 is an algorithm to construct Lyapunov functions for arbitrary switched systems possessing a uniformly asymptotically stable equilibrium is remarkably short.

Theorem 7.2 (Procedure 7.1 is an algorithm) *Consider the Switched System 1.9 where $\mathcal{P}$ is a finite set, let $a > 0$ be a constant such that $[-a, a]^n \subset \mathcal{U}$, and assume further, that at least one of the following two assumptions holds true:*

i) There exists a Lyapunov function $W \in \mathcal{C}^2(\mathbb{R}_{\geq 0} \times [-a, a]^n)$ for the Switched System 1.9.

ii) The functions $\mathbf{f}_p$ are all Lipschitz on the set $\mathbb{R}_{\geq 0} \times [-a, a]^n$, the origin is a uniformly asymptotically stable equilibrium point of the Switched System 1.9, and the set $[-a, a]^n$ is contained in its region of attraction.

Then, for every constants $0 \leq T' < T'' < +\infty$ and every neighborhood $\mathcal{N} \subset [-a, a]^n$ of the origin, no matter how small, the Procedure 7.1 delivers, in a finite number of steps, a CPWA *Lyapunov function V^{Lya},*

$$V^{Lya} : [T', T''] \times \big([-a, a]^n \setminus \mathcal{N}\big) \longrightarrow \mathbb{R},$$

for the Switched System 1.9.

Proof:
Follows directly from what we have shown in the last chapter. With the same notations as there, the linear programming problem

$$\mathbf{LP}(\{\mathbf{f}_p | p \in \mathcal{P}\}, [-a, a]^n, \mathbf{PS}, \mathbf{t},]-a2^{N^*-m}, a2^{N^*-m}[, \|\cdot\|)$$

possesses a feasible solution, when m is so large that

$$\frac{\max\{T'' - T', a\}}{2^m} \leq \frac{\sqrt{(A^*B^*)^2 + 4x^*\omega^*C^*} - A^*B^*}{2C^*}$$

and N^*, $0 \leq N^* \leq N$, is such that

$$[-\frac{a2^{N^*}}{2^m}, \frac{a2^{N^*}}{2^m}]^n \subset \{\mathbf{x} \in \mathbb{R}^n \big| \beta(\|\mathbf{x}\|) \leq \delta\} \cap \mathcal{N}.$$

■

Because we have already proved in Theorem 2.16 and Theorem 3.10 that the Switched System 1.9 possesses a Lyapunov function, if and only if an equilibrium of the system is uniformly asymptotically stable, this is equivalent to the statement:

> It is always possible, in a finite number of steps, to construct a Lyapunov function for the Switched System 1.9 with the methods presented in this thesis, whenever one exists.

7.3 The algorithm in the autonomous case

The procedure to construct Lyapunov functions for autonomous systems mimics Procedure 5.1.

Procedure 7.3 *Consider the Switched System 1.9 where $\mathcal{P}$ is a finite set, let $a > 0$ be a constant such that $[-a,a]^n \subset \mathcal{U}$, and let $\mathcal{N} \subset \mathcal{U}$ be an arbitrary neighborhood of the origin. Further, assume that the system is autonomous and let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$.*

First, we have to determine a constant B such that

$$B \geq \max_{\substack{p\in\mathcal{P}\\ i,r,s=1,2,\ldots,n\\ \mathbf{x}\in[-a,a]^n}} \left| \frac{\partial^2 f_{p,i}}{\partial x_r \partial x_s}(\mathbf{x}) \right|.$$

The procedure has two integer variables that have to be initialized, namely m and N. They should be initialized as follows: Set $N := 0$ and assign the smallest possible positive integer to m such that

$$]-a2^{-m}, a2^{-m}[^n \subset \mathcal{N}.$$

The procedure consists of the following steps:

i) Define the piecewise scaling function $\mathbf{PS} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ through

$$\mathbf{PS}(j_1, j_2, \ldots, j_n) := a2^{-m}(j_1, j_2, \ldots, j_n), \quad \text{for all } (j_1, j_2, \ldots, j_n) \in \mathbb{Z}^n.$$

ii) For every $N^* = 0, 1, \ldots, N$ we do the following:
Generate the linear programming problem

$$\mathbf{LP}(\{\mathbf{f}_p | p \in \mathcal{P}\}, [-a,a]^n, \mathbf{PS},]-a2^{N^*-m}, a2^{N^*-m}[, \|\cdot\|)$$

as defined in Definition 5.5 and check whether it possesses a feasible solution or not. If one of the linear programming problems possesses a feasible solution, then go to step iii). If none of them possesses a feasible solution, then assign $m := m+1$ and $N := N+1$ and go back to step i).

iii) Use the feasible solution to parameterize a CPWA *Lyapunov function for the Switched System 1.9 as described in Definition 5.6.*

□

The proof that Procedure 7.3 is an algorithm to construct time-invariant Lyapunov functions for arbitrary switched systems possessing an asymptotically stable equilibrium is essentially identical to the proof of Theorem 7.2, where the nonautonomous case is treated.

Theorem 7.4 (Procedure 7.3 is an algorithm) *Consider the Switched System 1.9 where $\mathcal{P}$ is a finite set and assume that it is autonomous. Let $a > 0$ be a constant such that $[-a,a]^n \subset \mathcal{U}$, and assume further, that at least one of the following two assumptions holds true:*

i) There exists a time-invariant Lyapunov function $W \in \mathcal{C}^2([-a,a]^n)$ for the Switched System 1.9.

ii) The functions $\mathbf{f}_p$ are all locally Lipschitz, the origin is a asymptotically stable equilibrium point of the Switched System 1.9, and the set $[-a,a]^n$ is contained in its region of attraction.

Then, for every neighborhood $\mathcal{N} \subset [-a,a]^n$ of the origin, no matter how small, the Procedure 7.3 delivers, in a finite number of steps, a time-invariant Lyapunov function V^{Lya} of class CPWA*,*

$$V^{Lya} : [-a,a]^n \setminus \mathcal{N} \longrightarrow \mathbb{R},$$

for the autonomous Switched System 1.9.

PROOF:
Almost identical to the proof of Theorem 7.2. With the same notations as in the last chapter, the linear programming problem

$$\mathbf{LP}(\{\mathbf{f}_p \big| p \in \mathcal{P}\}, [-a,a]^n, \mathbf{PS},]-a2^{N^*-m}, a2^{N^*-m}[, \|\cdot\|)$$

possesses a feasible solution, when m is so large that

$$\frac{a}{2^m} \leq \frac{\sqrt{(A^*B^*)^2 + 4x^*\omega^*C^*} - A^*B^*}{2C^*}$$

and N^*, $0 \leq N^* \leq N$, is such that

$$[-\frac{a2^{N^*}}{2^m}, \frac{a2^{N^*}}{2^m}]^n \subset \{\mathbf{x} \in \mathbb{R}^n \big| \beta(\|\mathbf{x}\|) \leq \delta\} \cap \mathcal{N}.$$

■

Because we have already proved in Theorem 2.16 and Theorem 3.10 that the autonomous Switched System 1.9 possesses a time-invariant Lyapunov function, if and only if an equilibrium of the system is asymptotically stable, this is equivalent to the statement:

> It is always possible, in a finite number of steps, to construct a time-invariant Lyapunov function for the autonomous Switched System 1.9 with the methods presented in this thesis, whenever one exists.

Part III

Examples of Lyapunov functions generated by linear programming

In this part we will give some examples of the construction of Lyapunov functions by the linear programming problem $\mathbf{LP}(\{\mathbf{f}_p \big| p \in \mathcal{P}\}, \mathcal{N}, \mathbf{PS}, \mathbf{t}, \mathcal{D}, \|\cdot\|)$ from Definition 5.1 and the linear programming problem $\mathbf{LP}(\{\mathbf{f}_p \big| p \in \mathcal{P}\}, \mathcal{N}, \mathbf{PS}, \mathcal{D}, \|\cdot\|)$ from Definition 5.5. In all the examples we will use the infinity norm, that is $\|\cdot\| := \|\cdot\|_\infty$, in the linear programming problems. Further, we will use piecewise scaling functions $\mathbf{PS}$, whose components PS_i are all odd functions, that is (recall that the i-th component PS_i of $\mathbf{PS}$ does only depend on the i-th variable x_i of the argument $\mathbf{x}$)

$$\mathrm{PS}_i(x_i) = -\mathrm{PS}_i(-x_i).$$

Because we will only be interested in the values of a piecewise scaling functions on compact subsets $[-m, m]^n \subset \mathbb{R}^n$, $m \in \mathbb{N}_{>0}$, this implies that we can define such a function by specifying n vectors $\mathbf{ps}_i := (\mathrm{ps}_{i,1}, \mathrm{ps}_{i,2}, \ldots, \mathrm{ps}_{i,m})$, $i = 1, 2, \ldots, n$. If we say that the piecewise scaling function $\mathbf{PS}$ is defined through the ordered vector tuple $(\mathbf{ps}_1, \mathbf{ps}_2, \ldots, \mathbf{ps}_n)$, we mean that $\mathbf{PS}(\mathbf{0}) := \mathbf{0}$ and that for every $i = 1, 2, \ldots, n$ and every $j = 1, 2, \ldots, m$, we have

$$\mathrm{PS_i}(j) := \mathrm{ps}_{i,j} \quad \text{and} \quad \mathrm{PS_i}(-j) := -\mathrm{ps}_{i,j}.$$

If we say that the piecewise scaling function $\mathbf{PS}$ is defined through the vector $\mathbf{ps}$, we mean that it is defined trough the vector tuple $(\mathbf{ps}_1, \mathbf{ps}_2, \ldots, \mathbf{ps}_n)$, where $\mathbf{ps}_i := \mathbf{ps}$ for all $i = 1, 2, \ldots, n$.

The linear programming problems were all solved by use of the GNU Linear programming kit (GLPK), version 4.8, developed by Andrew Makhorin. It is a free software that is available for download on the internet. The parameterized Lyapunov functions were drawn with gnuplot, version 3.7, developed by Thomas Williams and Colin Kelley. Just as GLPK, gnuplot is a free software that is available for download on the internet. The author is indebted to these developers.

Chapter 8

An autonomous system

As a fist example of the use of the linear programming problem from Definition 5.5 and Procedure 7.3 we consider the continuous system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \text{where} \quad \mathbf{f}(x, y) := \begin{pmatrix} x^3(y-1) \\ -\frac{x^4}{(1+x^2)^2} - \frac{y}{1+y^2} \end{pmatrix}. \tag{8.1}$$

This system is taken from Example 65 in Section 5.3 in [56]. The Jacobian of $\mathbf{f}$ at the origin has the eigenvalues 0 and -1. Hence, the origin is not an exponentially stable equilibrium point (see, for example, Theorem 4.4 in [24] or Theorem 15 in Section 5.5 in [56]). We initialize Procedure 7.3 with

$$a := \frac{8}{15} \quad \text{and} \quad \mathcal{N} := \left]-\frac{2}{15}, \frac{2}{15}\right[^2.$$

Further, with

$$x_{(\mathbf{z},\mathcal{J})} := \left|\mathbf{e}_1 \cdot \mathbf{PS}(\mathbf{R}^{\mathcal{J}}(\mathbf{z}+\mathbf{e}_1))\right| \quad \text{and} \quad y_{(\mathbf{z},\mathcal{J})} := \left|\mathbf{e}_2 \cdot \mathbf{PS}(\mathbf{R}^{\mathcal{J}}(\mathbf{z}+\mathbf{e}_2))\right|, \tag{8.2}$$

we set (note that for the constants $B_{p,rs}^{(\mathbf{z},\mathcal{J})}$ the index p is redundant because the system is non-switched)

$$\begin{aligned} B_{11}^{(\mathbf{z},\mathcal{J})} &:= 6x_{(\mathbf{z},\mathcal{J})}(1+y_{(\mathbf{z},\mathcal{J})}), \\ B_{12}^{(\mathbf{z},\mathcal{J})} &:= 3x_{(\mathbf{z},\mathcal{J})}^2, \\ B_{22}^{(\mathbf{z},\mathcal{J})} &:= \begin{cases} \frac{6y_{(\mathbf{z},\mathcal{J})}}{(1+y_{(\mathbf{z},\mathcal{J})}^2)^2} - \frac{8y_{(\mathbf{z},\mathcal{J})}^3}{(1+y_{(\mathbf{z},\mathcal{J})}^2)^3}, & \text{if } y_{(\mathbf{z},\mathcal{J})} \le \sqrt{2}-1, \\ 1.46, & \text{else,} \end{cases} \end{aligned}$$

for all $(\mathbf{z},\mathcal{J}) \in \mathcal{Z}$ in the linear programming problems. This is more effective than using one constant B larger than all $B_{p,rs}^{(\mathbf{z},\mathcal{J})}$ for all $(\mathbf{z},\mathcal{J}) \in \mathcal{Z}$ and all $r, s = 1, 2, \ldots, n$, as done to shorten the proof of Theorem 6.1.

Procedure 7.1 succeeds in finding a feasible solution to the linear programming problem with $m = 4$ and $D = 2$. The corresponding Lyapunov function of class CPWA is drawn in Figure 8.1. We used this Lyapunov function as a starting point to parameterize a CPWA Lyapunov function with a larger domain and succeeded with $\mathcal{N} := [-1,1]^2$, $\mathcal{D} :=]-0.133, 0.133[^2$, and $\mathbf{PS}$ defined through the vector

$$\mathbf{ps} := (0.033, 0.067, 0.1, 0.133, 0.18, 0.25, 0.3, 0.38, 0.45, 0.55, 0.7, 0.85, 0.93, 1)$$

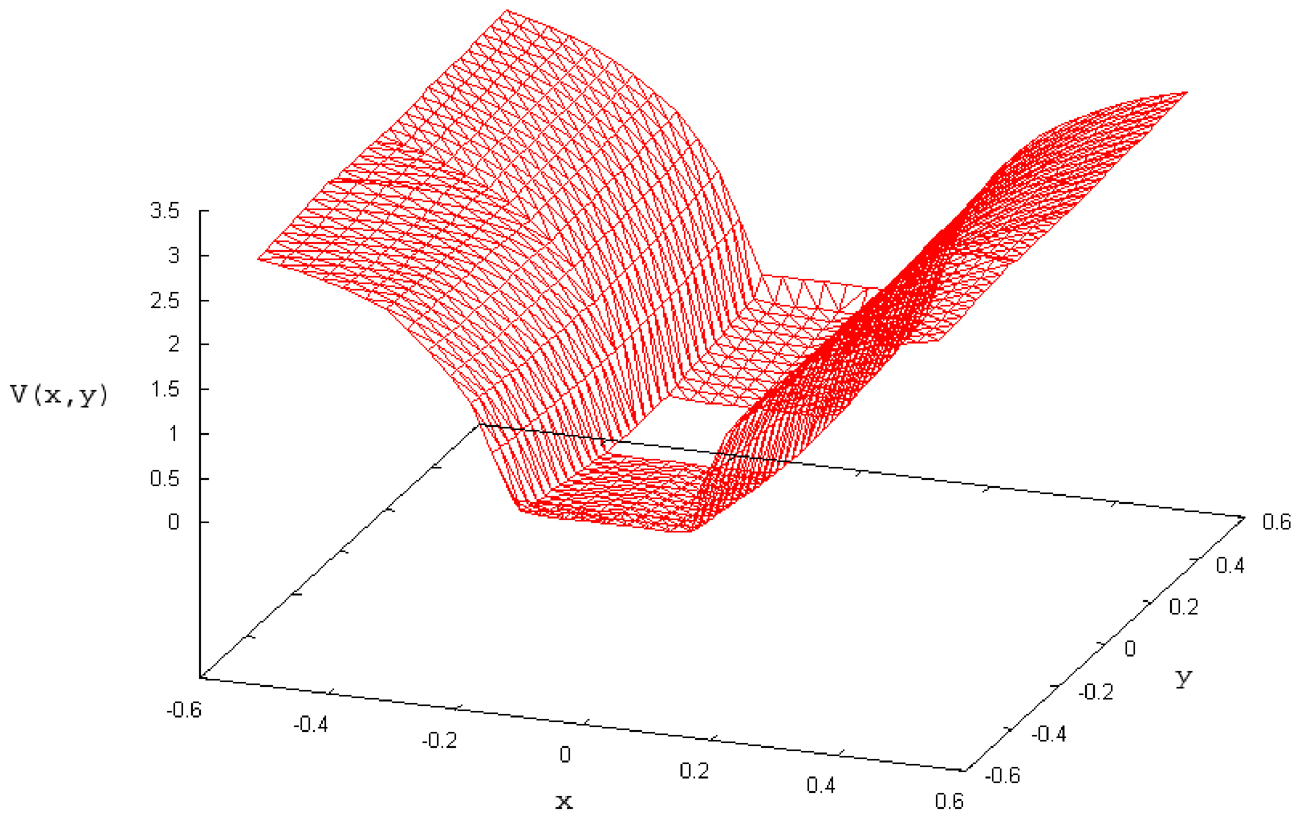


Figure 8.1: A Lyapunov function for the system (8.1) generated by Procedure 7.1.

as described at the beginning of this part. It is drawn in Figure 8.2.

Note that the domain of the Lyapunov function on Figure 8.1, where we used the Procedure 7.3 to scan the parameters of the linear programming problem from Definition 5.5, is much smaller than that of the Lyapunov function on Figure 8.2, where we used another trial-and-error procedure to scan the parameters. This is typical! The power of Procedure 7.3 and Theorem 7.4 is that they tell us that a systematic scan will lead to a success if there exists a Lyapunov function for the system. However, as Procedure 7.3 will not try to increase the distance between the points in the grid $\mathcal{G}$ of the linear programming problem far away from the equilibrium, it is not particularly well suited to parameterize Lyapunov functions with large domains. To actually parameterize Lyapunov functions a trial-and-error procedure that first tries to parameterize a Lyapunov function in a small neighborhood of the equilibrium, and if it succeeds it tries to extend the grid with larger grid-steps farther away from the equilibrium, is more suited.

In Figure 8.3 the sets $\mathcal{D}$, $\mathcal{T}$, and $\mathcal{A}$ from Lemma 5.9 are drawn for this particular Lyapunov function. The innermost square is the boundary of $\mathcal{D}$, the outmost figure is the boundary of the set $\mathcal{A}$, and in between the boundary of $\mathcal{T}$ is plotted. Every solution to the system (8.1) with an initial value $\boldsymbol{\xi}$ in $\mathcal{A}$ will reach the square $[-0.133, 0.133]^2$ in a finite time t' and will stay in the set $\mathcal{T}$ for all $t \geq t'$.

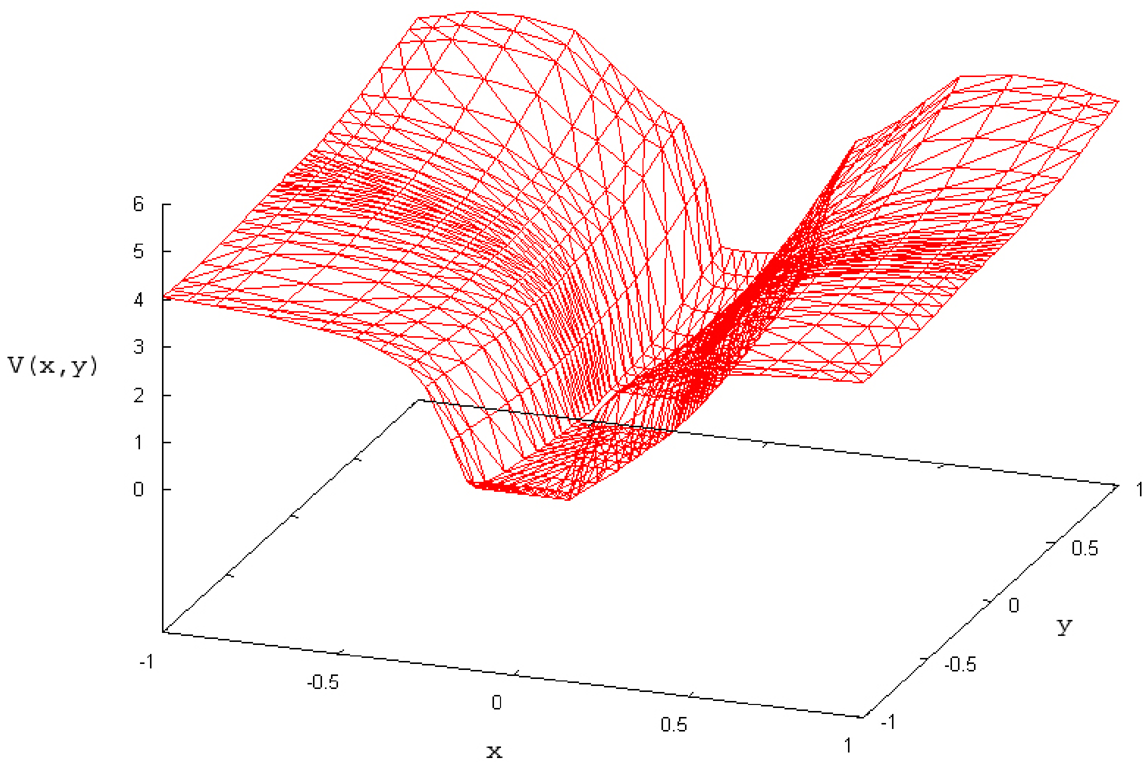


Figure 8.2: A Lyapunov function for the system (8.1), parameterized with the linear programming problem from Definition 5.5, with a larger domain than the Lyapunov function on Figure 8.1.

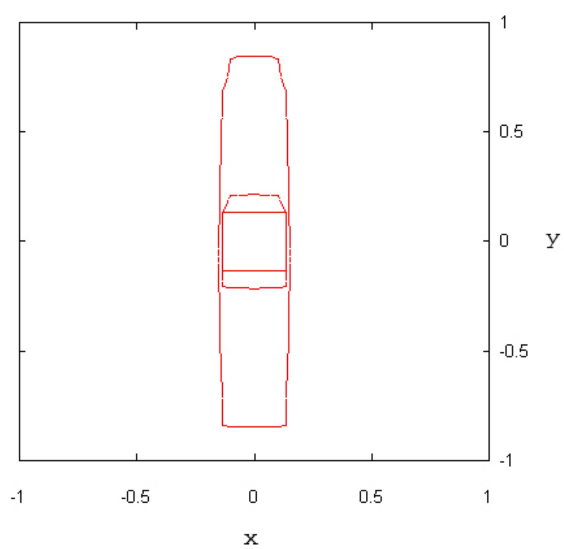


Figure 8.3: The sets $\mathcal{D}$, $\mathcal{T}$, and $\mathcal{A}$ from Lemma 5.4 for the Lyapunov function on Figure 8.2 for the system (8.1).

Chapter 9

An arbitrary switched autonomous system

Consider the autonomous systems

$$\dot{\mathbf{x}} = \mathbf{f}_1(\mathbf{x}), \quad \text{where} \quad \mathbf{f}_1(x,y) := \begin{pmatrix} -y \\ x - y(1 - x^2 + 0.1x^4) \end{pmatrix}, \tag{9.1}$$

$$\dot{\mathbf{x}} = \mathbf{f}_2(\mathbf{x}), \quad \text{where} \quad \mathbf{f}_2(x,y) := \begin{pmatrix} -y + x(x^2 + y^2 - 1) \\ x + y(x^2 + y^2 - 1) \end{pmatrix}, \tag{9.2}$$

and

$$\dot{\mathbf{x}} = \mathbf{f}_3(\mathbf{x}), \quad \text{where} \quad \mathbf{f}_3(x,y) := \begin{pmatrix} -1.5y \\ \frac{x}{1.5} + y\left(\left(\frac{x}{1.5}\right)^2 + y^2 - 1\right) \end{pmatrix}. \tag{9.3}$$

The systems (9.1) and (9.2) are taken from Exercise 1.16 in [24] and from page 194 in [50] respectively.

First, we used the linear programming problem from Definition 5.5 to parameterize a Lyapunov function for each of the systems (9.1), (9.2), and (9.3) individually. We define $x_{(\mathbf{z},\mathcal{J})}$ and $y_{(\mathbf{z},\mathcal{J})}$ as in formula (8.2) and for the system (9.1) we set

$$B_{1,11}^{(\mathbf{z},\mathcal{J})} := 2y_{(\mathbf{z},\mathcal{J})} + 1.2y_{(\mathbf{z},\mathcal{J})}x_{(\mathbf{z},\mathcal{J})}^2,$$
$$B_{1,12}^{(\mathbf{z},\mathcal{J})} := 2x_{(\mathbf{z},\mathcal{J})} + 0.4x_{(\mathbf{z},\mathcal{J})}^3,$$
$$B_{1,22}^{(\mathbf{z},\mathcal{J})} := 0,$$

for the system (9.2) we set

$$B_{2,11}^{(\mathbf{z},\mathcal{J})} := \max\{6x_{(\mathbf{z},\mathcal{J})}, 2y_{(\mathbf{z},\mathcal{J})}\},$$
$$B_{2,12}^{(\mathbf{z},\mathcal{J})} := \max\{2x_{(\mathbf{z},\mathcal{J})}, 2y_{(\mathbf{z},\mathcal{J})}\},$$
$$B_{2,22}^{(\mathbf{z},\mathcal{J})} := \max\{2x_{(\mathbf{z},\mathcal{J})}, 6y_{(\mathbf{z},\mathcal{J})}\},$$

and for the system (9.3) we set

$$B_{3,11}^{(\mathbf{z},\mathcal{J})} := \frac{8}{9}y_{(\mathbf{z},\mathcal{J})},$$
$$B_{3,12}^{(\mathbf{z},\mathcal{J})} := \frac{8}{9}x_{(\mathbf{z},\mathcal{J})},$$
$$B_{3,22}^{(\mathbf{z},\mathcal{J})} := 6y_{(\mathbf{z},\mathcal{J})}.$$

We parameterized a CPWA Lyapunov function for the system (9.1) by use of the linear programming problem from Definition 5.5 with $\mathcal{N} := [-1.337, 1.337]^2$, $\mathcal{D} := \emptyset$, and **PS** defined trough the vector

$$\mathbf{ps} := (0.0906, 0.316, 0.569, 0.695, 0.909, 1.016, 1.163, 1.236, 1.337)$$

as described at the beginning of this part. The Lyapunov function is depicted on Figure 9.1. The equilibrium's region of attraction, secured by this Lyapunov function, is drawn on Figure 9.2.

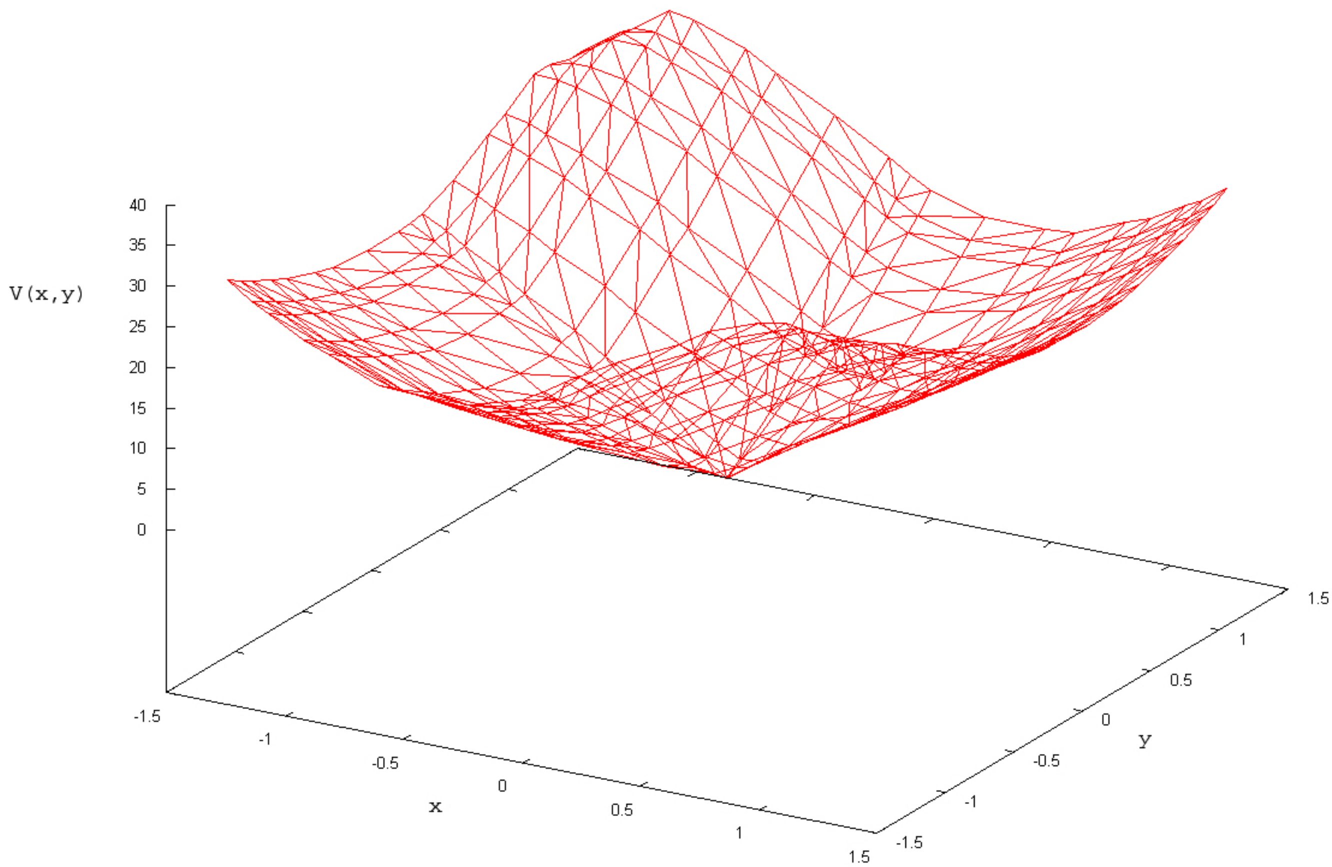


Figure 9.1: A Lyapunov function for the system (9.1) generated by the linear programming problem from Definition 5.5.

We parameterized a CPWA Lyapunov function for the system (9.2) by use of the linear programming problem from Definition 5.5 with $\mathcal{N} := [-0.818, 0.818]^2$, $\mathcal{D} := \emptyset$, and **PS** defined trough the vector

$$\mathbf{ps} := (0.188, 0.394, 0.497, 0.639, 0.8, 0.745, 0.794, 0.806, 0.818)$$

as described in the description at the beginning of this part. The Lyapunov function is depicted on Figure 9.3. The equilibrium's region of attraction, secured by this Lyapunov function, is drawn on Figure 9.4.

We parameterized a CPWA Lyapunov function for the system (9.3) by use of the linear programming problem from Definition 5.5 with $\mathcal{N} := [-0.506, 0.506]^2$, $\mathcal{D} :=]-0.01, 0.01[^2$, and **PS** defined trough the vector

$$\mathbf{ps} := (0.01, 0.0325, 0.0831, 0.197, 0.432, 0.461, 0.506)$$

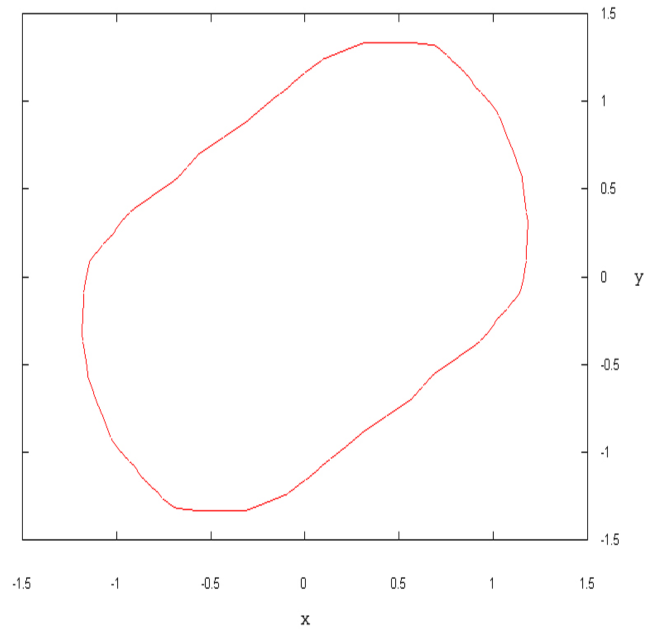


Figure 9.2: The region of attraction of the equilibrium at the origin secured by the Lyapunov function on Figure 9.1. All solution that start in this set are asymptotically attracted to the origin.

as described at the beginning of this part. The Lyapunov function is depicted on Figure 9.5. The equilibrium's region of attraction, secured by this Lyapunov function, and the set $\mathcal{D}$ are drawn on Figure 9.6. Every solution to the system (9.3) that starts in the larger set will reach the smaller set in a finite time.

Finally, we parameterized a CPWA Lyapunov function for the switched system

$$\dot{\mathbf{x}} = \mathbf{f}_p(\mathbf{x}), \quad p \in \{1, 2, 3\}, \tag{9.4}$$

where the functions $\mathbf{f}_1$, $\mathbf{f}_2$, and $\mathbf{f}_3$ are, of course, the functions from (9.1), (9.2), and (9.3), by use of the linear programming problem from Definition 5.1 with $\mathcal{N} := [-0.612, 0.612]^2$, $\mathcal{D} :=]-0.01, 0.01[^2$, and $\mathbf{PS}$ defined trough the vector

$$\mathbf{ps} := (0.01, 0.0325, 0.0831, 0.197, 0.354, 0.432, 0.535, 0.586, 0.612)$$

as described at the beginning of this part. The Lyapunov function is depicted on Figure 9.7. Note, that this Lyapunov function is a Lyapunov function for all of the systems (9.1), (9.2), and (9.3) individually. The equilibrium's region of attraction, secured by this Lyapunov function, and the set $\mathcal{D}$ are drawn on Figure 9.8. Every solution to the system (9.4) that starts in the larger set will reach the smaller set in a finite time.

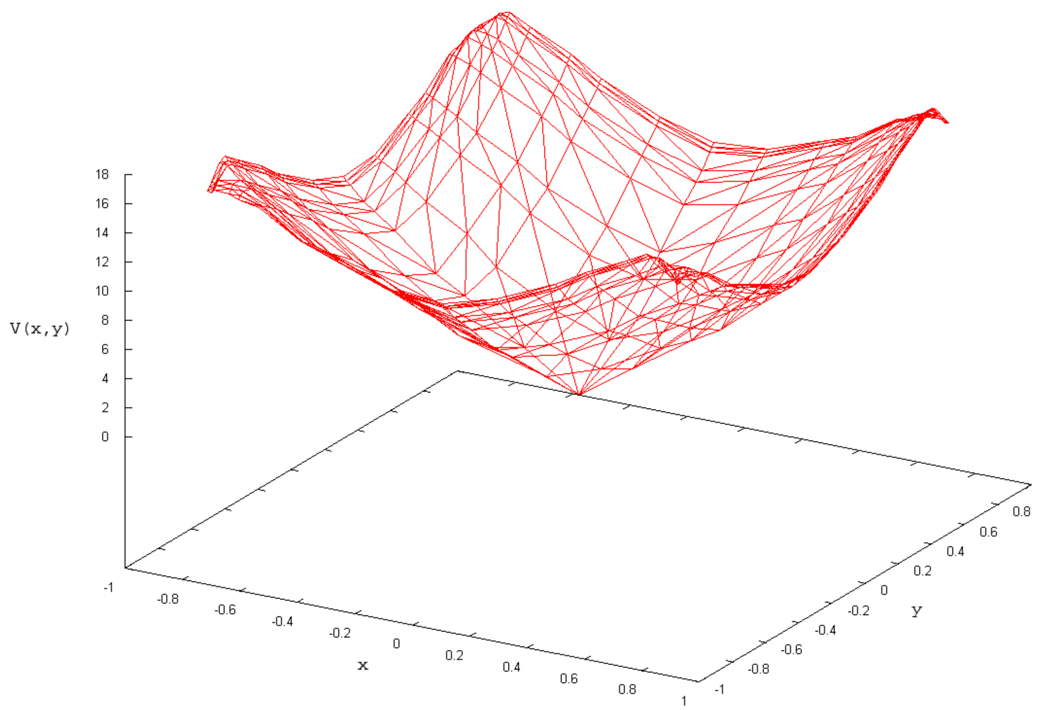


Figure 9.3: A Lyapunov function for the system (9.2) generated by the linear programming problem from Definition 5.5.

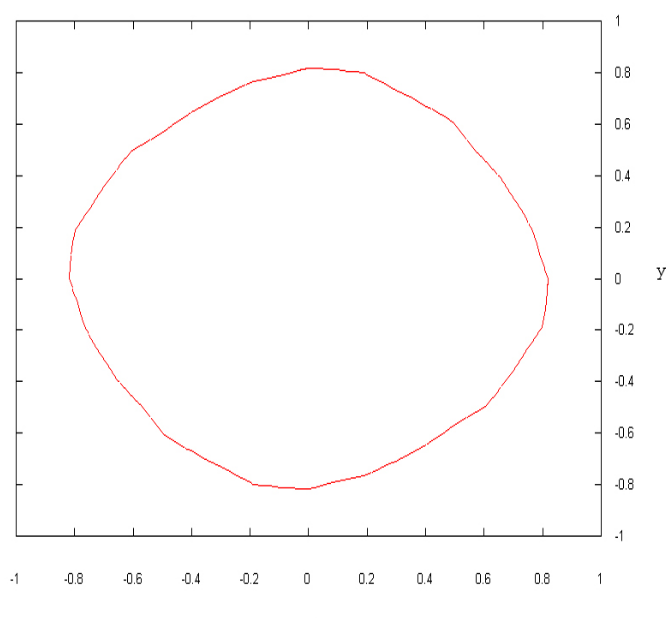


Figure 9.4: The region of attraction of the equilibrium at the origin secured by the Lyapunov function on Figure 9.3. All solution that start in this set are asymptotically attracted to the origin.

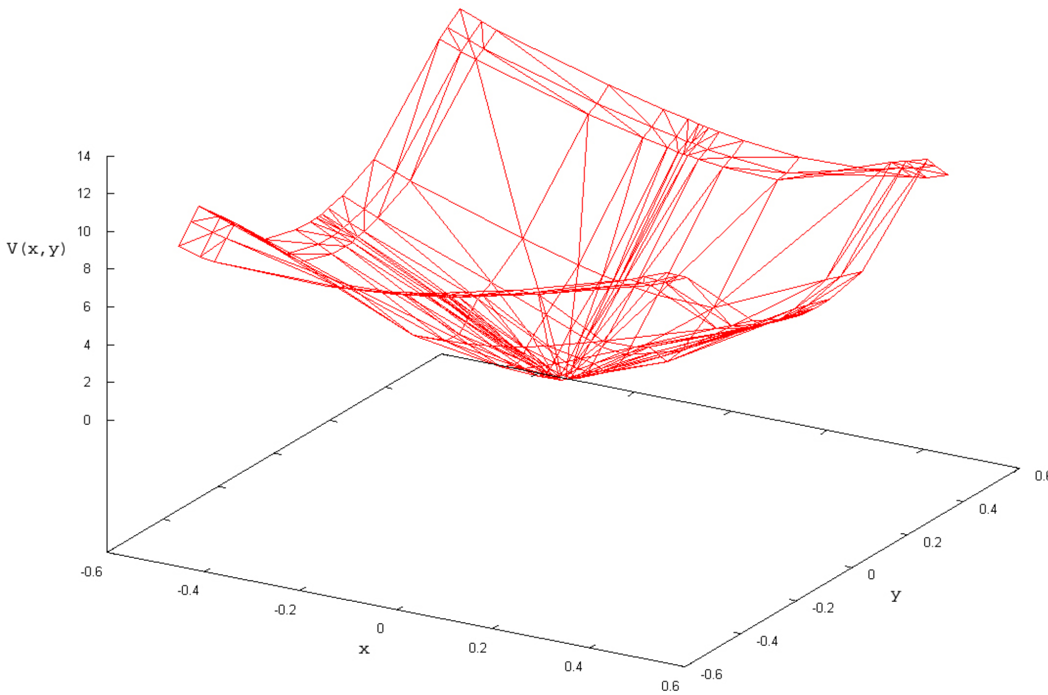


Figure 9.5: A Lyapunov function for the system (9.3) generated by the linear programming problem from Definition 5.5.

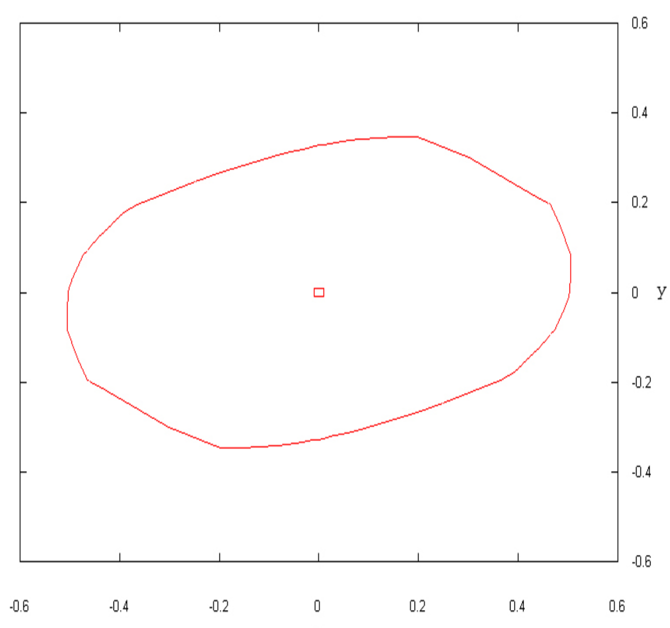


Figure 9.6: The region of attraction of the equilibrium at the origin secured by the Lyapunov function on Figure 9.5. All solution that start in the larger set are asymptotically attracted to the smaller set at the origin.

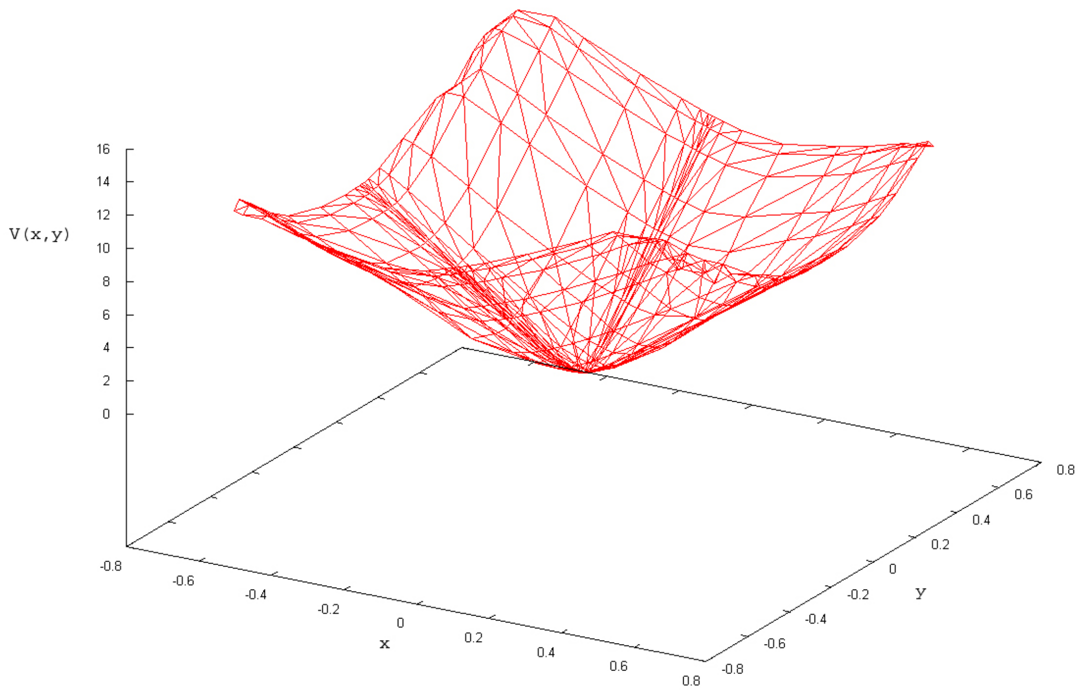


Figure 9.7: A Lyapunov function for the arbitrary switched system (9.4) generated by the linear programming problem from Definition 5.5.

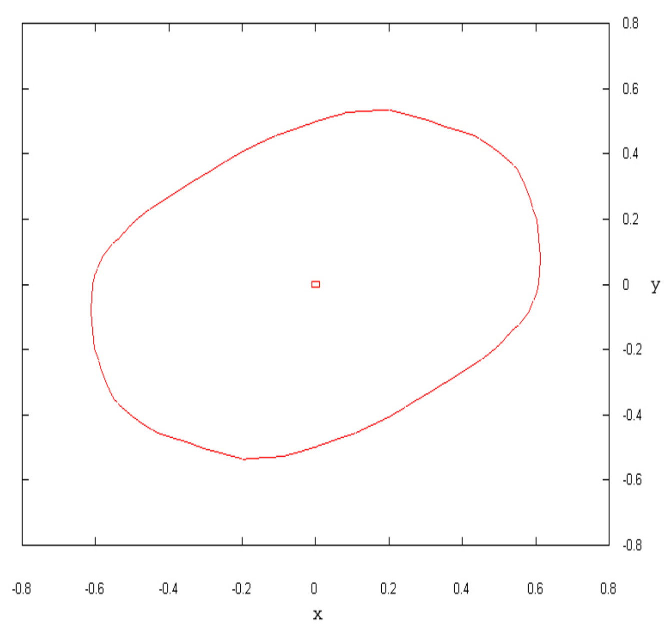


Figure 9.8: The region of attraction secured by the Lyapunov function in figure 9.7 for the switched system. All solution that start in the larger set are asymptotically attracted to the smaller set at the origin.

Chapter 10

A variable structure system

Consider the linear systems

$$\dot{\mathbf{x}} = A_1\mathbf{x}, \quad \text{where} \quad A_1 := \begin{pmatrix} 0.1 & -1 \\ 2 & 0.1 \end{pmatrix}, \tag{10.1}$$

and

$$\dot{\mathbf{x}} = A_2\mathbf{x}, \quad \text{where} \quad A_2 := \begin{pmatrix} 0.1 & -2 \\ 1 & 0.1 \end{pmatrix}. \tag{10.2}$$

These systems are taken from [29]. It is easy to verify that the matrices A_1 and A_2 both have the eigenvalues $\lambda_{\pm} = 0.1 \pm \sqrt{2}$. Therefore, by elementary linear stability theory, the systems (10.1) and (10.2) are both unstable. On Figure 10.1 and Figure 10.2 the trajectories of the systems (10.1) and

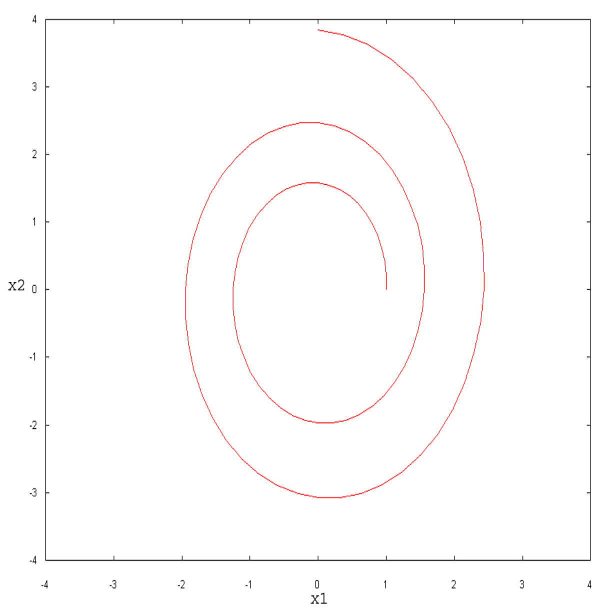


Figure 10.1: The trajectory of the system $\dot{\mathbf{x}} = A_1\mathbf{x}$ starting at $(1, 0)$.

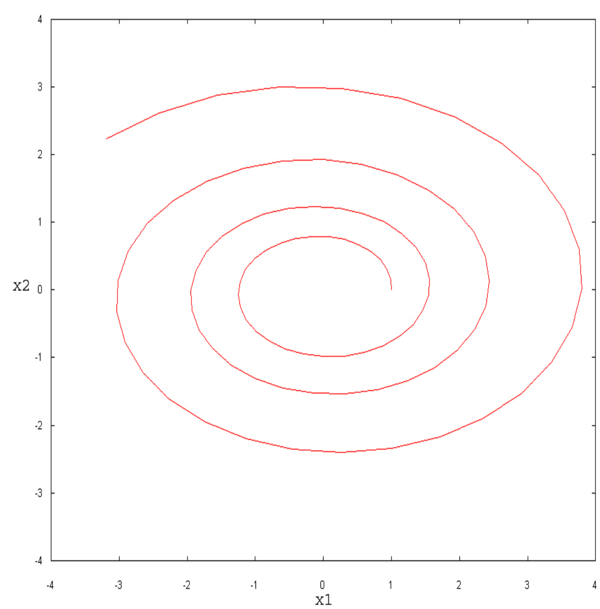


Figure 10.2: The trajectory of the system $\dot{\mathbf{x}} = A_2\mathbf{x}$ starting at $(1, 0)$.

(10.2) with the initial value $(1, 0)$ are depicted. That the norm of the solutions is growing with t in the long run is clear. However, it is equally clear, that the solution to (10.1) is decreasing on the sets

$$Q_2 := \{(x_1, x_2)\big|x_1 \leq 0 \text{ and } x_2 > 0\} \quad \text{and} \quad Q_4 := \{(x_1, x_2)\big|x_1 \geq 0 \text{ and } x_2 > 0\}$$

and that the solution to (10.2) is decreasing on the sets

$$Q_1 := \{(x_1, x_2)\big|x_1 > 0 \text{ and } x_2 \geq 0\} \quad \text{and} \quad Q_3 := \{(x_1, x_2)\big|x_1 < 0 \text{ and } x_2 \leq 0\}.$$

Now, consider the switched system

$$\dot{\mathbf{x}} = A_p \mathbf{x}, \quad p \in \{1, 2\}, \tag{10.3}$$

where the matrices A_1 and A_2 are the same as in (10.1) and (10.2). Obviously, this system is not stable under arbitrary switching, but, if we only consider solution trajectories $(t, \boldsymbol{\xi}) \mapsto \boldsymbol{\phi}_\sigma(t, \boldsymbol{\xi})$, such that

$$\boldsymbol{\phi}_\sigma(t, \boldsymbol{\xi}) \in Q_2 \cup Q_4, \quad \text{implies} \;\; \sigma(t) = 1 \tag{10.4}$$

and

$$\boldsymbol{\phi}_\sigma(t, \boldsymbol{\xi}) \in Q_1 \cup Q_3, \quad \text{implies} \;\; \sigma(t) = 2, \tag{10.5}$$

then we would expect all trajectories under consideration to be asymptotically attracted to the equilibrium. The switched system (10.3), together with the constraints (10.4) and (10.5), is said to be a *variable structure system*. The reason is quite obvious, the structure of the right-hand side of the system (10.3) depends on the current position in the state-space.

It is a simple task to modify the linear programming problem from Definition 5.5 to parameterize a Lyapunov function for the variable structure system. Usually, one would include the constraint **LC4a**, that is,

$$-\Gamma\big[\|\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}\|\big] \geq \sum_{j=1}^{n} \left(\frac{V[\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}] - V[\mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}]}{\mathbf{e}_{\sigma(j)} \cdot (\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})} - \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})})} \tilde{f}_{p,\sigma(j)}(\mathbf{y}_{\sigma,i}^{(\mathbf{z},\mathcal{J})}) + E_{p,\sigma,i}^{(\mathbf{z},\mathcal{J})} C[\{\mathbf{y}_{\sigma,j}^{(\mathbf{z},\mathcal{J})}, \mathbf{y}_{\sigma,j+1}^{(\mathbf{z},\mathcal{J})}\}] \right).$$

for every $p \in \mathcal{P}$, every $(\mathbf{z}, \mathcal{J}) \in \mathcal{Z}$, every $\sigma \in \text{Perm}[\{1, 2, \ldots, n\}]$, and every $i = 1, 2, \ldots, n+1$. In the modified linear programming problem however, we exclude the constraints for some values of p, $(\mathbf{z}, \mathcal{J})$, σ, and i. It goes as follows:

i) Whenever $p = 2$ and either $\mathcal{J} = \{1\}$ or $\mathcal{J} = \{2\}$, we do not include the constraint **LC4a**, for these particular values of p, $(\mathbf{z}, \mathcal{J})$, σ, and i, in the linear programming problem.

ii) Whenever $p = 1$ and either $\mathcal{J} = \emptyset$ or $\mathcal{J} = \{1, 2\}$, we do not include the constraints **LC4a**, for these particular values of p, $(\mathbf{z}, \mathcal{J})$, σ, and i, in the linear programming problem.

We parameterized a Lyapunov function for the variable structure system by use of this modified linear programming problem with $\mathcal{N} := [-1.152, 1.152]^2$, $\mathcal{D} :=]-0.01, 0.01[^2$, and **PS** defined trough the vector

$$\mathbf{ps} := (0.00333, 0.00667, 0.01, 0.0133, 0.0166, 0.0242, 0.0410, 0.0790, 0.157, 0.319, 0.652, 1.152)$$

as described at the beginning of this part. The Lyapunov function V^{Lya} is depicted on Figure 10.3.

Now, one might wonder, what information we can extract from this function V^{Lya}, which is parameterized by our modified linear programming problem. Denote by γ_a the function that is constructed from the variables $\Gamma_a[y_i]$ as in Definition 5.6. Then it is easy to see that for every

$$\mathbf{x} \in \big(\mathcal{N} \setminus \mathcal{D}\big) \cap \big(\mathcal{Q}_1 \cup \mathcal{Q}_3\big)$$

we have

$$\limsup_{h \to 0+} \frac{V^{Lya}(\mathbf{x} + hA_2\mathbf{x}) - V^{Lya}(\mathbf{x})}{h} \leq -\gamma_a(\|\mathbf{x}\|_\infty)$$

and for every

$$\mathbf{x} \in \big(\mathcal{N} \setminus \mathcal{D}\big) \cap \big(\mathcal{Q}_2 \cup \mathcal{Q}_4\big)$$

we have

$$\limsup_{h \to 0+} \frac{V^{Lya}(\mathbf{x} + hA_1\mathbf{x}) - V^{Lya}(\mathbf{x})}{h} \leq -\gamma_a(\|\mathbf{x}\|_\infty).$$

But this includes all trajectories of the system (10.3) that comply with the constraints (10.4) and (10.5) so

$$\limsup_{h \to 0+} \frac{V^{Lya}(\boldsymbol{\phi}_\sigma(t+h,\boldsymbol{\xi})) - V^{Lya}(\boldsymbol{\phi}_\sigma(t,\boldsymbol{\xi}))}{h} \leq -\gamma_a(\|\boldsymbol{\phi}_\sigma(t,\boldsymbol{\xi})\|_\infty)$$

for all $\boldsymbol{\phi}_\sigma(t,\boldsymbol{\xi})$ in the interior of $\mathcal{N} \setminus \mathcal{D}$ and all trajectories under consideration and therefore V^{Lya} is a Lyapunov function for the variable structure system.

The equilibrium's region of attraction, secured by this Lyapunov function, is drawn on Figure 10.4.

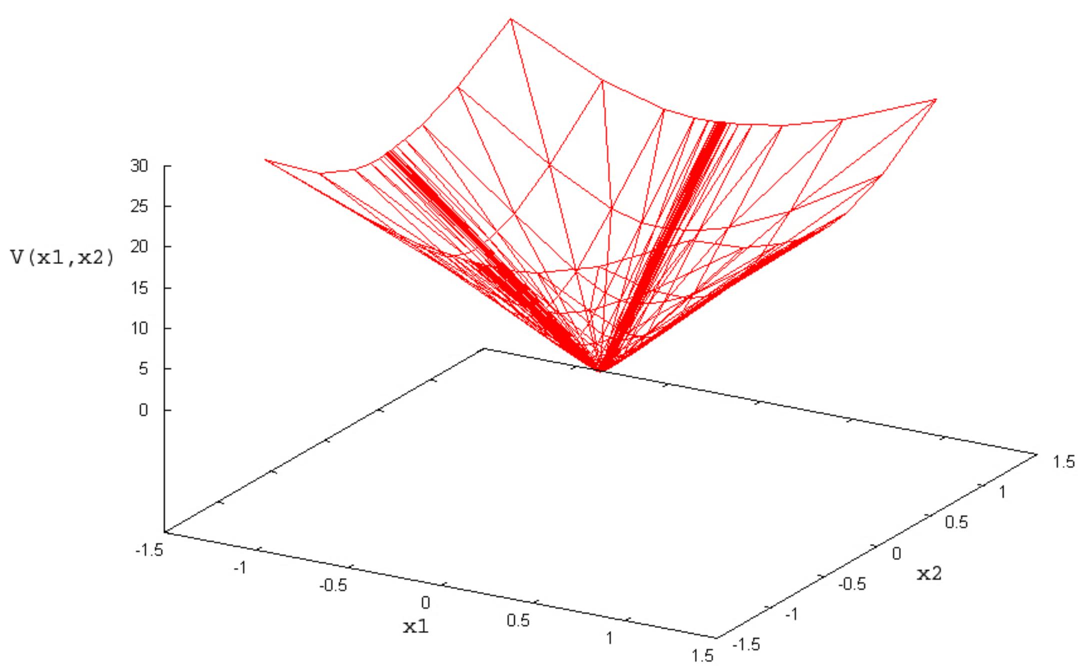


Figure 10.3: A Lyapunov function for the variable structure system (10.3) generated by an altered version of the linear programming problem from Definition 5.5.

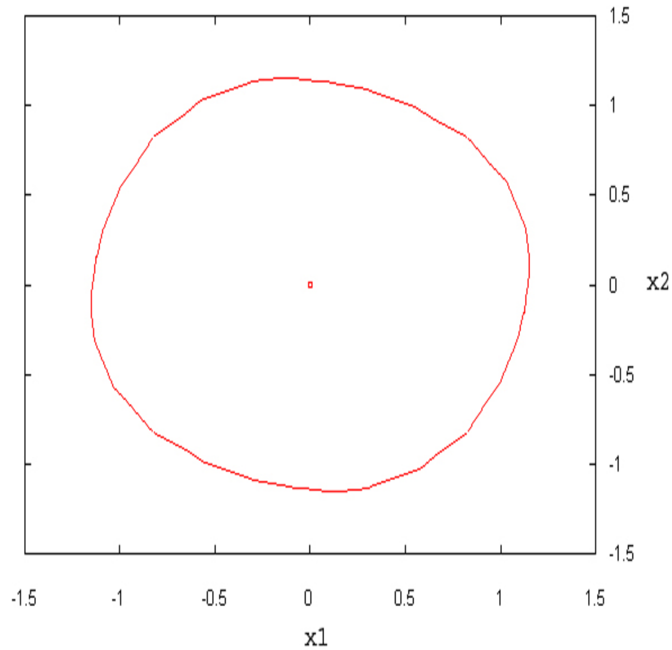


Figure 10.4: The region of attraction secured by the Lyapunov function in Figure 10.3 for the variable structure system. All solution that start in the larger set are asymptotically attracted to the smaller set at the origin.

Chapter 11

A variable structure system with sliding modes

Define the matrix A and the vector $\mathbf{p}$ through

$$A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{p} := \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and consider the systems

$$\dot{\mathbf{x}} = \mathbf{f}_1(\mathbf{x}), \quad \text{where} \quad \mathbf{f}_1(\mathbf{x}) := A\mathbf{x}, \tag{11.1}$$

$$\dot{\mathbf{x}} = \mathbf{f}_2(\mathbf{x}), \quad \text{where} \quad \mathbf{f}_2(\mathbf{x}) := -\mathbf{p}, \tag{11.2}$$

and

$$\dot{\mathbf{x}} = \mathbf{f}_3(\mathbf{x}), \quad \text{where} \quad \mathbf{f}_3(\mathbf{x}) := \mathbf{p}. \tag{11.3}$$

The eigenvalues of the matrix A in (11.1) are $\lambda_\pm = \pm 1$ and the equilibrium at the origin is therefore a saddle point of the system and is not stable. The direction field of the system (11.1) is drawn on Figure 11.1. The systems (11.2) and (11.3) do not even possess an equilibrium.

Let the sets Q_1, Q_2, Q_3, and Q_4 be defined as in the last chapter and consider the variable structure system where we use the system (11.1) in Q_2 and Q_4, the system (11.2) in Q_1, and the system (11.3) in Q_3. A look at Figure 11.1 suggests that this variable structure system might be stable, but the problem is that the system does not possess a properly defined solution compatible with our solution concept in Definition 1.8. The reason is that a trajectory, for example leaving Q_4 to Q_1, is sent straight back by the dynamics in Q_1 to Q_4, where it will, of course, be sent straight back to Q_1. This phenomena is often called *chattering* and the sets $\{\mathbf{x} \in \mathbb{R}^2 \big| x_1 = 0\}$ and $\{\mathbf{x} \in \mathbb{R}^2 \big| x_2 = 0\}$ are called the *sliding modes* of the dynamics. A solution concept for such variable structure systems has been developed by Filippov and others, see, for example [7], [8], and [51], or, for a brief review, [63].

Even though Filippov's solution trajectories are supposed to be close to the true trajectories if the switching is fast, we will use a simpler and more robust technique here to prove the stability of the system. Our approach is very simple, set $h := 0.005$ and define the sets

$$\mathcal{S}_{1,2} := \{\mathbf{x} \in \mathbb{R}^n \big| |x_1| < h \ \text{ and } \ x_2 > 0\}$$
$$\mathcal{S}_{2,3} := \{\mathbf{x} \in \mathbb{R}^n \big| \, x_1 < 0 \ \text{ and } \ |x_2| < h\}$$
$$\mathcal{S}_{3,4} := \{\mathbf{x} \in \mathbb{R}^n \big| |x_1| < h \ \text{ and } \ x_2 < 0\}$$
$$\mathcal{S}_{4,1} := \{\mathbf{x} \in \mathbb{R}^n \big| \, x_1 > 0 \ \text{ and } \ |x_2| < h\}$$

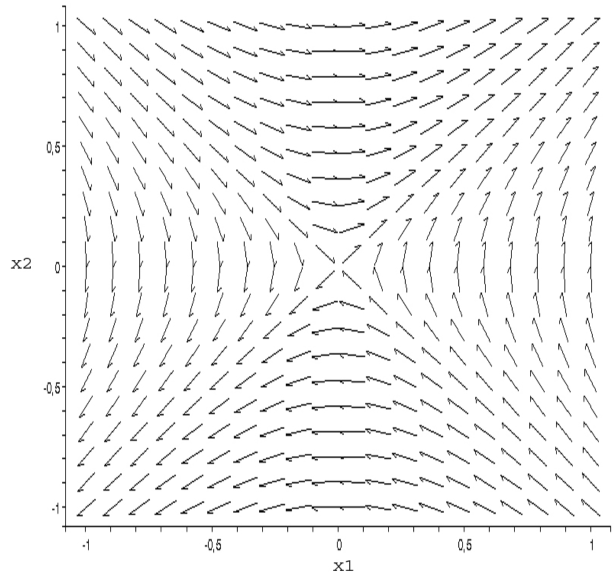


Figure 11.1: The direction field of $\dot{\mathbf{x}} = A\mathbf{x}$, where A is the matrix in (11.1).

and

$$\mathcal{D}' :=]-2h, 2h[^2.$$

We will generate a Lyapunov function for the variable structure system on $[-0.957, 0.957]^2$ so we can consider $\mathcal{D}$ to be a small neighborhood of the origin and the $\mathcal{S}_{i,j}$ to be thin stripes between Q_i and Q_j.

We parameterized a Lyapunov function V^{Lya} for the variable structure system by use of a modified linear programming problem with $\mathcal{N} := [-0,957, 0,957]^2$, $\mathcal{D} := \mathcal{D}'$, and **PS** defined trough the vector

$$\mathbf{ps} := (0.005, 0.01, 0.015, 0.0263, 0.052, 0.109, 0.237, 0.525, 0.957)$$

as described at the beginning of this part.

The modification we used to the linear programming problem in Definition 5.5, similar to the modification in the last chapter, was to only include the constraints **LC4a** in the linear programming problem for some sets of parameters

$$p \in \mathcal{P},\ (\mathbf{z}, \mathcal{J}) \in \mathcal{Z},\ \sigma \in \text{Perm}[\{1, 2, \dots, n\}],\ \text{and } i \in \{1, 2, \dots, n+1\}.$$

Exactly, for every simplex S in the simplicial partition, $S \cap \mathcal{D} = \emptyset$, we included the constraints **LC4a** for every vertex of the simplex, if and only if:

$$S \subset \mathcal{Q}_1 \setminus \big(\mathcal{S}_{1,2} \cup \mathcal{S}_{4,1}\big) \quad \text{and } p = 2,$$
$$S \subset \mathcal{Q}_2 \setminus \big(\mathcal{S}_{1,2} \cup \mathcal{S}_{2,3}\big) \bigcup \mathcal{Q}_4 \setminus \big(\mathcal{S}_{4,1} \cup \mathcal{S}_{3,4}\big) \quad \text{and } p = 1,$$
$$S \subset \mathcal{Q}_3 \setminus \big(\mathcal{S}_{2,3} \cup \mathcal{S}_{3,4}\big) \quad \text{and } p = 3,$$
$$S \subset \overline{\mathcal{S}_{1,2}} \cup \overline{\mathcal{S}_{4,1}} \quad \text{and } \big(p = 1 \text{ or } p = 2\big),$$
$$S \subset \overline{\mathcal{S}_{2,3}} \cup \overline{\mathcal{S}_{3,4}} \quad \text{and } \big(p = 1 \text{ or } p = 3\big).$$

This implies for the Lyapunov function V^{Lya}, where the function $\gamma_a \in \mathcal{K}$ is constructed from the variables $\Gamma_a[y_i]$ as in Definition 5.6, that:

V1) For every $\mathbf{x}$ in the interior of the set

$$\big(\mathcal{Q}_1 \cup \mathcal{S}_{1,2} \cup \mathcal{S}_{4,1}\big) \setminus \mathcal{D}$$

we have

$$\limsup_{h\to 0+} \frac{V^{Lya}(\mathbf{x}+h\mathbf{f}_2(\mathbf{x})) - V^{Lya}(\mathbf{x})}{h} \le -\gamma_a(\|\mathbf{x}\|_\infty).$$

V2) For every $\mathbf{x}$ in the interior of the set

$$\big(\mathcal{Q}_2 \cup \mathcal{Q}_4 \cup \mathcal{S}_{1,2} \cup \mathcal{S}_{2,3} \cup \mathcal{S}_{3,4} \cup \mathcal{S}_{4,1}\big) \setminus \mathcal{D}$$

we have

$$\limsup_{h\to 0+} \frac{V^{Lya}(\mathbf{x}+h\mathbf{f}_1(\mathbf{x})) - V^{Lya}(\mathbf{x})}{h} \le -\gamma_a(\|\mathbf{x}\|_\infty).$$

V3) For every $\mathbf{x}$ in the interior of the set

$$\big(\mathcal{Q}_3 \cup \mathcal{S}_{2,3} \cup \mathcal{S}_{3,4}\big) \setminus \mathcal{D}$$

we have

$$\limsup_{h\to 0+} \frac{V^{Lya}(\mathbf{x}+h\mathbf{f}_3(\mathbf{x})) - V^{Lya}(\mathbf{x})}{h} \le -\gamma_a(\|\mathbf{x}\|_\infty).$$

Now, let $\mathbf{f}_1$, $\mathbf{f}_2$, and $\mathbf{f}_3$ be the functions from the systems (11.1), (11.2), and (11.3) and consider the variable structure system

$$\dot{\mathbf{x}} = \mathbf{f}_p(\mathbf{x}), \qquad p \in \{1,2,3\}, \tag{11.4}$$

under the following constraints:

i) $\boldsymbol{\phi}_\sigma(t,\boldsymbol{\xi}) \in \mathcal{N} \setminus \mathcal{D}$ and $\boldsymbol{\phi}_\sigma(t,\boldsymbol{\xi})$ in the interior of $\mathcal{Q}_1 \setminus \big(\mathcal{S}_{1,2} \cup \mathcal{S}_{4,1}\big)$ implies $\sigma(t) = 2$.

ii) $\boldsymbol{\phi}_\sigma(t,\boldsymbol{\xi}) \in \mathcal{N} \setminus \mathcal{D}$ and $\boldsymbol{\phi}_\sigma(t,\boldsymbol{\xi})$ in the interior of $\mathcal{Q}_2 \setminus \big(\mathcal{S}_{1,2} \cup \mathcal{S}_{2,3}\big) \bigcup \mathcal{Q}_4 \setminus \big(\mathcal{S}_{4,1} \cup \mathcal{S}_{3,4}\big)$ implies $\sigma(t) = 1$.

iii) $\boldsymbol{\phi}_\sigma(t,\boldsymbol{\xi}) \in \mathcal{N} \setminus \mathcal{D}$ and $\boldsymbol{\phi}_\sigma(t,\boldsymbol{\xi})$ in the interior of $\mathcal{Q}_3 \setminus \big(\mathcal{S}_{2,3} \cup \mathcal{S}_{3,4}\big)$ implies $\sigma(t) = 3$.

iii) $\boldsymbol{\phi}_\sigma(t,\boldsymbol{\xi}) \in \mathcal{N} \setminus \mathcal{D}$ and $\boldsymbol{\phi}_\sigma(t,\boldsymbol{\xi})$ in the interior of $\mathcal{S}_{1,2} \cup \mathcal{S}_{4,1}$ implies $\sigma(t) \in \{1,2\}$.

iii) $\boldsymbol{\phi}_\sigma(t,\boldsymbol{\xi}) \in \mathcal{N} \setminus \mathcal{D}$ and $\boldsymbol{\phi}_\sigma(t,\boldsymbol{\xi})$ in the interior of $\mathcal{S}_{2,3} \cup \mathcal{S}_{3,4}$ implies $\sigma(t) \in \{1,3\}$.

One should make one self clear what these constraints imply. For example, if $\boldsymbol{\xi} \in \mathcal{Q}_2 \setminus \big(\overline{\mathcal{S}_{1,2}} \cup \overline{\mathcal{S}_{2,3}}\big)$, then we must use the dynamics $\dot{\mathbf{x}} = A\mathbf{x}$ until $t \mapsto \boldsymbol{\phi}_\sigma(t,\boldsymbol{\xi})$ leaves $\mathcal{Q}_2 \setminus \big(\overline{\mathcal{S}_{1,2}} \cup \overline{\mathcal{S}_{2,3}}\big)$. If then, for example, $\boldsymbol{\phi}_\sigma(t',\boldsymbol{\xi}) \in \mathcal{S}_{1,2}$ for some $t' > 0$, then every switching between the systems $\dot{\mathbf{x}} = A\mathbf{x}$ and $\dot{\mathbf{x}} = -\mathbf{p}$ is allowed as long as $t \mapsto \boldsymbol{\phi}_\sigma(t,\boldsymbol{\xi})$ stays in $\mathcal{S}_{1,2}$. However, if, for example, $\boldsymbol{\phi}_\sigma(t'',\boldsymbol{\xi}) \in \mathcal{Q}_1 \setminus \big(\overline{\mathcal{S}_{1,2}} \cup \overline{\mathcal{S}_{4,1}}\big)$ for some $t'' > t'$, then we must use the dynamics $\dot{\mathbf{x}} = -\mathbf{p}$ until $t \mapsto \boldsymbol{\phi}_\sigma(t,\mathbf{x})$ leaves $\mathcal{Q}_1 \setminus \big(\overline{\mathcal{S}_{1,2}} \cup \overline{\mathcal{S}_{4,1}}\big)$.

By **V1**, **V2**, and **V3** we have for every trajectory $t \mapsto \boldsymbol{\phi}_\sigma(t,\boldsymbol{\xi})$ under consideration that

$$\limsup_{h\to 0+} \frac{V^{Lya}(\boldsymbol{\phi}_\sigma(t+h,\boldsymbol{\xi})) - V^{Lya}(\boldsymbol{\phi}_\sigma(t,\boldsymbol{\xi}))}{h} \le -\gamma_a(\|\boldsymbol{\phi}_\sigma(t,\boldsymbol{\xi})\|_\infty),$$

so the function V^{Lya} is a Lyapunov function for this system.

The parameterized Lyapunov function V^{Lya} for the system (11.4) is depicted on Figure 11.2 and its region of attraction on Figure 11.3. Because it is difficult to recognize the structure of the Lyapunov function close to the origin, a Lyapunov function for the same system, but with a much smaller domain, is depicted on Figure 11.4.

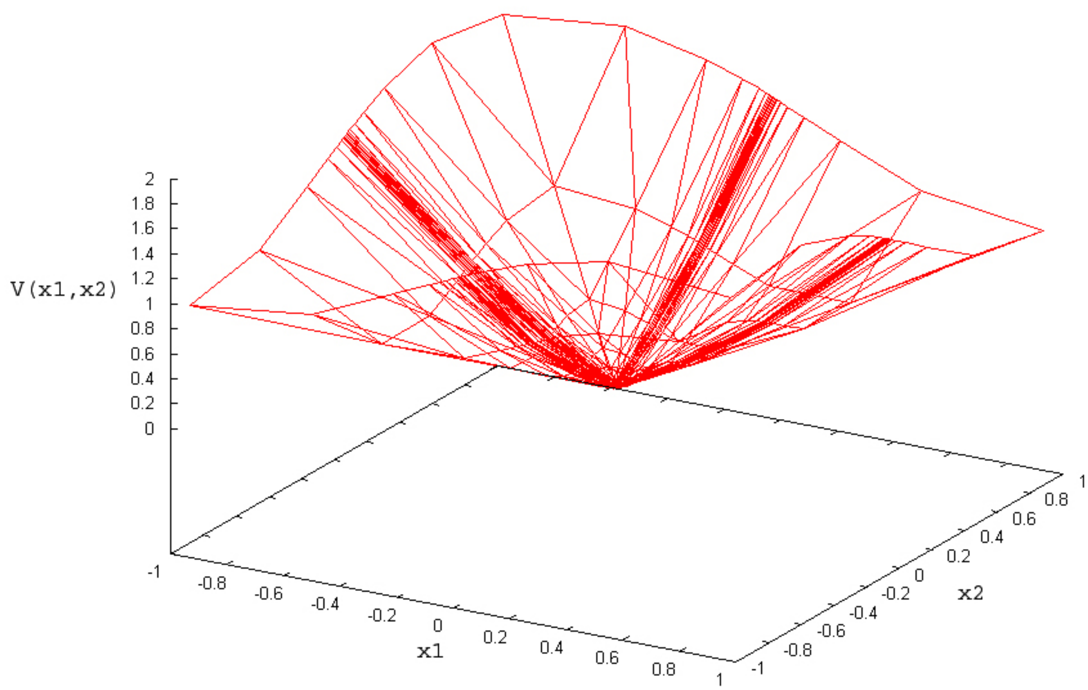


Figure 11.2: A Lyapunov function for the variable structure system (11.4) generated by an altered version of the linear programming problem from Definition 5.5.

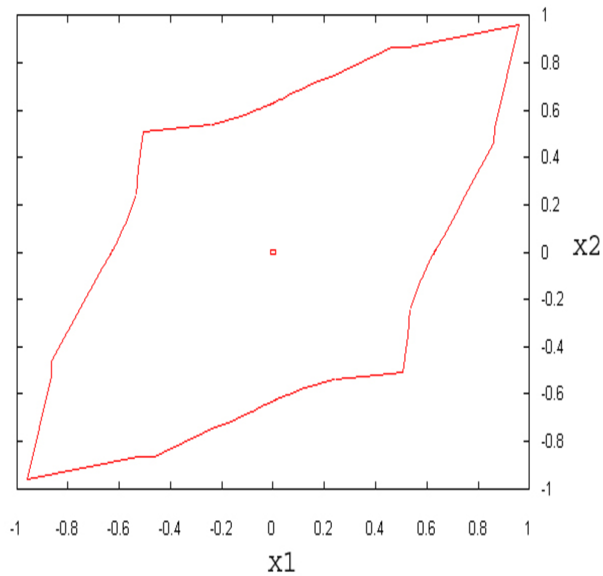


Figure 11.3: The region of attraction secured by the Lyapunov function in Figure 11.2 for the variable structure system. All solution that start in the larger set are asymptotically attracted to the smaller set at the origin.

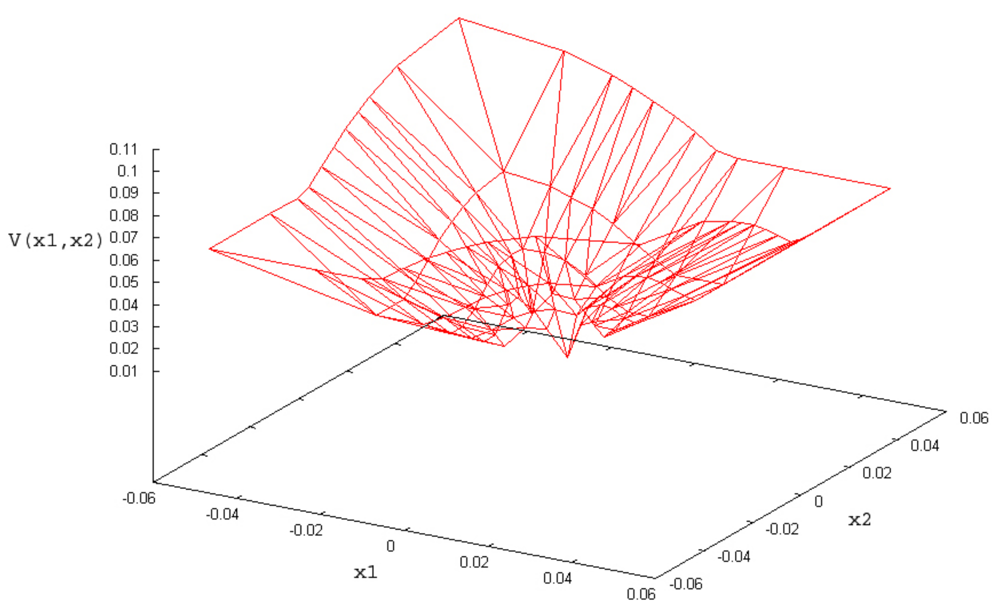


Figure 11.4: A Lyapunov function for the variable structure system (11.4) generated by an altered version of the linear programming problem from Definition 5.5.

Chapter 12

A one-dimensional nonautonomous switched system

Consider the one-dimensional systems

$$\dot{x} = f_1(t,x), \quad \text{where} \quad f_1(t,x) := -\frac{x}{1+t} \tag{12.1}$$

and

$$\dot{x} = f_2(t,x), \quad \text{where} \quad f_2(t,x) := -\frac{tx}{1+t}. \tag{12.2}$$

The system (12.1) has the closed-form solution

$$\phi(t,t_0,\xi) = \xi\frac{1+t_0}{1+t}$$

and the system (12.2) has the closed-form solution

$$\phi(t,t_0,\xi) = \xi e^{-(t-t_0)}\frac{1+t}{1+t_0}.$$

The origin in the state-space is therefore, for every fixed t_0, an asymptotically stable equilibrium point of the system (12.1) and, because

$$|\xi|e^{-(t-t_0)}\frac{1+t}{1+t_0} \leq 2|\xi|e^{-\frac{t-t_0}{2}},$$

a uniformly exponentially stable equilibrium point of the system (12.2). However, as can easily be verified, it is not a uniformly asymptotically stable equilibrium point of the system (12.1). This implies that the system (12.1) cannot possess a Lyapunov function that is defined for all $t \geq 0$. Note however, that this does not imply that we cannot parameterize a Lyapunov-like function on a compact time interval for the system (12.1).

We set

$$t_{(\mathbf{z},\mathcal{J})} := \mathbf{e}_0 \cdot \widetilde{\mathbf{PS}}(\mathbf{z}) \quad \text{and} \quad x_{(\mathbf{z},\mathcal{J})} := |\mathbf{e}_1 \cdot \widetilde{\mathbf{PS}}(\mathbf{R}^{\mathcal{J}}(\mathbf{z}+\mathbf{e}_1))|$$

and define the constants $B_{p,rs}^{(\mathbf{z},\mathcal{J})}$ from the linear programming problem from Definition 5.1 by

$$B_{p,00}^{(\mathbf{z},\mathcal{J})} := \frac{2x_{(\mathbf{z},\mathcal{J})}}{(1+t_{(\mathbf{z},\mathcal{J})})^3},$$

$$B_{p,01}^{(\mathbf{z},\mathcal{J})} := \frac{1}{(1+t_{(\mathbf{z},\mathcal{J})})^2},$$

$$B_{p,11}^{(\mathbf{z},\mathcal{J})} := 0$$

for $p \in \{1,2\}$.

We parameterized a CPWA Lyapunov function for the system (12.1), the system (12.2), and the switched system

$$\dot{x} = f_p(t,x), \qquad p \in \{1,2\} \tag{12.3}$$

by use of the linear programming problem from Definition 5.1 with $\mathcal{N} :=]-1.1, 1.1[$, $\mathcal{D} :=]-0.11, 0.11[$, **PS** defined through the vector

$$\mathbf{ps} := (0.11, 0.22, 0.33, 0.44, 0.55, 0.66, 0.77, 0.88, 0.99, 1.1)$$

as described at the beginning of this part, and the vector

$$\mathbf{t} := (0, 0.194, 0.444, 0.75, 1.111, 1.528, 2, 2.528, 3.111, 3.75, 4.444, 5.194, 6, \\ 6.861, 7.778, 8.75, 9.778, 10.861, 12, 13.194, 14.444, 15.75, 17.111, 18.528, 20).$$

The Lyapunov function for the system (12.1) is depicted on Figure 12.1, the Lyapunov function for the system (12.2) on Figure 12.1, and the Lyapunov function for the arbitrary switched system (12.3) on Figure 12.3.

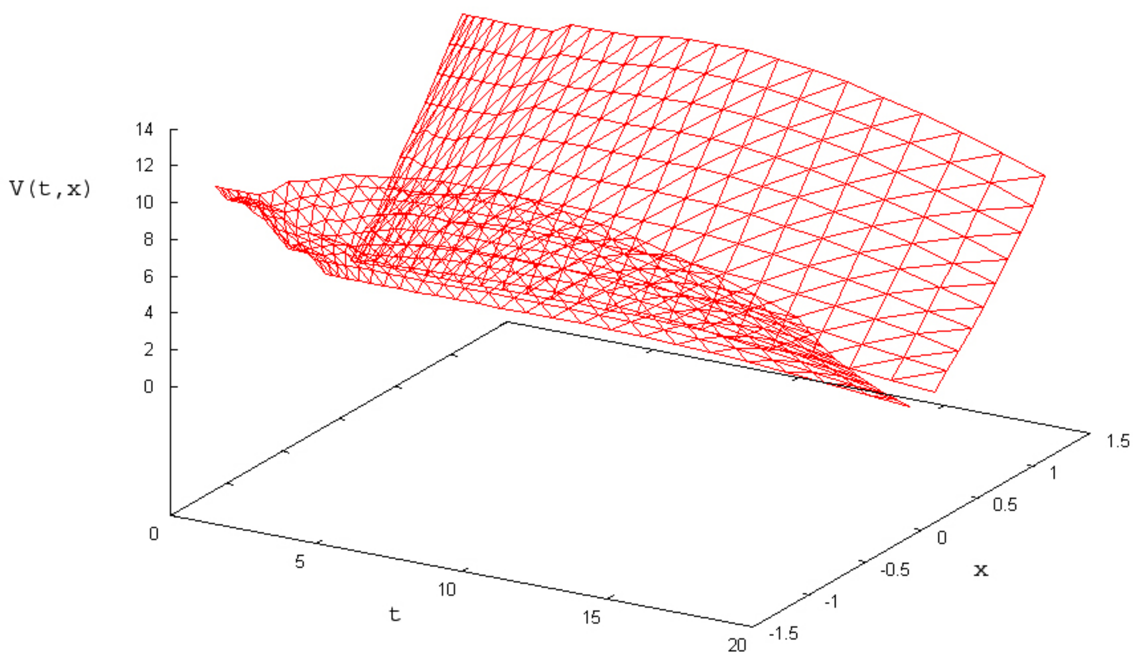


Figure 12.1: A Lyapunov function for the nonautonomous system (12.1) generated by the linear programming problem from Definition 5.1.

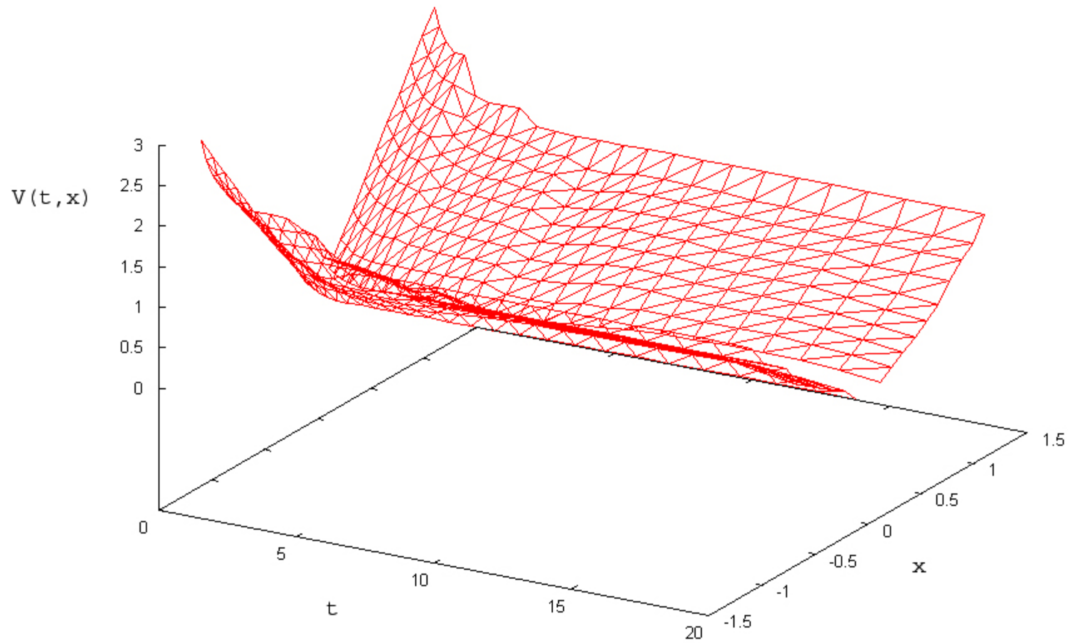


Figure 12.2: A Lyapunov function for the nonautonomous system (12.2) generated by the linear programming problem from Definition 5.1.

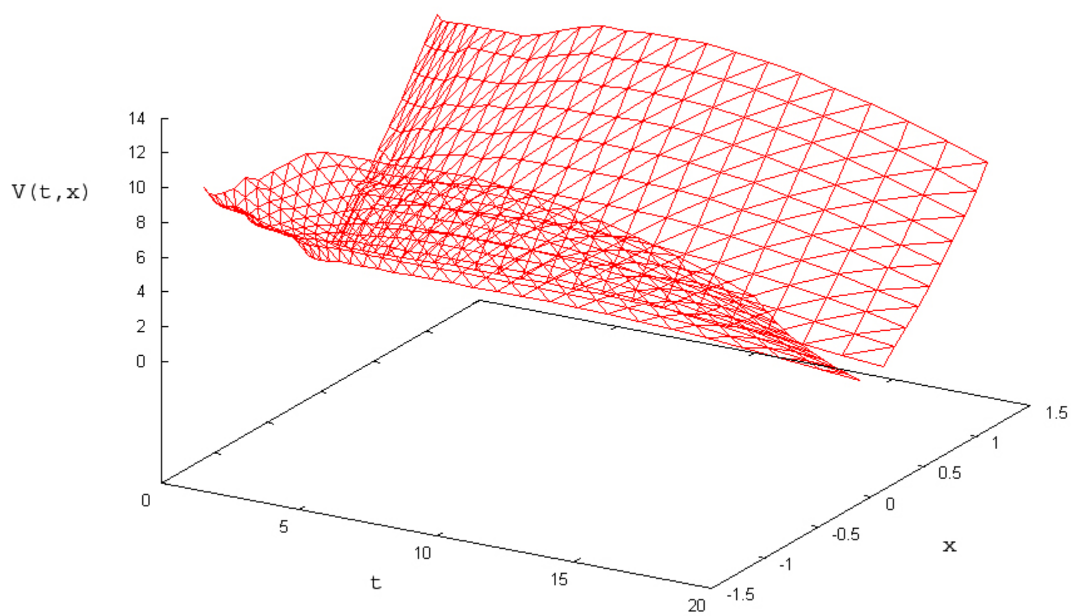


Figure 12.3: A Lyapunov function for the switched nonautonomous system (12.3) generated by the linear programming problem from Definition 5.1.

Chapter 13

A two-dimensional nonautonomous switched system

Consider the two-dimensional systems

$$\dot{\mathbf{x}} = \mathbf{f}_1(t, \mathbf{x}), \quad \text{where} \quad \mathbf{f}_1(t, x, y) := \begin{pmatrix} -2x + y\cos(t) \\ x\cos(t) - 2y \end{pmatrix} \tag{13.1}$$

and

$$\dot{\mathbf{x}} = \mathbf{f}_2(t, \mathbf{x}), \quad \text{where} \quad \mathbf{f}_2(t, x, y) := \begin{pmatrix} -2x + y\sin(t) \\ x\sin(t) - 2y \end{pmatrix}. \tag{13.2}$$

We set

$$x_{(\mathbf{z},\mathcal{J})} := |\mathbf{e}_1 \cdot \widetilde{\mathbf{PS}}(\mathbf{R}^{\mathcal{J}}(\mathbf{z} + \mathbf{e}_1))| \quad \text{and} \quad y_{(\mathbf{z},\mathcal{J})} := |\mathbf{e}_2 \cdot \widetilde{\mathbf{PS}}(\mathbf{R}^{\mathcal{J}}(\mathbf{z} + \mathbf{e}_2))|$$

and assign values to the constants $B_{p,rs}^{(\mathbf{z},\mathcal{J})}$ from the linear programming problem in Definition 5.1 as follows:

$$\begin{aligned}
B_{p,00}^{(\mathbf{z},\mathcal{J})} &:= \max\{x_{(\mathbf{z},\mathcal{J})}, y_{(\mathbf{z},\mathcal{J})}\}, \\
B_{p,11}^{(\mathbf{z},\mathcal{J})} &:= 0, \\
B_{p,22}^{(\mathbf{z},\mathcal{J})} &:= 0, \\
B_{p,01}^{(\mathbf{z},\mathcal{J})} &:= 1, \\
B_{p,02}^{(\mathbf{z},\mathcal{J})} &:= 1, \\
B_{p,12}^{(\mathbf{z},\mathcal{J})} &:= 0
\end{aligned}$$

for $p \in \{1, 2\}$.

We parameterized a Lyapunov function for the system (13.1), the system (13.2), and the switched system

$$\dot{\mathbf{x}} = \mathbf{f}_p(t, \mathbf{x}), \qquad p \in \{1, 2\} \tag{13.3}$$

by use of the linear programming problem from Definition 5.1 with $\mathcal{N} :=]-0.55, 0.55[^2$, $\mathcal{D} :=]-0.11, 0.11[^2$, $\mathbf{PS}$ defined through the vector

$$\mathbf{ps} := (0.11, 0.22, 0.33, 0.44, 0.55)$$

as described at the beginning of this part, and the vector

$$\mathbf{t} := (0, 0.3125, 0.75, 1.3125, 2).$$

Because the Lyapunov functions are functions from $\mathbb{R} \times \mathbb{R}^2$ into $\mathbb{R}$ it is hardly possible to draw them in any sensible way on a two-dimensional sheet. Therefore, we draw them for the fixed time-values $t := 0$, $t := 0.3125$, $t := 0.75$, $t := 1.3125$, and $t := 2$. From the definition of the function spaces CPWA it should be clear how to interpolate the drawings to get the full Lyapunov functions.

On figures 13.1, 13.2, 13.3, 13.4, and 13.5 the state-space dependence of the parameterized Lyapunov function for the system (13.1) for the times $t := 0$, $t := 0.3125$, $t := 0.75$, $t := 1.3125$, and $t := 2$ respectively is depicted.

On figures 13.6, 13.7, 13.13, 13.14, and 13.15 the state dependence of the parameterized Lyapunov function for the system (13.2) for the times $t := 0$, $t := 0.3125$, $t := 0.75$, $t := 1.3125$, and $t := 2$ respectively is depicted.

On figures 13.11, 13.12, 13.13, 13.14, and 13.15 the state dependence of the parameterized Lyapunov function for the switched system (13.3) for the times $t := 0$, $t := 0.3125$, $t := 0.75$, $t := 1.3125$, and $t := 2$ respectively is depicted.

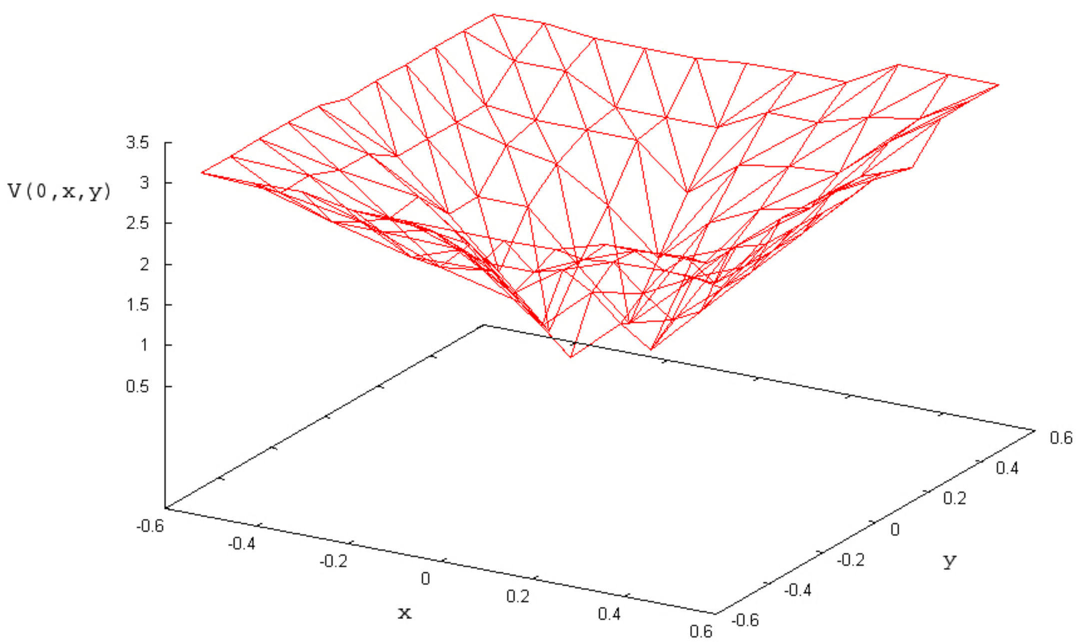


Figure 13.1: The function $(x, y) \mapsto V(0, x, y)$, where $V(t, x, y)$ is the parameterized Lyapunov function for the nonautonomous system (13.1).

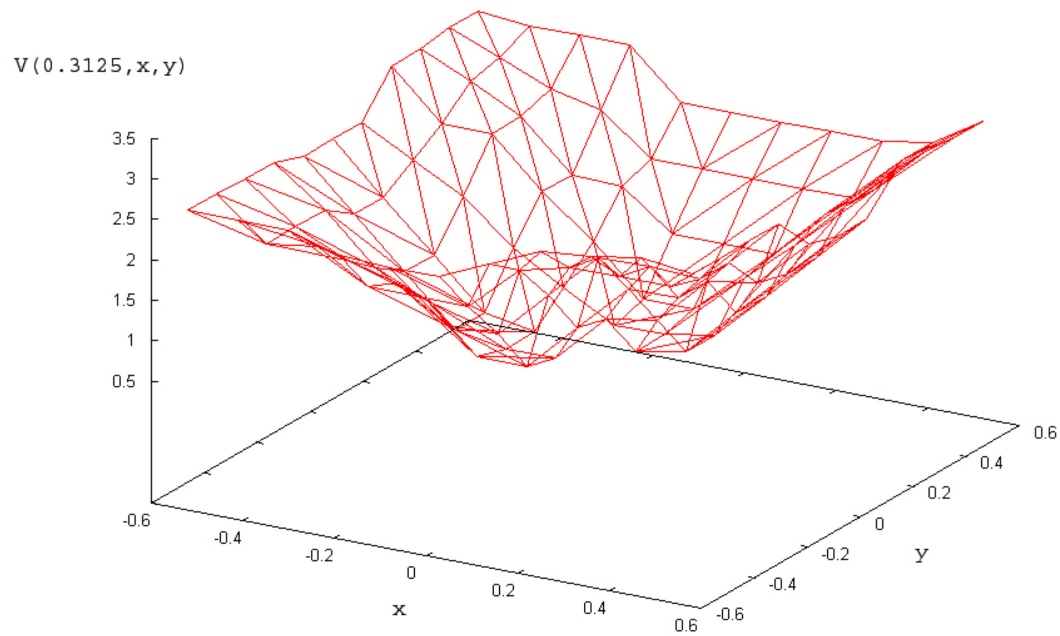


Figure 13.2: The function $(x, y) \mapsto V(0.3125, x, y)$, where $V(t, x, y)$ is the parameterized Lyapunov function for the nonautonomous system (13.1).

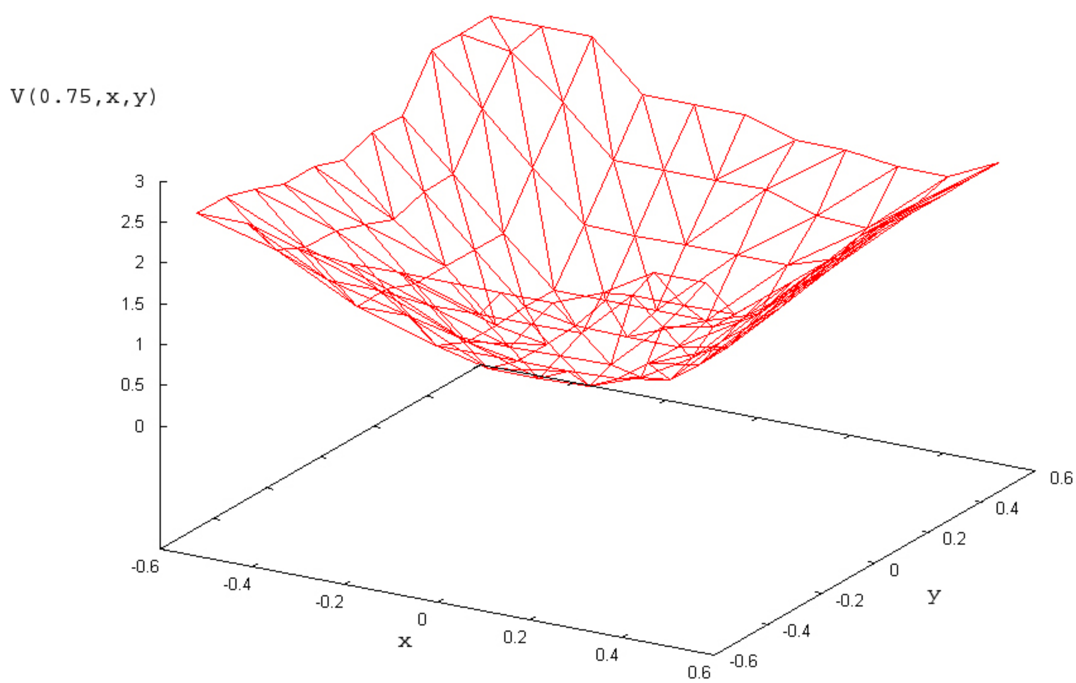


Figure 13.3: The function $(x, y) \mapsto V(0.75, x, y)$, where $V(t, x, y)$ is the parameterized Lyapunov function for the nonautonomous system (13.1).

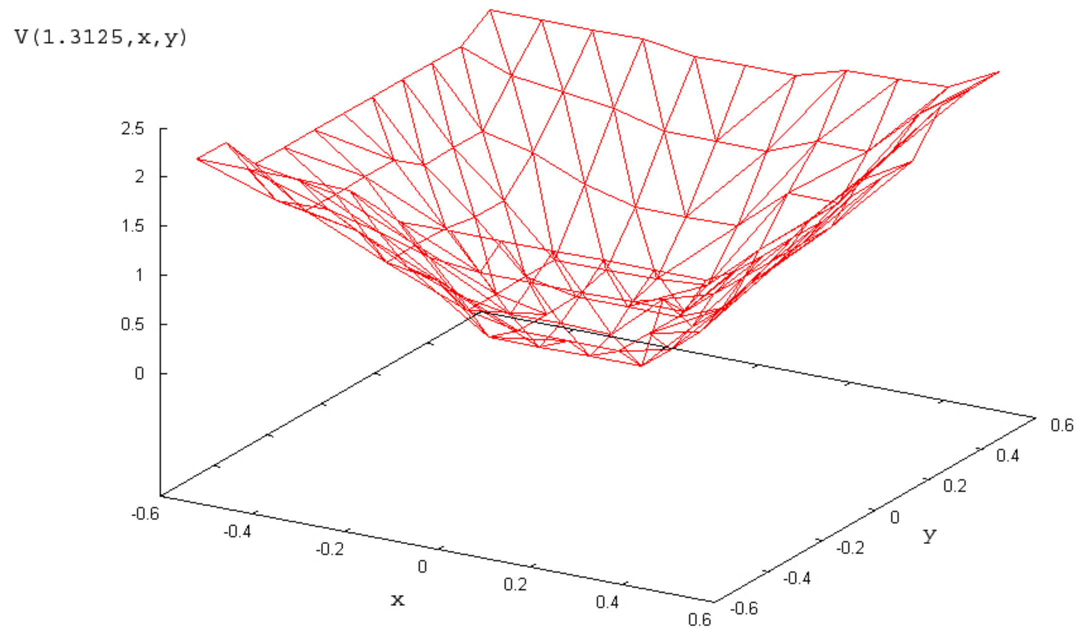


Figure 13.4: The function $(x, y) \mapsto V(1.3125, x, y)$, where $V(t, x, y)$ is the parameterized Lyapunov function for the nonautonomous system (13.1).

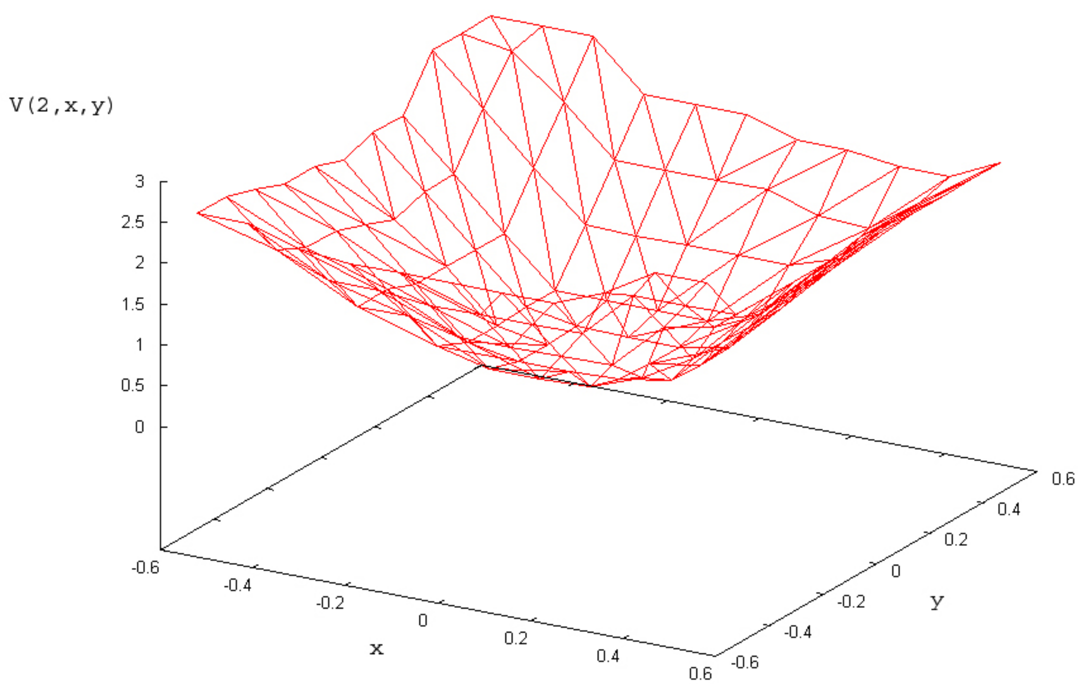


Figure 13.5: The function $(x, y) \mapsto V(2, x, y)$, where $V(t, x, y)$ is the parameterized Lyapunov function for the nonautonomous system (13.1).

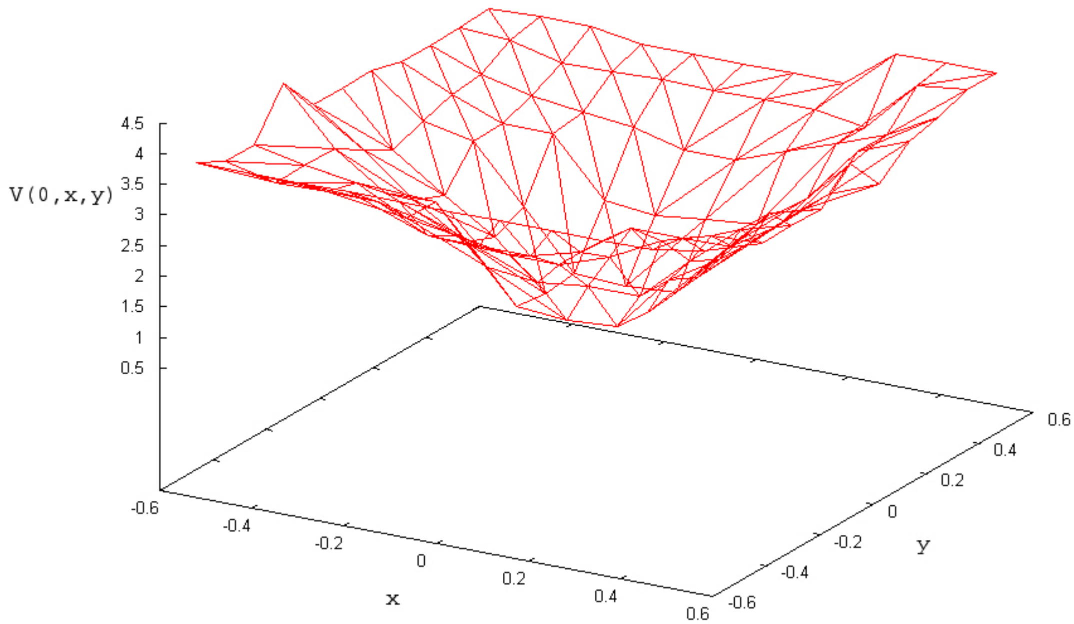


Figure 13.6: The function $(x, y) \mapsto V(0, x, y)$, where $V(t, x, y)$ is the parameterized Lyapunov function for the nonautonomous system (13.2).

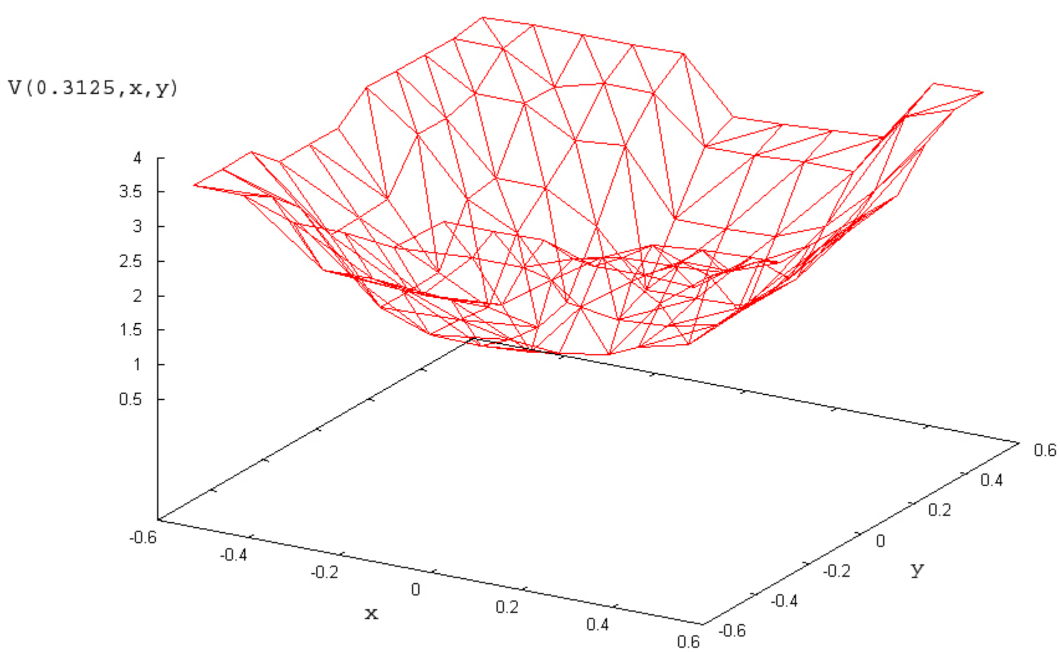


Figure 13.7: The function $(x, y) \mapsto V(0.3125, x, y)$, where $V(t, x, y)$ is the parameterized Lyapunov function for the nonautonomous system (13.2).

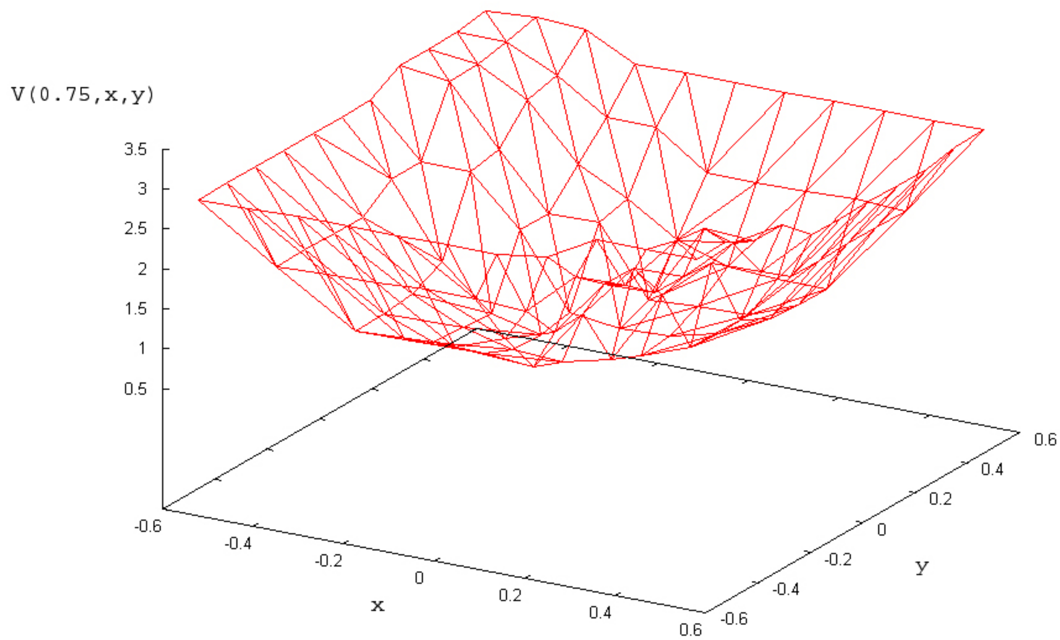


Figure 13.8: The function $(x, y) \mapsto V(0.75, x, y)$, where $V(t, x, y)$ is the parameterized Lyapunov function for the nonautonomous system (13.2).

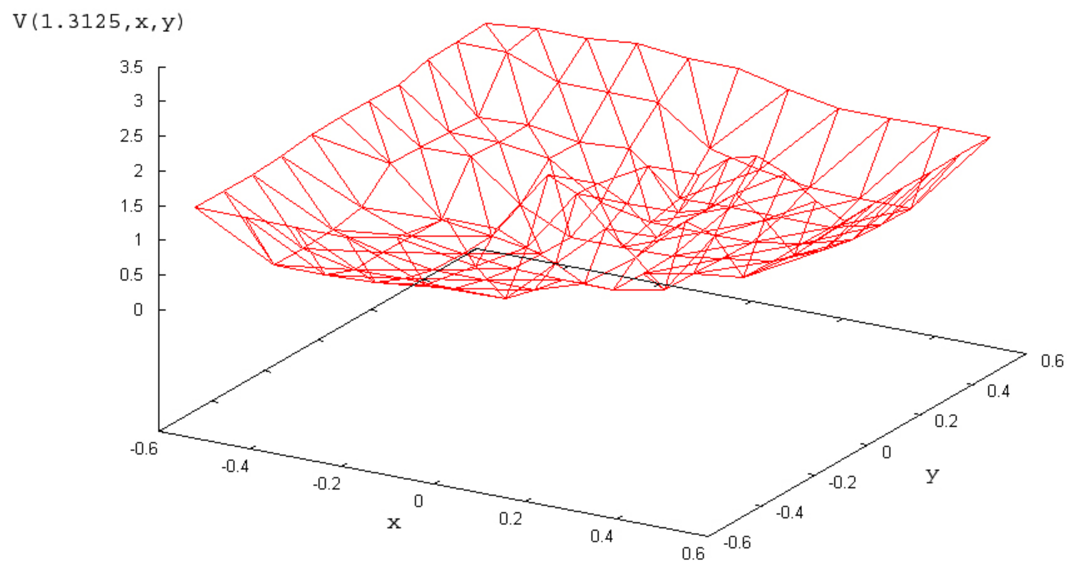


Figure 13.9: The function $(x, y) \mapsto V(1.3125, x, y)$, where $V(t, x, y)$ is the parameterized Lyapunov function for the nonautonomous system (13.2).

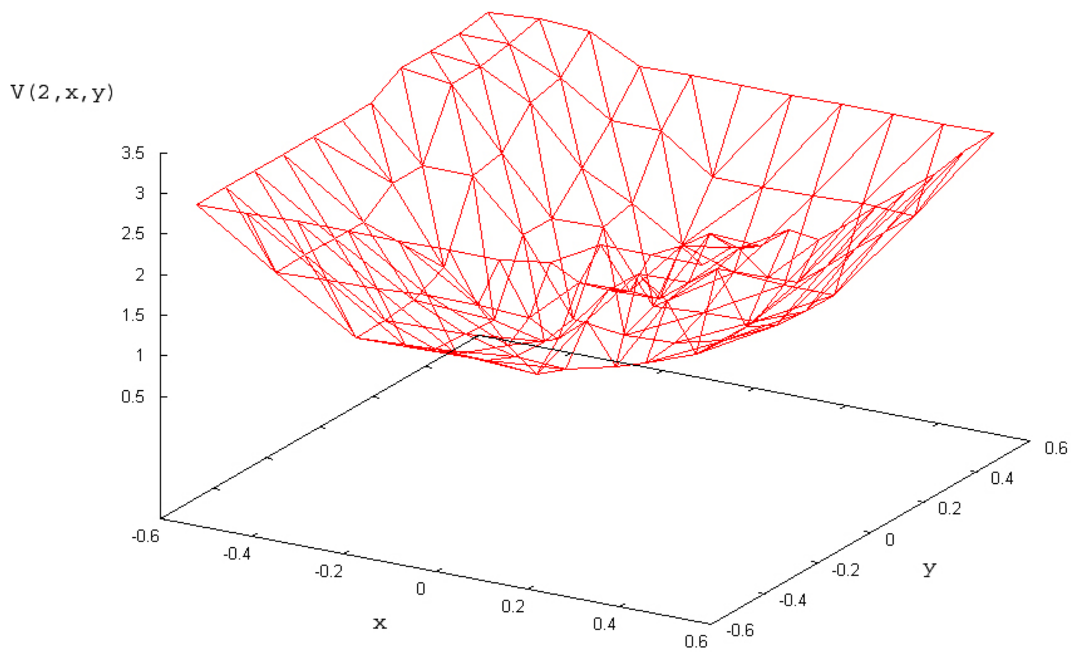


Figure 13.10: The function $(x, y) \mapsto V(2, x, y)$, where $V(t, x, y)$ is the parameterized Lyapunov function for the nonautonomous system (13.2).

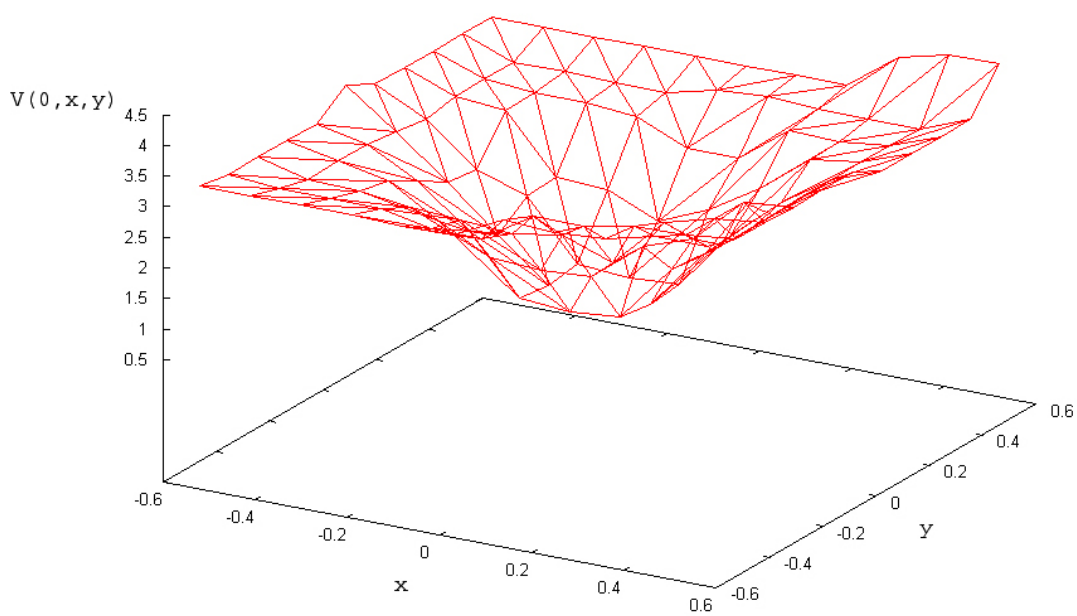


Figure 13.11: The function $(x, y) \mapsto V(0, x, y)$, where $V(t, x, y)$ is the parameterized Lyapunov function for the nonautonomous system (13.3).

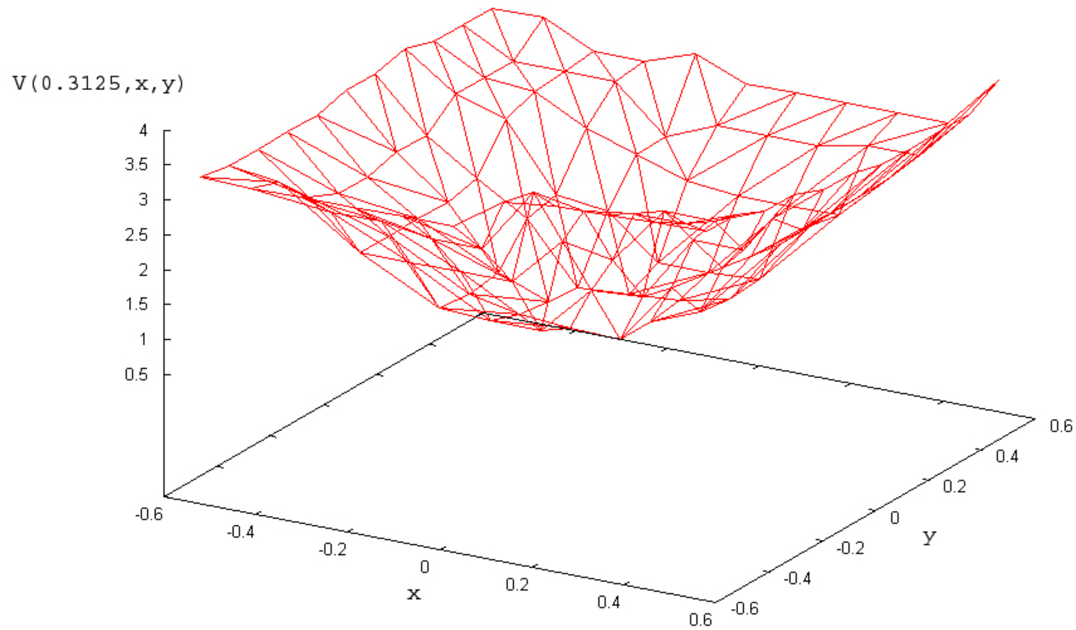


Figure 13.12: The function $(x, y) \mapsto V(0.3125, x, y)$, where $V(t, x, y)$ is the parameterized Lyapunov function for the nonautonomous system (13.3).

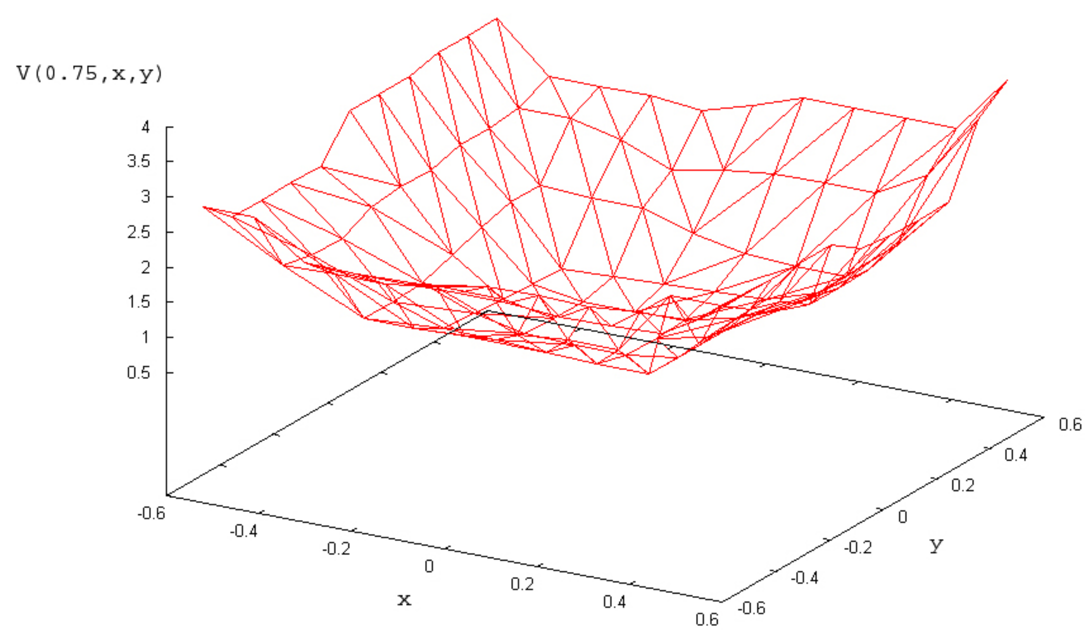


Figure 13.13: The function $(x, y) \mapsto V(0.75, x, y)$, where $V(t, x, y)$ is the parameterized Lyapunov function for the nonautonomous system (13.3).

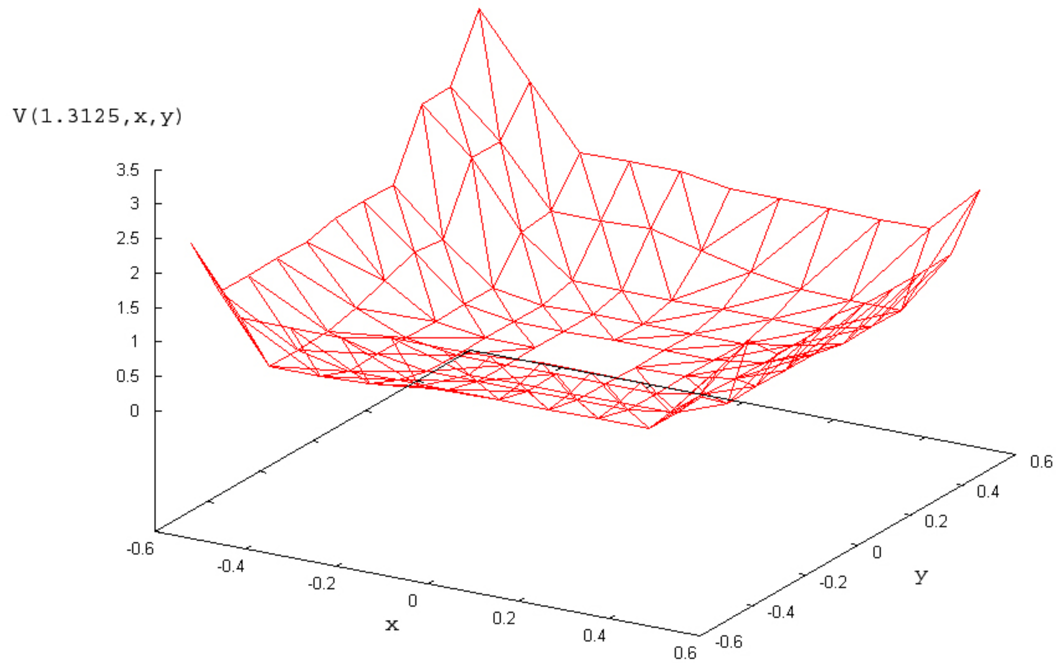


Figure 13.14: The function $(x, y) \mapsto V(1.3125, x, y)$, where $V(t, x, y)$ is the parameterized Lyapunov function for the nonautonomous system (13.3).

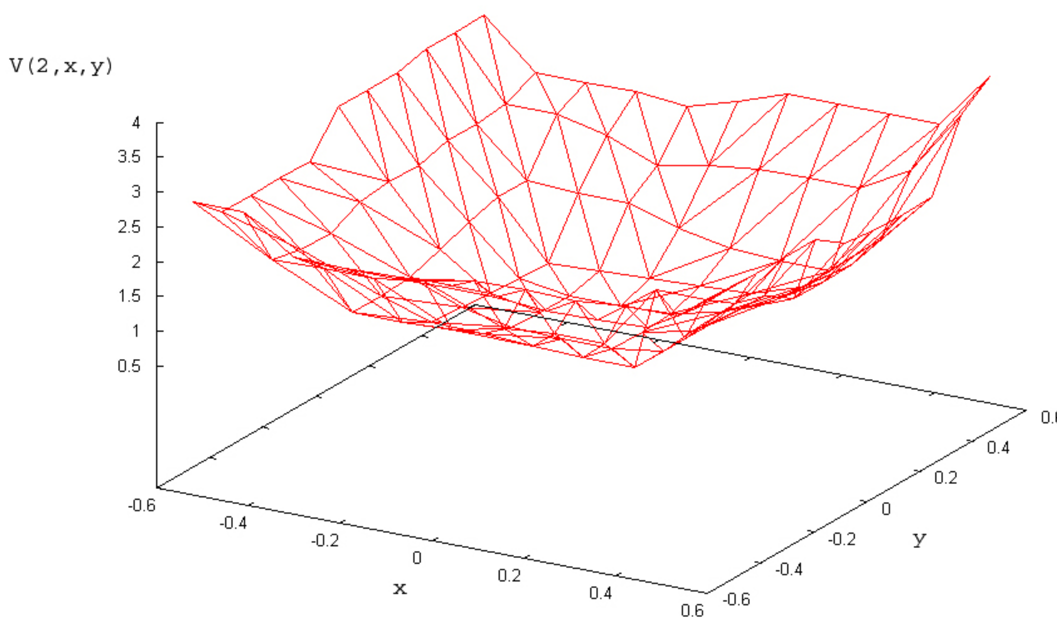


Figure 13.15: The function $(x, y) \mapsto V(2, x, y)$, where $V(t, x, y)$ is the parameterized Lyapunov function for the nonautonomous system (13.3).

Chapter 14

Final words

Because we have already thoroughly discussed the reasons for and the usefulness of the results of this thesis in *Historical background* and *The contributions of this work* at the beginning, there is no reason to repeat it here. Therefore, we let a very short review suffice.

In this work we developed an algorithm to construct Lyapunov functions for nonlinear, nonautonomous, arbitrary switched continuous systems. The necessary stability theory of switched systems, including a converse Lyapunov theorem for uniformly asymptotically stable arbitrary switched nonlinear, nonautonomous, Lipschitz systems (Theorem 3.10), was developed in Part I. In Part II we presented a linear programming problem in Definition 5.1 that can be constructed from a finite set of nonlinear and nonautonomous differential equations $\dot{\mathbf{x}} = \mathbf{f}_p(t, \mathbf{x})$, $p \in \mathcal{P}$, where the components of the $\mathbf{f}_p$ are $\mathcal{C}^2$, and we proved that every feasible solution to the linear programming problem can be used to parameterize a common Lyapunov function for the systems. Further, we proved that if the origin in the state-space is a uniformly asymptotically stable equilibrium of the switched system $\dot{\mathbf{x}} = \mathbf{f}_p(t, \mathbf{x})$, $p \in \mathcal{P}$, then Procedure 7.1, which uses the linear programming problem from Definition 5.1, is an algorithm to construct a Lyapunov function for the switched system. Finally, in Part III, we gave several examples of Lyapunov functions that we generated by use of the linear programming problem. Especially, we generated Lyapunov functions for variable structure systems with sliding modes.

It is the belief of the author that this work is a considerable advance in the Lyapunov stability theory of dynamical systems and he hopes to have convinced the reader that the numerical construction of Lyapunov functions, even for arbitrary switched, nonlinear, nonautonomous, continuous systems, is not only a theoretical possibility, but is capable of being developed to a standard tool in system analysis software in the future.

Thus, the new algorithm presented in this work should give system engineers a considerable advantage in comparison to the traditional approach of linearization and pure local analysis.

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