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On stationary and cycle-stationary sequences

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Abstract. Consider random sequences in two-sided time split into cycles by the visits to a recurrent set of states A. The two Palm dualities between stationary sequences Z and sequences with stationary cycles Z° are constructed using the change-of-measure and change-of-origin method. The first duality has the standard interpretation that Z° behaves like Z conditioned on $Z_0 \in A$. The second duality has the less known but no less important interpretation that Z° behaves like Z seen from a typical visit to A.

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1 Introduction

Toss a coin at each integer time n to obtain a doubly infinite sequence of i.i.d. coin tosses. Note that if we observe these coin tosses from the heads, then the sequence splits into independent cycles, each cycle starting at a head and continuing with tails until the next cycle starts with a head. These cycles are all of geometric length, except the cycle starting at the last head strictly before time zero and ending just before the first head at or after time zero. This particular cycle has a length which is the sum of two independent geometrics minus 1. This is the so-called "inspection (or waiting-time) paradox". Thus one of the cycles differs in distribution from the others, that is, the heads do *not* split the coin tosses into i.i.d. cycles.

Now suppose we observe that there is a head at time zero. Then conditionally on this observed fact, the coin tosses are no longer i.i.d. (unless we remove the head at the origin). On the other hand, the heads now split the coin tosses into i.i.d. *cycles* (of geometric length). Hence, conditioning on a head at the origin turns an i.i.d. sequence into a sequence with i.i.d. cycles.

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In this paper we consider a general stationary sequence $Z = (Z_k)_{k\in\mathbf{Z}}$ rather than an i.i.d. heads-and-tails sequence. The counterpart of "heads" are the time-points when the sequence is in a particular recurrent set of states A. These time-points we call simply "points" and denote the nth point by S_n with the convention that $S_{-1} < 0 \leq S_0$. The points split the process into a two-sided sequence of cycles. The conditioning on a point at time zero (that is, on $Z_0 \in A$) turns a stationary sequence Z into a sequence $Z^\circ = (Z_k^\circ)_{k\in\mathbf{Z}}$ with stationary cycles, that is, into a *cycle-stationary sequence*. We shall present the distributional relationship between these two types of sequences, stationary and cycle-stationary.

In fact, there is not only one but rather two important relationships between stationary and cycle-stationary sequences. The first is the *point-at-zero* duality already mentioned above, which says that the cycle-stationary sequence behaves as the stationary one when there happens to be a point at time zero. The second is the *randomized-origin* duality which says that the cycle-stationary sequence behaves as the stationary one with origin shifted to a point picked uniformly at random from all the points (this uniform picking should of course be understood in a limit sense). The two dualities coincide for instance in the ergodic case. An intuitive explanation of the second duality can be found at the end of Section 3.

This is a particular approach to the so-called Palm theory of stationary discrete-time sequences. The continuous-time counterpart is presented in Chapter 8 of Thorisson (2000); see the notes and references of that book for the historical background. The discrete-time case is considerably simplified by the fact that the conditioning on $S_0 = 0$ is elementary and need not be motivated by a limit theorem as in the continuous-time case (where the probability of $S_0 = 0$ is zero). On the other hand, the discrete-time case is slightly complicated by the fact that S_0 is only "conditionally uniform" and does not become "uniform and independent" when divided by $S_0 - S_{-1}$ (as in the continuous-time case).

The plan of the paper is as follows. Section 2 presents the point-at-zero duality, and Section 3 the randomized-origin duality, while Section 4 contains the main equivalence theorem on which these dualities rely.

2 The Point-at-Zero Duality

The hard part of the proof of Theorems 1 and 2 below is the equivalence between (a) and (e) in Theorem 6 in the final section.

Theorem 1. (From Stationarity to Cycle-Stationarity). *Let* $Z = (Z_k)_{k\in\mathbf{Z}}$ *be a random sequence with a general state space* $(E, \mathcal{E})$ *and supported by a probability space* $(\Omega, \mathcal{F}, \mathbf{P})$. *Let* $A \in \mathcal{E}$ *be such that the events* $\{Z_n \in A\}$

happen for infinitely many positive and infinitely many negative times n*, call such times "points". Let* $S = (S_k)_{k\in\mathbf{Z}}$ *be the increasing two-sided sequence of points with the convention that* $S_{-1} < 0 \leq S_0$*. Think of the sequence* S *as splitting* Z *into cycles and put*

$$X_n = S_n - S_{n-1} = nth\ cycle\ length\,.$$

In particular, $X_0 = S_0 - S_{-1}$ *is the length of the cycle straddling* 0*. Put*

$$Z^\circ = \theta_{S_0} Z$$

Suppose Z *is stationary under* $\mathbf{P}$*, that is,*

$$\textit{under}\ \mathbf{P}:\ \ \theta_n Z =_D Z,\ \ n \in \mathbf{Z}\,,$$

where

$$\theta_n Z = (Z_{n+k})_{k\in\mathbf{Z}}\,.$$

Then conditionally on Z° *the distribution of* S_0 *is uniform on* $\{0, \ldots, X_0 - 1\}$ *and moreover, if we define a new probability measure* $\mathbf{P}^\circ$ *on* $(\Omega, \mathcal{F})$ *by*

$$\mathbf{dP}^\circ = 1/(X_0\mathbf{E}[1/X_0])\,\mathbf{dP} \quad \textit{(length-debiasing } \mathbf{P}\textit{)}$$

then firstly:

$$\mathbf{E}^\circ[X_0] = 1/\mathbf{E}[1/X_0] < \infty$$

and

$$\mathbf{P}(S_0 = k) = \mathbf{P}^\circ(X_0 > k)/\mathbf{E}^\circ[X_0]\,;$$

secondly: Z° *is cycle-stationary under* $\mathbf{P}^\circ$*, that is, with* $S_n^\circ = S_n - S_0$ *we have* $Z_0^\circ \in A$ *and*

$$\textit{under}\ \mathbf{P}^\circ:\quad \theta_{S_n^\circ} Z^\circ =_D Z^\circ, \quad n \in \mathbf{Z}\,;$$

thirdly:

$$\mathbf{P}(Z^\circ \in \cdot | S_0 = k) = \mathbf{P}^\circ(Z^\circ \in \cdot | X_0 > k), \quad k \geq 0\,; \tag{1}$$

and finally: conditionally on a point at zero, Z *under* $\mathbf{P}$ *behaves as* Z° *under* $\mathbf{P}^\circ$*, that is,*

$$\mathbf{P}(Z \in \cdot | Z_0 \in A) = \mathbf{P}^\circ(Z^\circ \in \cdot)\,. \tag{2}$$

Proof. The conditional uniformity of S_0 under $\mathbf{P}$ and the cycle-stationarity of Z° under $\mathbf{P}^\circ$ follows from the equivalence of (*a*) and (*e*) in Theorem 6 below and the definition of $\mathbf{P}^\circ$. The claim $\mathbf{E}^\circ[X_0] = 1/\mathbf{E}[1/X_0] < \infty$ is obvious from the definition of $\mathbf{P}^\circ$. That $\mathbf{P}(S_0 = k) = \mathbf{P}^\circ(X_0 > k)/\mathbf{E}^\circ[X_0]$ can be seen as follows:

$$\begin{aligned}
\mathbf{P}(S_0 = k) &= \mathbf{E}[\mathbf{P}(S_0 = k|Z^\circ)] \\
&= \mathbf{E}[1_{\{X_0>k\}}/X_0] \qquad \text{(conditional uniformity of } S_0\text{)} \\
&= \mathbf{E}^\circ[1_{\{X_0>k\}}]/\mathbf{E}^\circ[X_0] \quad \text{(definition of } \mathbf{P}^\circ\text{)}
\end{aligned}$$

In order to prove (1), let f be a bounded measurable function from $(E, \mathcal{E})$ to $(\mathbf{R}, \mathcal{B})$. Since S_0 is uniform on $\{0, \dots, X_0 - 1\}$ conditionally on Z°, we have

$$\mathbf{E}[1_{\{S_0=k\}}|Z^\circ] = 1_{\{X_0>k\}}/X_0\,.$$

This yields the second step in

$$\begin{aligned}
\mathbf{E}[f(Z^\circ)1_{\{S_0=k\}}] &= \mathbf{E}[f(Z^\circ)\mathbf{E}[1_{\{S_0=k\}}|Z^\circ]] \\
&= \mathbf{E}[f(Z^\circ)1_{\{X_0>k\}}/X_0]\,.
\end{aligned}$$

By the definition of $\mathbf{P}^\circ$ this yields

$$\mathbf{E}[f(Z^\circ)1_{\{S_0=k\}}] = \mathbf{E}^\circ[f(Z^\circ)1_{\{X_0>k\}}]/\mathbf{E}^\circ[X_0]\,.$$

Divide by $\mathbf{P}(S_0 = k) = \mathbf{P}^\circ(X_0 > k)/\mathbf{E}^\circ[X_0]$ to obtain (1). We get (2) from (1) by taking $k = 0$, since $Z^\circ = Z$ on $\{Z_0 \in A\} = \{S_0 = 0\}$, and $1_{\{X_0>0\}} = 1$. □

Theorem 1 above associates to each stationary sequence a cycle-stationary sequence with cycle lengths having finite mean. This is done by a length-*debiasing* change-of-measure and by shifting the origin to a point. The next theorem turns this around: it associates a stationary sequence to each cycle-stationary sequence with cycle lengths having finite mean. This is done by the reversed length-*biasing* change-of-measure and a uniform shift of the origin from the point at zero into the interval ending at time zero. Due to (2), we call this the *point-at-zero duality*.

Theorem 2. (From Cycle-Stationarity to Stationarity). *Let $Z^\circ = (Z_k^\circ)_{k\in\mathbf{Z}}$ be a random sequence with a general state space $(E, \mathcal{E})$ and supported by a probability space $(\Omega, \mathcal{F}, \mathbf{P}^\circ)$. Let U be a random variable uniformly distributed on $[0, 1)$ and independent of Z°. Let $A \in \mathcal{E}$ be such that $Z_0^\circ \in A$ and such that the events $\{Z_n^\circ \in A\}$ happen for infinitely many positive and infinitely many negative times n, call such times "points". Let $S^\circ = (S_k^\circ)_{k\in Z}$ be the increasing* *two-sided sequence of points with the convention that* $S_0^\circ = 0$*. Think of the sequence* S° *as splitting* Z° *into cycles and put*

$$X_n = S_n^\circ - S_{n-1}^\circ = nth \text{ cycle length}\,.$$

In particular, $X_0 = -S_{-1}^\circ$ *is the length of the cycle ending at time zero. Put*

$$\begin{aligned} Z &= \theta_{-[UX_0]} Z^\circ \\ &= \textit{the sequence seen from a location uniformly placed in a cycle}. \end{aligned}$$

Suppose Z° *is cycle-stationary under* $\mathbf{P}^\circ$ *and* $\mathbf{E}^\circ[X_0] < \infty$*, and define a probability measure* $\mathbf{P}$ *on* $(\Omega, \mathcal{F})$ *by*

$$\mathbf{dP} = (X_0/\mathbf{E}^\circ[X_0])\mathbf{dP}^\circ \quad \textit{(length-biasing } \mathbf{P}^\circ\textit{)}$$

Then firstly:

$$\mathbf{E}[1/X_0] = 1/\mathbf{E}^\circ[X_0] > 0$$

and with $S = (S_k)_{k\in\mathbf{Z}}$ *the points of* Z*,*

$$\mathbf{P}(S_0 = k) = \mathbf{P}^\circ(X_0 > k)/\mathbf{E}^\circ[X_0]\,;$$

secondly: Z *is stationary under* $\mathbf{P}$*; thirdly:*

$$\mathbf{P}(Z^\circ \in \cdot | S_0 = k) = \mathbf{P}^\circ(Z^\circ \in \cdot | X_0 > k), \quad k \geq 0\,;$$

and finally: conditionally on a point at zero, Z *under* $\mathbf{P}$ *behaves as* Z° *under* $\mathbf{P}^\circ$*, that is,*

$$\mathbf{P}(Z \in \cdot | Z_0 \in A) = \mathbf{P}^\circ(Z^\circ \in \cdot)\,.$$

Proof. The first point of Z at or after time zero is at $S_0 = [UX_0]$ which is uniform on $\{0, \ldots, X_0 - 1\}$ conditionally on Z°. This, the cycle-stationarity of Z° under $\mathbf{P}^\circ$, the equivalence of (a) and (e) in Theorem 6 below, and the definition of $\mathbf{P}$ yields the stationarity of Z under $\mathbf{P}$. The claim $\mathbf{E}[1/X_0] = 1/\mathbf{E}^\circ[X_0]$ is obvious from the definition of $\mathbf{P}$. The remaining claims follow from Theorem 1, since Z is stationary under $\mathbf{P}$ and since the definition of $\mathbf{P}$ in Theorem 2 is the reversal of the definition of $\mathbf{P}^\circ$ in Theorem 1. □

3 The Randomized-Origin Duality

We are now at the second duality between stationarity and cycle-stationarity. In this case the hard part of the proof of Theorems 4 and 5 below is not only the equivalence between (a) and (e) in Theorem 6 in the final section, but also the following shift-coupling theorem, which we will not prove here.

Theorem 3. (Shift-Coupling). *Let $Z = (Z_k)_{k\in\mathbf{Z}}$ and $Z' = (Z'_k)_{k\in\mathbf{Z}}$ be two random sequences with a general state space $(E, \mathcal{E})$ and supported by a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. For each integer $n > 0$, let U_n be uniform on $\{-n, \dots, n\}$ and independent of Z and Z'. Let $\mathcal{I}$ be the invariant sub-σ-algebra of $\mathcal{E}^{\mathbf{Z}}$, that is,*

$$\mathcal{I} = \{B \in \mathcal{E}^{\mathbf{Z}} : \theta_n B = B \text{ for all } n \in \mathbf{Z}\}\,.$$

Let $||\cdot||$ denote the total variation norm for bounded signed measures; in particular, if P and Q are two probability measures then

$$||P - Q|| = 2 \sup_{B\in\mathcal{E}^{\mathbf{Z}}} |P(B) - Q(B)|\,.$$

Then the following claims are equivalent:

(a) $\mathbf{P}(Z \in B) = \mathbf{P}(Z' \in B), \quad B \in \mathcal{I}\,.$

(b) $||\mathbf{P}(\theta_{U_n} Z \in \cdot) - \mathbf{P}(\theta_{U_n} Z' \in \cdot)|| \to 0, \quad n \to \infty\,.$

(c) The probability space $(\Omega, \mathcal{F}, \mathbf{P})$ can be extended to support a finite random time T such that $\theta_T Z =_D Z'$.

For proof, see Thorisson (2000), Section 7.4 in Chapter 7.

The randomized-origin duality is presented in the following two theorems. Although this duality is not as elementary as the other one, it has a relatively simple intuitive explanation which is given at the end of this section.

Theorem 4. (From Stationarity to Cycle-Stationarity). *Assume the conditions of Theorem 1, in particular suppose Z is stationary under $\mathbf{P}$. Let U_n be uniform on $\{-n, \dots, n\}$ and independent of Z. Let $\mathcal{J}$ be the invariant σ-algebra of Z and Z°, that is,*

$$\mathcal{J} = \{\{Z \in B\} : B \in \mathcal{I}\} = \{\{Z^\circ \in B\} : B \in \mathcal{I}\}$$

Define a new probability measure $\mathbf{P}^\circ$ on $(\Omega, \mathcal{F})$ by

$$\mathbf{dP}^\circ = 1/(X_0\mathbf{E}[1/X_0|\mathcal{J}])\,\mathbf{dP} \quad \textit{(length-debiasing } \mathbf{P} \textit{ conditionally on } \mathcal{J}\textit{)}$$

Then Z° is cycle-stationary under $\mathbf{P}^\circ$,

$$\mathbf{E}^\circ[X_0|\mathcal{J}] = 1/\mathbf{E}[1/X_0|\mathcal{J}] < \infty\,,$$

$$\mathbf{P}(Z \in B) = \mathbf{P}^\circ(Z^\circ \in B), \quad B \in \mathcal{I}\,, \tag{3}$$

$$||\mathbf{P}(\theta_{S_{U_n}} Z \in \cdot) - \mathbf{P}^\circ(Z^\circ \in \cdot)|| \to 0, \quad n \to \infty\,, \tag{4}$$

and the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ can be extended to support a finite random integer M such that

$$\mathbf{P}(\theta_{S_M} Z \in \cdot) = \mathbf{P}^\circ(Z^\circ \in \cdot)\,.$$

Proof. We obtain (3) as follows: for $B \in \mathcal{J}$,

$$\begin{aligned}
\mathbf{P}^\circ(B) &= \mathbf{E}^\circ[1_B] = \mathbf{E}[1_B/(X_0\mathbf{E}[1/X_0|\mathcal{J}])] && \text{(by the definition of } \mathbf{P}^\circ\text{)}\\
&= \mathbf{E}[\mathbf{E}[1_B/(X_0\mathbf{E}[1/X_0|\mathcal{J}])|\mathcal{J}]] && \text{(conditioning on } \mathcal{J}\text{)}\\
&= \mathbf{E}[(1_B/\mathbf{E}[1/X_0|\mathcal{J}])\mathbf{E}[1/X_0|\mathcal{J}]] && \text{(moving out functions in } \mathcal{J}\text{)}\\
&= \mathbf{E}[1_B] = \mathbf{P}(B)\,.
\end{aligned}$$

We next prove that for all nonnegative random variables Y it holds that

$$\mathbf{E}^\circ[Y|\mathcal{J}] = \mathbf{E}[Y/X_0|\mathcal{J}]/\mathbf{E}[1/X_0|\mathcal{J}]\,. \tag{5}$$

To establish (5) take $B \in \mathcal{J}$ to obtain

$$\begin{aligned}
\mathbf{E}^\circ[1_B Y] &= \mathbf{E}[1_B Y/(X_0\mathbf{E}[1/X_0|\mathcal{J}])] && \text{(by the definition of } \mathbf{P}^\circ\text{)}\\
&= \mathbf{E}[\mathbf{E}[1_B Y/(X_0\mathbf{E}[1/X_0|\mathcal{J}])|\mathcal{J}]] && \text{(conditioning on } \mathcal{J}\text{)}\\
&= \mathbf{E}[(1_B/\mathbf{E}[1/X_0|\mathcal{J}])\mathbf{E}[Y/X_0|\mathcal{J}]] && \text{(moving out functions in } \mathcal{J}\text{)}\\
&= \mathbf{E}^\circ[(1_B/\mathbf{E}[1/X_0|\mathcal{J}])\mathbf{E}[Y/X_0|\mathcal{J}]] && \text{(due to (3))}
\end{aligned}$$

which is equivalent to (5). Due to the equivalence of (a) and (e) in Theorem 6 below, we have that for all nonnegative measurable functions f and all integers i,

$$\mathbf{E}[f(\theta_{S_i} Z)/X_0] = \mathbf{E}[f(Z^\circ)/X_0]\,.$$

Take $B \in \mathcal{I}$, note that $\{\theta_{S_i} Z \in B\} = \{Z^\circ \in B\}$, and replace $f(\theta_{S_i} Z)$ and $f(Z^\circ)$ by $f(\theta_{S_i} Z)1_{\{Z^\circ \in B\}}$ and $f(Z^\circ)1_{\{Z^\circ \in B\}}$ to obtain

$$\mathbf{E}[f(\theta_{S_i} Z)/X_0|\mathcal{J}] = \mathbf{E}[f(Z^\circ)/X_0|\mathcal{J}]\,.$$

Apply (5) with $Y = f(\theta_{S_i} Z)$ and $Y = f(Z^\circ)$ and divide by $\mathbf{E}[1/X_0|\mathcal{J}]$ to obtain from this that

$$\mathbf{E}^\circ[f(\theta_{S_i} Z)|\mathcal{J}] = \mathbf{E}^\circ[f(Z^\circ)|\mathcal{J}]\,.$$

Take expectation to obtain that Z° is cycle-stationary under $\mathbf{P}^\circ$. The rest of the theorem follows from Theorem 3 (noting that the shift-coupling time must be a point). □

Theorem 4 above associates to each stationary sequence a cycle-stationary sequence with cycle lengths having finite conditional mean. This is done by a conditional length-*debiasing* change-of-measure and by shifting the origin to a point. The next theorem turns this around: it associates a stationary sequence to each cycle-stationary sequence with cycle lengths having finite conditional mean. This is done by the reversed conditional length-*biasing* change-of-measure and a uniform shift of the origin from the point at zero into the interval ending at time zero. Due to (4) and (7), we call this the randomized origin duality.

Theorem 5. (From Cycle-Stationarity to Stationarity). *Assume the conditions of Theorem 2, in particular suppose Z° is cycle-stationary under $\mathbf{P}^\circ$. Let U_n be uniform on $\{-n, \dots, n\}$ and independent of Z°. Let $\mathcal{J}$ be the invariant σ-algebra of Z and Z°. Assume that $\mathbf{E}^\circ[X_0|\mathcal{J}] < \infty$ and define a new probability measure $\mathbf{P}$ on $(\Omega, \mathcal{F})$ by*

$$\mathbf{dP} = (X_0/\mathbf{E}^\circ[X_0|\mathcal{J}])\mathbf{dP}^\circ \qquad \textit{(length-biasing } \mathbf{P}^\circ\textit{)}\,.$$

Then Z is stationary under $\mathbf{P}$,

$$\mathbf{P}(Z \in B) = \mathbf{P}^\circ(Z^\circ \in B), \quad B \in \mathcal{I}\,, \tag{6}$$

$$||\mathbf{P}(Z \in \cdot) - \mathbf{P}^\circ(\theta_{U_n} Z^\circ \in \cdot)|| \to 0, \quad n \to \infty\,, \tag{7}$$

and the probability space $(\Omega, \mathcal{F}, \mathbf{P}^\circ)$ can be extended to support a finite random time T such that

$$\mathbf{P}(Z \in \cdot) = \mathbf{P}^\circ(\theta_T Z^\circ \in \cdot)\,.$$

Proof. Due to the cycle-stationarity of Z° under $\mathbf{P}^\circ$, we have for all nonnegative measurable functions f and all integers i,

$$\mathbf{E}^\circ[f(\theta_{S_i} Z)] = \mathbf{E}^\circ[f(Z^\circ)]\,.$$

Take $B \in \mathcal{I}$, note that $\{\theta_{S_i} Z \in B\} = \{Z^\circ \in B\}$, and replace $f(\theta_{S_i} Z)$ and $f(Z^\circ)$ by $f(\theta_{S_i} Z)1_{\{Z^\circ \in B\}}$ and $f(Z^\circ)1_{\{Z^\circ \in B\}}$ to obtain $\mathbf{E}^\circ[f(\theta_{S_i} Z)1_{\{Z^\circ \in B\}}] = \mathbf{E}^\circ[f(Z^\circ)1_{\{Z^\circ \in B\}}]$. Thus

$$\mathbf{E}^\circ[f(\theta_{S_i} Z)|\mathcal{J}] = \mathbf{E}^\circ[f(Z^\circ)|\mathcal{J}]\,.$$

In the same way as we obtained (5) we get $\mathbf{E}[Y/X_0|\mathcal{J}] = \mathbf{E}^\circ[Y|\mathcal{J}]/\mathbf{E}^\circ[X_0|\mathcal{J}]$. Apply this with $Y = f(\theta_{S_i} Z)$ and $Y = f(Z^\circ)$ and take expectations to obtain

$$\mathbf{E}[f(\theta_{S_i} Z)/X_0] = \mathbf{E}[f(Z^\circ)/X_0]\,.$$

Thus (e) in Theorem 6 below holds, and the equivalence of (e) and (a) yields the stationarity of Z under $\mathbf{P}$. We obtain (6) in the same way as (3) and the rest of the theorem follows from Theorem 3. □

Here is an intuitive explanation to the above theorem. Consider a cycle-stationary sequence Z° and suppose we can pick an integer at random on the whole line. Then firstly, the process seen from there should be stationary. Secondly, the position of the integer in the interval where it lands should be uniform. Thirdly, a particular interval of length k will be picked with probability proportional to k and, due to the ergodic theorem, conditionally on $\mathcal{J}$ the number of such intervals is proportional to $\mathbf{P}^\circ(X_0 = k|\mathcal{J})$; thus some interval of length k is picked with probability proportional to $k\mathbf{P}^\circ(X_0 = k|\mathcal{J})$ which is exactly the result of the length-biasing change of measure in Theorem 5.

4 Key Equivalence Theorem

The following theorem was the key to the above dualities.

Theorem 6. *Let $Z = (Z_k)_{k\in\mathbf{Z}}$ be a random sequence with a general state space $(E, \mathcal{E})$ and supported by a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Let $A \in \mathcal{E}$ be such that the events $\{Z_n \in A\}$ happen for infinitely many positive and infinitely many negative times n, call such times "points". Let $S = (S_k)_{k\in\mathbf{Z}}$ be the increasing two-sided sequence of points with the convention that $S_{-1} < 0 \le S_0$. Put $Z^\circ = \theta_{S_0} Z$. For nonnegative integers k, put $N_k = \inf\{n \ge 0 : S_n \ge k\}$. Then the following statements are equivalent:*

(a) Z is stationary under $\mathbf{P}$.

(b) For all nonnegative measurable functions f and nonnegative integers n,

$$\mathbf{E}[\sum_k 1_{\{S_0 < k \le S_{N_n}\}} f(\theta_k Z)/X_{N_k}] = n\,\mathbf{E}[f(Z)/X_0]\,. \qquad (8)$$

(c) For all nonnegative measurable functions f *and all nonnegative integers* n *and* m,

$$\mathbf{E}[\sum_{1\le i\le N_n} f(\theta_{S_i}Z)1_{\{X_i>m\}}/X_i] = n\mathbf{E}[1_{\{S_0=m\}}f(Z^\circ)/X_0]\,. \tag{9}$$

(d) For all nonnegative measurable functions f *and nonnegative integers* n,

$$\mathbf{E}[\sum_{1\le i\le N_n} f(\theta_{S_i}Z)] = n\,\mathbf{E}[f(Z^\circ)/X_0]\,, \tag{10}$$

and conditionally on Z° *the variable* S_0 *is uniform on* $\{0,\dots,X_0-1\}$, *that is,*

$$\mathbf{E}[1_{\{S_0=m\}}f(Z^\circ)] = \mathbf{E}[1_{\{X_0>m\}}f(Z^\circ)/X_0], \quad m\ge 0\,.$$

(e) For all nonnegative measurable functions f *and integers* i

$$\mathbf{E}[f(\theta_{S_i}Z)/X_0] = \mathbf{E}[f(Z^\circ)/X_0] \tag{11}$$

and conditionally on Z° *the variable* S_0 *is uniform on* $\{0,\dots,X_0-1\}$.

We prove this theorem circularly.

Proof that *(a)* implies *(b)*. Assume that (a) holds. First suppose f is bounded, say $f\le a$. Since $N_k=0$ for $0<k\le S_0$ and since $S_0<X_0$ we have

$$\sum_k 1_{\{0<k\le S_0\}}f(\theta_k Z)/X_{N_k}] \le aS_0/X_0\le a\,.$$

Thus the expectation of the left-hand side is finite which allows us to split the expectation in the final step in the following calculation

$$\begin{aligned} n\mathbf{E}[f(Z)/X_0] &= \sum_{1\le k\le n}\mathbf{E}[f(\theta_k Z)/X_{N_k}] \qquad \text{(by stationarity)}\\ &= \mathbf{E}[\sum_{1\le k\le n} f(\theta_k Z)/X_{N_k}]\\ &= \mathbf{E}[\sum_k 1_{\{0<k\le S_0\}}f(\theta_k Z)/X_{N_k}]\\ &\quad +\mathbf{E}[1_{\{S_0<n\}}\sum_k 1_{\{S_0<k\le n\}}f(\theta_k Z)/X_{N_k}]\\ &\quad -\mathbf{E}[1_{\{S_0\ge n\}}\sum_k 1_{\{n<k\le S_0\}}f(\theta_k Z)/X_{N_k}] \end{aligned}$$

Applying stationarity to the first term on the right yields the first step in

$$\begin{aligned}
&\mathbf{E}[\sum_k 1_{\{0<k\le S_0\}} f(\theta_k Z)/X_{N_k}] \\
&\quad= \mathbf{E}[\sum_k 1_{\{n<k\le S_{N_n}\}} f(\theta_k Z)/X_{N_k}] \\
&\quad= \mathbf{E}[1_{\{S_0<n\}} \sum_k 1_{\{n<k\le S_{N_n}\}} f(\theta_k Z)/X_{N_k}] \\
&\qquad\quad + \mathbf{E}[1_{\{S_0\ge n\}} \sum_k 1_{\{n<k\le S_{N_n}\}} f(\theta_k Z)/X_{N_k}]
\end{aligned}$$

Combining these two calculations yields

$$\begin{aligned}
n\,\mathbf{E}[f(Z)/X_0] &= \mathbf{E}[1_{\{S_0<n\}} \sum_k 1_{\{n<k\le S_{N_n}\}} f(\theta_k Z)/X_{N_k}] \\
&\quad + \mathbf{E}[1_{\{S_0\ge n\}} \sum_k 1_{\{n<k\le S_{N_n}\}} f(\theta_k Z)/X_{N_k}] \\
&\quad + \mathbf{E}[1_{\{S_0<n\}} \sum_k 1_{\{S_0<k\le n\}} f(\theta_k Z)/X_{N_k}] \\
&\quad - \mathbf{E}[1_{\{S_0\ge n\}} \sum_k 1_{\{n<k\le S_0\}} f(\theta_k Z)/X_{N_k}]\,.
\end{aligned}$$

If $S_0 \ge n$ then $S_0 = S_{N_n}$ and thus the second and fourth term on the right are identical but have different signs and thus cancel. Adding the two remaining terms on the right yields

$$n\mathbf{E}[f(Z)/X_0] = \mathbf{E}[1_{\{S_0<n\}} \sum_k 1_{\{S_0<k\le S_{N_n}\}} f(\theta_k Z)/X_{N_k}]\,. \qquad (12)$$

Since $S_0 \ge n$ implies $S_0 = S_{N_n}$ we have

$$\mathbf{E}[1_{\{S_0\ge n\}} \sum_k 1_{\{S_0<k\le S_{N_n}\}} f(\theta_k Z)/X_{N_k}] = 0\,.$$

Add this to (12) to obtain (8) for f bounded. In order to remove the boundedness restriction replace f by $f \wedge a$ in (8) and apply monotone convergence once on the left hand side and twice on the right hand side to obtain that (*a*) implies (*b*).

Proof that *(b)* implies *(c)*. Assume that (*b*) holds. Since

$$1_{\{S_{N_k}-k=m\}} f(\theta_{S_{N_k}} Z) \qquad [\text{which equals } 1_{\{S_0=m\}} f(Z^\circ) \text{ when } k=0]$$

is the same mapping of $\theta_k Z$ for all k, we obtain from (b) the first equality in

$$
\begin{aligned}
&n\mathbf{E}[1_{\{S_0=m\}} f(Z^\circ)/X_0] \\
&\quad = \mathbf{E}[\sum_k 1_{\{S_0<k\le S_{N_n}\}} 1_{\{S_{N_k}-k=m\}} f(\theta_{S_{N_k}} Z)/X_{N_k}] \\
&\quad = \mathbf{E}[\sum_{1\le i\le N_n} \sum_k 1_{\{S_{i-1}<k\le S_i\}} 1_{\{S_i-k=m\}} f(\theta_{S_i} Z)/X_i] \\
&\quad = \mathbf{E}[\sum_{1\le i\le N_n} \sum_k 1_{\{0<k\le X_i\}} 1_{\{X_i-k=m\}} f(\theta_{S_i} Z)/X_i] \\
&\quad = \mathbf{E}[\sum_{1\le i\le N_n} f(\theta_{S_i} Z) 1_{\{X_i>m\}}/X_i]\,.
\end{aligned}
$$

Thus (c) holds.

Proof that *(c)* implies *(d)*. Assume that (c) holds. Summing over m in (9) yields (10) since

$$\sum_{m\ge 0} 1_{\{X_i>m\}} = X_i \qquad \text{and} \qquad \sum_{m\ge 0} 1_{\{S_0=m\}} = 1\,.$$

In order to establish the conditional uniformity of S_0, note that if we replace $f(\theta_{S_i} Z)$ in (9) by $f(\theta_{S_i} Z)X_i$ [and $f(Z^\circ)$ by $f(Z^\circ)X_0$] then we obtain

$$\mathbf{E}[\sum_{1\le i\le N_n} f(\theta_{S_i} Z) 1_{\{X_i>m\}}] = n\mathbf{E}[1_{\{S_0=m\}} f(Z^\circ)]$$

while if we replace $f(\theta_{S_i} Z)$ in (10) by $f(\theta_{S_i} Z)1_{\{X_i>m\}}$ and $f(Z^\circ)$ by $f(Z^\circ)\ 1_{\{X_0>m\}}$ then we obtain

$$\mathbf{E}[\sum_{1\le i\le N_n} f(\theta_{S_i} Z) 1_{\{X_i>m\}}] = n\mathbf{E}[1_{\{X_0>m\}} f(Z^\circ)/X_0]\,.$$

Since the left-hand sides of the last two identities are identical, so are the right-hand sides and we obtain the following: for all nonnegative measurable functions f and all integers $m \ge 0$ it holds that

$$\mathbf{E}[1_{\{S_0=m\}} f(Z^\circ)] = \mathbf{E}[1_{\{X_0>m\}} f(Z^\circ)/X_0]\,.$$

This is the definition of the claim that conditionally on Z° the distribution of S_0 is uniform on $\{0, \dots, X_0 - 1\}$.

Proof that *(d)* implies *(e)*. We obtain (e) from (d) if we can show that (10) implies (11). For that purpose assume that (10) holds for all nonnegative measurable f and all nonnegative integers n. Let j be an arbitrary integer and apply (10) with $f(\theta_{S_i}Z)$ replaced by $f(\theta_{S_{i+j}}Z)$ [and thus $f(Z)$ replaced by $f(\theta_{S_j}Z)$] to obtain the first equality in

$$
\begin{aligned}
n\mathbf{E}[f(\theta_{S_j}Z)/X_0] &= \mathbf{E}[\sum_{1\le i\le N_n} f(\theta_{S_{i+j}}Z)] \\
&= \mathbf{E}[\sum_{1\le i\le N_n} f(\theta_{S_i}Z)] - \mathbf{E}[\sum_{1\le i\le j} f(\theta_{S_{i+j}}Z)] + \mathbf{E}[\sum_{N_n+1\le i\le N_n+j} f(\theta_{S_i}Z)]\,.
\end{aligned}
$$

Let f be bounded, say $f \le a$, divide by n and apply (10) to the first term on the right to obtain

$$
|\mathbf{E}[f(\theta_{S_j}Z)/X_0] - \mathbf{E}[f(Z^\circ)/X_0]| \le a|j|/n\,.
$$

Send n to infinity to obtain (11) for f bounded. Apply monotone convergence to remove the boundedness of f.

Proof that *(e)* implies *(a)*. Assume that (e) holds. Then the conditional uniformity of S_0 yields the second step in

$$
\begin{aligned}
\mathbf{E}[f(\theta_n Z)] &= \mathbf{E}[f(\theta_{n-S_0}Z^\circ)] \\
&= \mathbf{E}[\sum_k 1_{\{n-X_0<k\le n\}} f(\theta_k Z^\circ)/X_0] \qquad (13)\\
&= \sum_i \mathbf{E}[\sum_k 1_{\{n-X_0<k\le n\}} 1_{\{S^\circ_{-i-1}<k\le S^\circ_{-i}\}} f(\theta_k Z^\circ)/X_0]\,.
\end{aligned}
$$

Apply (11) with $f(Z^\circ)$ replaced by

$$
1_{\{n-X_0<k\le n\}} 1_{\{S^\circ_{-i-1}<k\le S^\circ_{-i}\}} f(\theta_k Z^\circ)
$$

and $f(\theta_{S_i}Z)$ by $1_{\{n-X_i<k\le n\}}\ 1_{\{-S_i^\circ-X_0<k\le -S_i^\circ\}}\ f(\theta_k\theta_{S_i}Z)$ to obtain

$$
\begin{aligned}
&\mathbf{E}[1_{\{n-X_0<k\le n\}} 1_{\{S^\circ_{-i-1}<k\le S^\circ_{-i}\}} f(\theta_k Z^\circ)/X_0] \\
&\quad = \mathbf{E}[1_{\{n-X_i<k\le n\}} 1_{\{-S_i^\circ-X_0<k\le -S_i^\circ\}} f(\theta_k\theta_{S_i}Z)/X_0]\,.
\end{aligned}
$$

Sum over k, move the sum inside the expectation, and make the variable substitution $m = k + S_i^\circ$ in the sum on the right-hand side to obtain

$$
\begin{aligned}
&\mathbf{E}[\sum_k 1_{\{n-X_0<k\le n\}} 1_{\{S^\circ_{-i-1}<k\le S^\circ_{-i}\}} f(\theta_k Z^\circ)/X_0] \\
&\quad = \mathbf{E}[\sum_m 1_{\{n+S^\circ_{i-1}<m\le n+S_i^\circ\}} 1_{\{-X_0<m\le 0\}} f(\theta_m Z^\circ)/X_0]\,.
\end{aligned}
$$

Sum over i and apply (13) to obtain

$$\mathbf{E}[f(\theta_n Z)] = \sum_i \mathbf{E}[\sum_m 1_{\{n+S_{i-1}<m\leq n+S_i^\circ\}} 1_{\{-X_0<m\leq 0\}} f(\theta_m Z^\circ)/X_0] \, .$$

Since $\sum_i 1_{\{n+S_{i-1}^\circ<m\leq n+S_i^\circ\}} = 1$ we obtain

$$\mathbf{E}[f(\theta_n Z)] = \mathbf{E}[\sum_m 1_{\{-X_0<m\leq 0\}} f(\theta_m Z^\circ)/X_0] \, .$$

Thus $\mathbf{E}[f(\theta_n Z)]$ does not depend on n. Thus Z is stationary, that is, (e) implies (a) and the proof of Theorem 6 is complete. □

References

[1] Thorisson, H. *Coupling, Stationarity, and Regeneration*. Springer, New York, (2000).

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