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DISTRICT HEAT DISTRIBUTION NETWORKS

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ABSTRACT

This paper treats the mathematics for calculating flow, pressure and temperature in a heat distribution network, when the network has loops. A looped network cannot be calculated directly, and the flow and temperature solution has to be obtained by iteration of the non-linear system equations. Network theory is used to reduce the number of equations which are iterated. The thermal solution of the network is then found without requiring iteration.

An example is given of an analysis of the Balcova district heating network in Turkey.

1. MICROSCOPIC MODELS

The goal is to calculate temperature, heat flow, pressure and water flow for the distribution network. A district heating model has to be able to:

- Calculate water flow in all system elements; and
- Calculate head at all nodes.

Where

- Some elements have known flow; and
- Some nodes have known head.

The unknowns are:

- The element flow; and
- The head at the nodes.

The constraints are:

- Kirchhoff's current law;
- Kirchhoff's voltage law; and
- Elements (branch) relations.

1.1 Kirchhoff's current law

The sum of the mass flows at any node equals 0 at any time. This results in one equation for each node:

$$\sum_{j=1}^{n_n} a_{ij}\ m_j = 0 \tag{1}$$

1.2 Kirchhoff's voltage law

The sum of all voltage (potential) differences along any closed path (loop) in the network is zero. This results in one equation for each loop:

$$\sum_{j=1}^{n_n} b_{ij}\ h_j = 0 \tag{2}$$

1.3 Elements relations

The element relations add one equation for each element, relating flow and head loss:

$$h_j = f(m_j) \tag{3}$$

This is the so-called resistance formulation, where $f(m_j)$ is a non-linear head loss function. This equation can be inverted in order to give the conductivity formulation:

$$m_j = g(h_j) \tag{4}$$

1.4 Direct mass flow solution

A solution of these two sets of equations will give the flow in all elements. The head change and subsequentially the nodal head can be found from the element relations. The resistance formulation is used here.

$$\sum_{j=1}^{n_n} a_{ij}\ m_j = 0 \tag{5}$$

$$\sum_{j=1}^{n_n} b_{ij}\ f(m_j) = 0 \tag{6}$$

1.5 Direct head loss solution

A solution of these two sets of equations will give the head loss in all elements. The flow and subsequentially the nodal head can be found from the element relations. The conductivity formulation is used here.

$$\sum_{j=1}^{n_n} a_{ij} h^{-1}(h_j) = 0 \tag{7}$$

$$\sum_{j=1}^{n_l} b_{ij} h_j = 0 \tag{8}$$

1.6 Previous methods

Three linearization and solution methods have been traditionally applied. These are the Hardy – Cross method, the Newton – Raphson gradient iteration and the Wood and Charles linearization (Figure 1).

The Hardy-Cross method is an error correction method. An initial guess value is set for all elements. The head losses are calculated and added along the loops in the system (which is to sum to zero according to Kirchhoff's voltage law), and the error is calculated. Then a flow change in all the loop elements, necessary to make the error zero is found, and a new set of element flows is defined. This is done for all the loops, and repeated until the error vanishes. This method is stable, but requires high number of iterations.

The Newton-Raphson method is a linearization method, the non-linear equations are linearized by the gradient corresponding to the guess value, and a new value calculated according to the solution of the linearized equation system. This method does converge in a few iterations, if it converges at all. A good set of guess values is needed for the method to work. Many commercial programs use Hardy-Cross to obtain a good set of guess values for the Newton-Raphson method.

The Wood and Charles method is as well a linearization method, but the linearization is made by a chord going through origo instead of a tangent as in the Newton-Raphson method. This method converges almost as quickly as Newton-Raphson, but is stable, and can be formulated so, that a set of good initial guess values is generated automatically. The head difference in the pipe branches is usually a quadratic function of the flow. The Newton-Raphson (classical) linearization can give results that cause problems in the iteration, particularly if the flow becomes less than one half of the flow value on which the linearization is based. In that case the head difference will become negative. For the quadratic flow resistance, the slope of the Wood and Charles linearization will be one half of that resulting from the classical linearization. This is indeed a crude approximation, but will result in a robust iteration and have acceptable convergence by averaging two successive iterations.

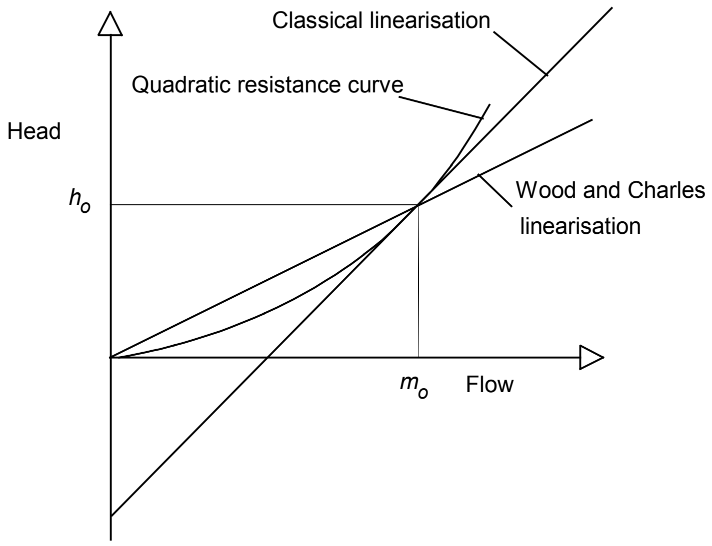


FIGURE 1: Comparison of linearization methods

2. GRAPH THEORY FOR DISTRICT HEATING NETWORK MODELING

In the modelling of district heating network these basic laws have to be fulfilled:

- Conservation of mass;
- Conservation of momentum; and
- Conservation of energy.

The graph theory considers a network to be a composite concept of:

- A set of nodes (x, y, z);
- A set of branches; and
- A connectivity relation (ni, nj).

2.1 Definitions

The general network analysis presented here follows the terminology commonly used in network theory.

Path: A set of branches b_1 ... b_n in the graph G_n is a path between nodes V_j and V_k if consecutive branches b_i and b_i+1 have a common endpoint, no node of G_n is an endpoint of more than two of the branches in the set, and V_i as well as V_j are endpoints of exactly one branch of in the set.
Connected graph: A graph G_n is connected if there is a path between any two nodes of the graph.

Loop: A subgraph G_s is a loop if G_s is connected and every branch of G_s has exactly two nodes of G_s incident at it. Associated with the loop is a direction specified by the direction of a given datum branch in the loop.

Tree: A subgraph G_s of the connected graph G_n is a tree if it is connected and G_s has no loops.

Spanning tree: A subgraph G_s of the connected graph G_n is a spanning tree if it is connected, G_s contains all nodes of G_n and G_s has no loops.

Cutset: A set of branches of a connected graph G_n (not their endpoints) is a cutset if the removal of these branches results in a graph that is not connected, and the restoration of any one of these branches results in the graph being connected again. The cutset can be seen as a border going through the graph. Associated with the cutset is a direction specified by the direction of a given datum branch in the cutset. The separate graphs obtained by removing the branches of the cutset are called components of the graph with respect to the cutset. The net flow over the cutset must be zero in order to conserve the mass in each of the components.

Link: The branches not belonging to a tree T are called links.

Cotree: The set of links in a network with a tree T is named cotree with respect to the tree T.

2.2 Element types

The flow solution of a network has three element types:

p: pipes;
m: flow elements; and
h: head elements.

2.2.1 Pipes

Here the word "pipe" is used in a general sense that is a conduit carrying a fluid from one point in space to another, and can have many elements, pumps, valves, etc. A pipe element is simply a set of serially connected physical element in the network having some relation between flow and head change.

2.2.2 Flow elements

Flow elements have a constant, known flow. They are usually used to define consumption point in the network, and have than one end connected to a datum or zero point.

2.2.3 Head elements

Head elements have a constant, known head difference between the element connection points. They are often used to define a supply point, and have than one end connected to a datum or zero point.

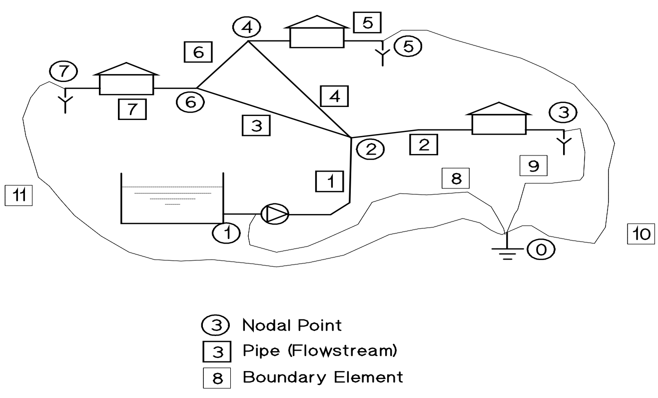
2.3 The connectivity relation

The incidence or connectivity relation relates each branch to a pair of nodes, the node where the branch originates and the node where it ends. A distribution system can be treated as a connected graph, where the pipes correspond to branches and the nodes to points where the pipes divide or are united, or convey the flow to the consumer. In network theory an incidence (or connectivity) matrix must be defined in order to describe the above mentioned connectivity relation for a network with n_n nodes and n_f branches:

Matrix **A** is an $n_n \cdot n_f$ matrix, with entries a_{ij} where:

a_{ij} = 1 if pipe j starts at node i;
a_{ij} = -1 if pipe j ends at node j; and
a_{ij} = 0 otherwise.

The connectivity matrix as defined above has one column for each flow stream in the system, and one row for each node. Each column can only have two non-zero entries, -1 and 1, as the flow stream has to originate somewhere and end at some other location. A simple district heating system, containing typical elements of such a system is shown in Figure 2 along with the associated connectivity matrix.



Boundary Elements

$$\underline{\underline{A}} = \left[\begin{array}{ccccccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

FIGURE 2: The connectivity matrix for a simple district heating system

2.4 Continuity equation (Kirchhoff's current law)

Continuity for the mass in a pipe network can be defined by reference to the current law of Kirchhoff: *The sum of the mass flows at any node equals 0 at any time.*

The connectivity matrix has a row for every node in the system. In each row all entries of 1 represent an outgoing flow stream from that node, and entries of -1 an incoming flow stream. The system flow can conveniently be stated by means of a column vector with n_j entries, each stating the flow in the corresponding flow stream. A positive flow indicates flow in the same direction as defined in the connectivity matrix, a minus signs an opposite flow direction. By using the connectivity matrix this becomes:

$$\mathbf{A}\,\mathbf{m} = \mathbf{0} \tag{9}$$

2.5 Momentum equation (Kirchhoff's voltage law)

The node piezometric head is conveniently stated in the column vector $\mathbf{h}_n$ with n_n entries, each stating the head at the corresponding node. As the connectivity matrix contains information on which flow streams connect to each node in the corresponding row, it is possible to calculate the head difference between the ends of all pipes in a vector form:

$$\mathbf{A}^T \mathbf{h}_n = \mathbf{h} \tag{10}$$

2.6 Definition of spanning tree

The choice of a spanning tree is usually based on a certain order of preference in electrical circuit analysis. The following normal tree algorithm can be used to define the spanning tree used for the network calculations:

1. Sort the network branches in the following order:

   h: Head sources;
   p: Pipes; and
   m: Flow sources.

2. Consider the next branch in the sorted list.

3. Check if the new branch will form a loop in the network. (If one and only one node of the new branch nodes is incident at a tree branch, the new branch will not form a loop). If yes, then do not add it to the tree *T*, but to the cotree L. If no, add it to the tree *T*.

Go back to step 2.

Repeat this until all nodes in the network are covered by tree branches.

Note that the check on whether a new branch will not form a loop is specified. This is because that it is not easy to check whether the new branch forms a loop, as specified in the references mentioned. One might expect that if both nodes of a new branch are already incident at tree branches, then the new branch will form a loop. However, this will only be the case when the tree is a connected graph. There is no guarantee that this will be true. It is obvious that if the new branch is not at all connected to the tree, it will be added to it, as it will not form a loop. Then the tree is not a connected graph anymore. The branch that finally connects the components of the tree will have both nodes incident at tree branches, and will therefore be wrongly assumed to form a loop.

The check on the new branches used in the algorithms above assures that the tree will be correctly formed, but assures at the same time that it will be a connected graph at any stage during the selection of the spanning tree.

The tree algorithm assumes that the number of resistors is low compared to the number of pipes, and that no loop will be formed only by resistors, storage tanks and head sources. It is possible to reduce this condition to that of prohibiting only loops composed of head sources, by treating the resistors and the storage tanks in a manner similar to that for the pipes. This will complicate the analysis, and is not relevant to a district heating system, where the majority of elements are pipes. A loop made only of head sources is in violation of the voltage law of Kirchhoff, as the heads around the loop do not necessarily sum up to zero. At least one element in the loop must be such that the head difference is not prescribed in order to fulfill this law.

Cutsets of flow sources are also prohibited. A cutset made up only of flow sources is in violation of the current law of Kirchhoff, as the flow in the cutset branches does not necessarily sum up to zero. Therefore at least one branch in the cutset must be such that the flow is not prescribed in order to fulfill this law.

The graph for the piping system is closed as all boundary points of the physical system are connected with the datum point by some combination of sources and resistors. Therefore, this condition corresponds to requiring that at least one boundary node of the physical system be attached to a head source. That is quite reasonable, because otherwise the pressure level of the network cannot be determined.

The connectivity matrix **A** can be rearranged with respect to a spanning tree *T* containing n_T branches by splitting it into two sub-matrices $\mathbf{A}_T$ and $\mathbf{A}_L$ in the following manner:

$$\mathbf{A} = \left[\mathbf{A}_T \mid \mathbf{A}_L\right] \tag{11}$$

The sub-matrix $\mathbf{A}_T$ is the $n_n \cdot n_T$ connectivity matrix for the branches of the spanning tree, and the matrix $\mathbf{A}_L$ is the $n_n \cdot n_L$ connectivity matrix for the links, where n_L denotes the number of links. The sum of n_T and n_L is n_f, the total number of branches in the network. As the datum point is not included in the connectivity matrix, and the sub-matrix $\mathbf{A}^T$ is based on a spanning tree, $n_n = n_T$. Therefore $\mathbf{A}^T$ is a square invertible matrix.

2.7 The cutset matrix

A cutset matrix is a matrix with one row for a cutset in the network, and one column for every branch. The entries of the cutset matrix are as follows:

- d_{ij} = 1 denotes that branch *j* is a member of the cutset *i* with same direction;
- d_{ij} = -1 that branch *j* is a member with opposite direction; and
- d_{ij} = 0 that branch *j* is not member of cutset *i*.

It follows from the definition of a spanning tree, that every tree branch is member of one and only one cutset, together with some (or no) links, but no other tree branches. Such cutsets are called fundamental cutsets with respect to the spanning tree *T*. The fundamental cutset matrix **D** is an $n_T \cdot n_f$ matrix, partitioned as follows:

$$\mathbf{D} = \mathbf{A}_T^{-1}\mathbf{A} = \mathbf{A}_T^{-1}\left[\mathbf{A}_T \mid \mathbf{A}_L\right] = \left[\mathbf{I} \mid \mathbf{A}_T^{-1}\mathbf{A}_L\right] \tag{12}$$

As the tree branches are members of and only one fundamental cutset, the tree part of the matrix is the identity matrix. The submatrix $\mathbf{D}_L$ reflects the membership of the links in every fundamental cutset.

2.8 The loop matrix

A loop matrix is a matrix with one row for each loop in the network, and one column for each branch. The entries of the loop matrix are as follows:

b_{ij} = 1 denotes that branch *j* is a member of the loop *i* with same direction;
b_{ij} = -1 that branch *j* is a member with opposite direction; and
b_{ij} = 0 that branch *j* is not a member of loop *i*.

It follows from the definition of a spanning tree, that every link is a member of one and only one loop together with some tree branches, but no other links. Such loops are called fundamental loops with respect to the cotree *L*.

The fundamental loop matrix **B** is an $n_L \cdot n_f$ matrix, partitioned as follows:

$$\mathbf{B} = \left[\mathbf{B}_T \middle| \mathbf{I}\right] \tag{13}$$

As the links are members of one and only one fundamental loop, the link part of the matrix is the identity matrix. The sub-matrix $\mathbf{B}_T$ reflects the membership of the tree branches in every fundamental loop.

2.9 Loop and cutset relations

If the cutset gets into a loop, it has to go out of the loop again. The number of elements common both to the loop and the cutset will thus always have an even number. At one intersection of loop and cutset, the directions will coincide, but be opposite at the other (Figure 3).

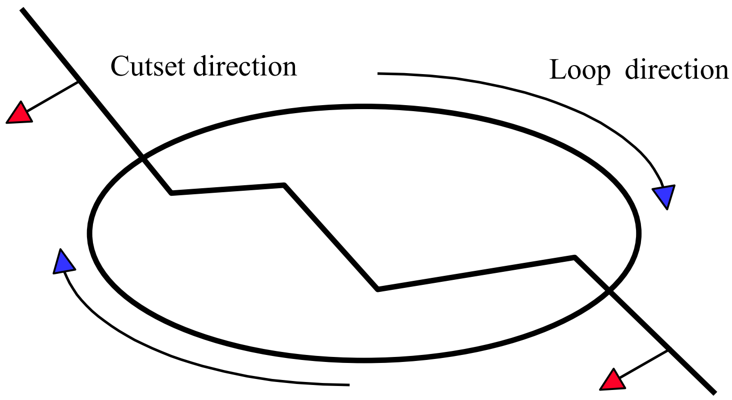


FIGURE 3: The loop and cutset relations

Both the **B** and **D** matrices have one column for every branch in the graph. If both matrices are arranged in the same column order, the following relationship holds:

$$[\mathbf{I} \mid \mathbf{D}_L] \cdot \left[\frac{\mathbf{B}_T^T}{\mathbf{I}} \right] = \mathbf{0}$$

From this, it can be seen that:

$$\mathbf{B}_T^T = -\mathbf{D}_L \tag{14}$$

Combining this with equation (12), the **B** matrix is calculated as:

$$\mathbf{B} = \left[-\mathbf{D}_L^T \middle| \mathbf{I} \right] = \left[-\left(\mathbf{A}_T^{-1} \mathbf{A}_L \right)^T \middle| \mathbf{I} \right] \tag{15}$$

2.10 Flow elements grouping

The fluid flow vector is divided into four groups. The flow elements, where the flow is known, the head elements, where the head is known, and the pipes, where neither flow nor head is known. All the head elements are members of the spanning tree, and all the flow elements of the cotree. The pipes are divided into tree pipes and cotree pipes.

$$\mathbf{m} = \begin{bmatrix} \mathbf{m}_{hT} \\ \mathbf{m}_{pT} \\ \hline \mathbf{m}_{pL} \\ \mathbf{m}_{mL} \end{bmatrix} \tag{16}$$

The current law of Kirchhoff now looks a little bit different:

$$[\mathbf{A}_T \mid \mathbf{A}_L] \begin{bmatrix} \mathbf{m}_{hT} \\ \mathbf{m}_{pT} \\ \hline \mathbf{m}_{pL} \\ \mathbf{m}_{mL} \end{bmatrix} = 0 \tag{17}$$

2.11 The partitioned cutset equation

The cutset matrix is calculated from the connectivity matrix. The connectivity matrix can be used to calculate the net flow at every node, which has to equal zero. In the same way, the net flow in every cutset equals zero, and the cutset or connectivity matrices can either be used for establishing the mass conservation in the network. Mass conservation (Kirchhoff's current law) by the cutset matrix is:

$$[\mathbf{I} \mid \mathbf{D}_L] \left[\frac{\mathbf{m}_T}{\mathbf{m}_L} \right] = \mathbf{0} \tag{18}$$

The cutset matrix is then partitioned into submatrices according to the various branch groups. The partition lines indicate the partitioning between the tree and the cotree as shown in equation (16). Note that there cannot be any flow sources among the tree branches and only pipes and flow sources can occur among the link branches.

$$\underbrace{\begin{bmatrix} \mathbf{I}_{hT} & 0 \\ 0 & \mathbf{I}_{pT} \end{bmatrix}}_{\mathbf{I}} \underbrace{\begin{bmatrix} \mathbf{F}_{11} & \mathbf{F}_{12} \\ \mathbf{F}_{21} & \mathbf{F}_{22} \end{bmatrix}}_{\mathbf{D}_L} \begin{bmatrix} \mathbf{m}_{hT} \\ \mathbf{m}_{pT} \\ \hline \mathbf{m}_{pL} \\ \mathbf{m}_{mL} \end{bmatrix} = \mathbf{0} \tag{19}$$

2.12 The partitioned loop equation

The nodal analysis does not require a specific treatment of the voltage law, as the system heads (pressures) are only expressed at the nodes. The head differences over the loop branches can then be calculated from the nodal heads, and will sum up to zero for any closed path in the network. The voltage law specifies that the sum of voltage (head) differences for any loop in the network shall be zero. This can be written for a pipe network using the fundamental loop matrix as:

$$\mathbf{B}\,\mathbf{h} = \mathbf{0} \tag{20}$$

The loop matrix can then be partitioned into submatrices according to the various branch categories. The partition lines indicate the partitioning between the tree and the cotree as shown in equation (16). The submatrices in the loop matrix tree part are obtained from equation (19) by equation (14).

$$\left[\begin{array}{cc|cc} -\mathbf{F}_{11}^T & -\mathbf{F}_{21}^T & \mathbf{I}_{pL} & \mathbf{0} \\ -\mathbf{F}_{12}^T & -\mathbf{F}_{22}^T & \mathbf{0} & \mathbf{I}_{mL} \end{array}\right] \begin{bmatrix} \mathbf{h}_{hT} \\ \mathbf{h}_{pT} \\ \hline \mathbf{h}_{pL} \\ \mathbf{h}_{mL} \end{bmatrix} = \mathbf{0} \tag{21}$$

2.13 Element relations

The pipes in the network have relation between the head loss and the flow. The matrix notation of the resistance formulation is:

Tree pipe head vector:

$$\mathbf{h}_{pT} = \mathbf{r}_{pT}(\mathbf{m}_{pT}) \tag{22}$$

Link pipe head vector:

$$\mathbf{h}_{pL} = \mathbf{r}_{pL}(\mathbf{m}_{pL}) \tag{23}$$

When an appropriate linearization method has been used, the element relations can be used in order to solve for the network flow. The Wood and Charles linearization changes equations (22) and (23) into linear matrix equations, using a diagonal resistance matrix to relate flow and head loss:

$$\mathbf{h}_{pT} = \mathbf{R}_{pT}\mathbf{m}_{pT} \tag{24}$$

$$\mathbf{h}_{pL} = \mathbf{R}_{pL}\mathbf{m}_{pL} \tag{25}$$

3. BRANCH CHARACTERISTICS – RESISTORS

The resistance function relates the head loss to the flow and the element parameters (diameter, surface roughness etc). The function is defined both for a single pipe (scalar) as *r*(*m,parameters*) and a set of pipes (vector valued function) **r**(**m**,**parameters**). The resistance matrix is then defined by the Wood and Charles linearization as:

$$\mathbf{R} = diag\left(\frac{r_j\left(m_j, parameters_j\right)}{m_j}\right) \tag{26}$$

The resistance matrix is a diagonal matrix, with the linearized resistance factors on the diagonal.

3.1 Pipes

The pipes have a resistance defined by the Darcy-Weisbach equation, which is written as:

$$h = \frac{V^2}{2g}\frac{L}{D}f = \frac{8m^2Lf}{D^5\rho^2\pi^2 g} \tag{27}$$

The friction factor can be calculated directly from Colebrook - White equation:

$$\frac{1}{\sqrt{f}} = \left(\frac{a}{\mathrm{Re}\sqrt{f}} + \frac{b}{kD}\right)^2 \tag{28}$$

3.2 Valves

$$h = k_L\frac{V^2}{2g} = k_L m^2 = \frac{k_{L\min}}{x}m^2 \tag{29}$$

The factor k_L is a property of the valve or fitting, and is dependent on the valve position when referring to a valve, but is constant for a fitting.

$k_{L\min}$	- loss factor at x =1; and
x	- valve position (0…1).

3.3 Pumps

The negative resistance function of a pump can be determined from performance measurements of the pump. A common form of such a function is:

$$h_{pump} = -\left(h_o - k\cdot m^3\right) \tag{30}$$

The factors h_o and *k* describe pump properties, and depend on the pump speed.

4. STEADY STATE FLOW SOLUTION

4.1 Stepwise solution with back-substitution

The equations which have to be solved together are the cutset, loop and linearized element equations (19), (21), (24) and (25). Recall:

$$\left[\begin{array}{cc|cc} \mathbf{I}_{hT} & 0 & \mathbf{F}_{11} & \mathbf{F}_{12} \\ 0 & \mathbf{I}_{pT} & \mathbf{F}_{21} & \mathbf{F}_{22} \end{array}\right] \left[\begin{array}{c} \mathbf{m}_{hT} \\ \mathbf{m}_{pT} \\ \hline \mathbf{m}_{pL} \\ \mathbf{m}_{mL} \end{array}\right] = \mathbf{0} \tag{19}$$

$$\left[\begin{array}{cc|cc} -\mathbf{F}_{11}^T & -\mathbf{F}_{21}^T & \mathbf{I}_{pL} & \mathbf{0} \\ -\mathbf{F}_{12}^T & -\mathbf{F}_{22}^T & \mathbf{0} & \mathbf{I}_{mL} \end{array}\right] \left[\begin{array}{c} \mathbf{h}_{hT} \\ \mathbf{h}_{pT} \\ \hline \mathbf{h}_{pL} \\ \mathbf{h}_{mL} \end{array}\right] = \mathbf{0} \tag{21}$$

$$\mathbf{h}_{pT} = \mathbf{R}_{pT}\mathbf{m}_{pT} \tag{24}$$

$$\mathbf{h}_{pL} = \mathbf{R}_{pL}\mathbf{m}_{pL} \tag{25}$$

The know vectors (inputs) are the head element head vector $\mathbf{h}_{hT}$ and the flow element flow vector $\mathbf{m}_{mL}$. Desired are the vectors of head loss and flow in the pipes, $\mathbf{m}_{pT}$, $\mathbf{m}_{pL}$, $\mathbf{h}_{pT}$ and $\mathbf{h}_{pL}$. The flow element head vector $\mathbf{h}_{mL}$ and the head element flow vector $\mathbf{m}_{hT}$ are of secondary interest, the show only what head is required to keep the input flow for the flow element as well as what flow is needed to keep the input head for the head element.

The tree pipe flow vector is found in the second row of equation (19):

$$\mathbf{m}_{pT} = -\mathbf{F}_{21}\mathbf{m}_{pL} - \mathbf{F}_{22}\mathbf{m}_{mL} \tag{31}$$

The cotree pipe head vector is in the second row of equation (21):

$$\mathbf{h}_{pL} = \mathbf{F}_{11}^T\mathbf{h}_{hT} + \mathbf{F}_{21}^T\mathbf{h}_{pT} \tag{32}$$

Inserting equation (24):

$$\mathbf{h}_{pL} = \mathbf{F}_{11}^T\mathbf{h}_{hT} + \mathbf{F}_{21}^T\mathbf{R}_{pT}\mathbf{m}_{pT} \tag{33}$$

Inserting equations (25) and (26):

$$\mathbf{R}_{pL}\mathbf{m}_{pL} = \mathbf{F}_{11}^T\mathbf{h}_{hT} - \mathbf{F}_{21}^T\mathbf{R}_{pT}\mathbf{F}_{21}\mathbf{m}_{pL} - \mathbf{F}_{21}^T\mathbf{R}_{pT}\mathbf{F}_{22}\mathbf{m}_{mL} \tag{34}$$

Regrouping:

$$\left(\mathbf{R}_{pL} + \mathbf{F}_{21}^T \mathbf{R}_{pT} \mathbf{F}_{21}\right)\mathbf{m}_{pL} = \mathbf{F}_{11}^T \mathbf{h}_{hT} - \mathbf{F}_{21}^T \mathbf{R}_{pT} \mathbf{F}_{22} \mathbf{m}_{mL} \tag{35}$$

and solving:

$$\mathbf{m}_{pL} = \left(\mathbf{R}_{pL} + \mathbf{F}_{21}^T \mathbf{R}_{pT} \mathbf{F}_{21}\right)^{-1} \left(\mathbf{F}_{11}^T \mathbf{h}_{hT} - \mathbf{F}_{21}^T \mathbf{R}_{pT} \mathbf{F}_{22} \mathbf{m}_{mL}\right) \tag{36}$$

Equation (36) has to be solved by iteration. It relates the cotree pipe flow vector to both the input vectors. The real degree of freedom for the network is the cotree pipe flow, so when this vector has been determined, the flow solution has been found. It has one row for every loop in the network, so the number of equations which have to be iterated is reduced considerably compared to the traditional methods. When the iteration has converged, all remaining flows and heads in the network can be found by back-substitution.

This allows all system flows to be calculated in terms of the flows in the flow source elements and the flow in the pipes in the cotree:

$$\begin{bmatrix} \mathbf{m}_{hT} \\ \mathbf{m}_{pT} \end{bmatrix} = -\begin{bmatrix} \mathbf{F}_{11} & \mathbf{F}_{12} \\ \mathbf{F}_{21} & \mathbf{F}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{m}_{pL} \\ \mathbf{m}_{mL} \end{bmatrix} \tag{37}$$

All head losses can now be found from the branch equations (Equations (24) and (25) recalled):

$$\mathbf{h}_{pT} = \mathbf{R}_{pT} \mathbf{m}_{pT} \tag{24}$$

$$\mathbf{h}_{pL} = \mathbf{R}_{pL} \mathbf{m}_{pL} \tag{25}$$

$$\mathbf{h}_{mL} = \mathbf{F}_{12}^T \mathbf{h}_{hT} + \mathbf{F}_{22}^T \mathbf{h}_{pT} \tag{38}$$

This solution approach has the advantage that the calculation effort within the iteration is kept low. A direct matrix solution may be more interesting, but it will require more effort within the iteration loop.

4.2 Direct matrix solution

Rearrange equations (19) and (21) in order to have only known variables on the left hand side:

$$\left[\begin{array}{cc|c} \mathbf{I}_{hT} & 0 & \mathbf{F}_{11} \\ 0 & \mathbf{I}_{pT} & \mathbf{F}_{21} \end{array}\right] \left[\begin{array}{c} \mathbf{m}_{hT} \\ \mathbf{m}_{pT} \\ \hline \mathbf{m}_{pL} \end{array}\right] = -\begin{bmatrix} \mathbf{F}_{12} \\ \mathbf{F}_{22} \end{bmatrix} \mathbf{m}_{mL} \tag{39}$$

$$\left[\begin{array}{c|cc} -\mathbf{F}_{21}^T & \mathbf{I}_{pL} & 0 \\ -\mathbf{F}_{22}^T & 0 & \mathbf{I}_{mL} \end{array}\right] \left[\begin{array}{c} \mathbf{h}_{pT} \\ \hline \mathbf{h}_{pL} \\ \mathbf{h}_{mL} \end{array}\right] = -\begin{bmatrix} -\mathbf{F}_{11}^T \\ -\mathbf{F}_{12}^T \end{bmatrix} \mathbf{h}_{hT} \tag{40}$$

Now eliminate the pipe head vectors from equation (40) by equations (24) and (25):

$$\left[\begin{array}{c|cc} -\mathbf{F}_{21}^T\mathbf{R}_{pT} & \mathbf{R}_{pL} & 0 \\ -\mathbf{F}_{22}^T\mathbf{R}_{pT} & 0 & \mathbf{I}_{mL} \end{array}\right]\left[\begin{array}{c} \mathbf{m}_{pT} \\ \hline \mathbf{m}_{pL} \\ \mathbf{h}_{mL} \end{array}\right] = -\begin{bmatrix} -\mathbf{F}_{11}^T \\ -\mathbf{F}_{12}^T \end{bmatrix}\mathbf{h}_{hT} \tag{41}$$

The row equations from equations (39) and (41) are:

$$\mathbf{m}_{hT} + \mathbf{F}_{11}\mathbf{m}_{pL} = -\mathbf{F}_{12}\mathbf{m}_{mL}$$

$$\mathbf{m}_{pT} + \mathbf{F}_{21}\mathbf{m}_{pL} = -\mathbf{F}_{22}\mathbf{m}_{mL}$$

$$-\mathbf{F}_{21}^T\mathbf{R}_{pT}\mathbf{m}_{pT} + \mathbf{R}_{pL}\mathbf{m}_{pL} = \mathbf{F}_{11}^T\mathbf{h}_{hT}$$

$$-\mathbf{F}_{22}^T\mathbf{R}_{pT}\mathbf{m}_{pT} + \mathbf{h}_{mL} = \mathbf{F}_{21}^T\mathbf{h}_{hT}$$

The three first equations are sufficient to calculate all flows:

$$\begin{bmatrix} \mathbf{I}_{hT} & \mathbf{0} & \mathbf{F}_{11} \\ \mathbf{0} & \mathbf{I}_{pT} & \mathbf{F}_{21} \\ \mathbf{0} & -\mathbf{F}_{21}^T\mathbf{R}_{pT} & \mathbf{R}_{pL} \end{bmatrix}\begin{bmatrix} \mathbf{m}_{hT} \\ \mathbf{m}_{pT} \\ \mathbf{m}_{pL} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{F}_{12} \\ \mathbf{0} & -\mathbf{F}_{22} \\ \mathbf{F}_{11}^T & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{h}_{hT} \\ \mathbf{m}_{mL} \end{bmatrix} \tag{42}$$

or:

$$\begin{bmatrix} \mathbf{m}_{hT} \\ \mathbf{m}_{pT} \\ \mathbf{m}_{pL} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{hT} & \mathbf{0} & \mathbf{F}_{11} \\ \mathbf{0} & \mathbf{I}_{pT} & \mathbf{F}_{21} \\ \mathbf{0} & -\mathbf{F}_{21}^T\mathbf{R}_{pT} & \mathbf{R}_{pL} \end{bmatrix}^{-1}\begin{bmatrix} \mathbf{0} & -\mathbf{F}_{12} \\ \mathbf{0} & -\mathbf{F}_{22} \\ \mathbf{F}_{11}^T & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{h}_{hT} \\ \mathbf{m}_{mL} \end{bmatrix} \tag{43}$$

The head losses are found by substituting equations (24) and (25). Then the three needed row equations are:

$$\mathbf{R}_{pT}^{-1}\mathbf{h}_{pT} + \mathbf{F}_{21}\mathbf{R}_{pL}^{-1}\mathbf{h}_{pL} = -\mathbf{F}_{22}\mathbf{m}_{mL}$$

$$-\mathbf{F}_{21}^T\mathbf{h}_{pT} + \mathbf{h}_{pL} = \mathbf{F}_{11}^T\mathbf{h}_{hT}$$

$$-\mathbf{F}_{22}^T\mathbf{h}_{pT} + \mathbf{h}_{mL} = \mathbf{F}_{21}^T\mathbf{h}_{hT}$$

The head loss matrix equation is then:

$$\begin{bmatrix} \mathbf{R}_{pT}^{-1} & \mathbf{F}_{21}\mathbf{R}_{pL}^{-1} & \mathbf{0} \\ -\mathbf{F}_{21}^T & \mathbf{I}_{pT} & \mathbf{0} \\ -\mathbf{F}_{22}^T & \mathbf{0} & \mathbf{I}_{mL} \end{bmatrix}\begin{bmatrix} \mathbf{h}_{pT} \\ \mathbf{h}_{pL} \\ \mathbf{h}_{mL} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{F}_{22} \\ \mathbf{F}_{11}^T & \mathbf{0} \\ \mathbf{F}_{21}^T & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{h}_{hT} \\ \mathbf{m}_{mL} \end{bmatrix} \tag{44}$$

or:

$$\begin{bmatrix} \mathbf{h}_{pT} \\ \mathbf{h}_{pL} \\ \mathbf{h}_{mL} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{pT}^{-1} & \mathbf{F}_{21}\mathbf{R}_{pL}^{-1} & \mathbf{0} \\ -\mathbf{F}_{21}^{T} & \mathbf{I}_{pT} & \mathbf{0} \\ -\mathbf{F}_{22}^{T} & \mathbf{0} & \mathbf{I}_{mL} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} & -\mathbf{F}_{22} \\ \mathbf{F}_{11}^{T} & \mathbf{0} \\ \mathbf{F}_{21}^{T} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{h}_{hT} \\ \mathbf{m}_{mL} \end{bmatrix} \tag{45}$$

5. THERMAL SOLUTION

The calculation of temperatures and heat flow in the network is based on the flow solution. Heat is transferred through the pipes of the network by the fluid, so similar methods have to be used to ensure that the conservation of energy for the network is fulfilled, as what was done for the flow solution.

First of all, the connectivity matrix has to be modified. Now the direction of flow in every element does matter, and the connectivity matrix has to be corrected, so that the direction of the elements corresponds with the flow direction. If the connectivity matrix is multiplied from the left hand side with a diagonal matrix containing the sign of the flow on the diagonal, each column in the connectivity matrix will either be multiplied by 1 (if the flow direction is the same as the element direction) or by -1 (if the flow direction is opposite to the element direction). The corrected connectivity matrix is named element flow connectivity matrix:

$$\mathbf{A}_f = \mathbf{A} \cdot diag(sign(\mathbf{m})) \tag{46}$$

The heat flow in a pipe is only dependent on the inflow condition of the fluid. The temperature of the fluid at the inflow end will solely determine the heat flow in the pipe. So a new variant of the connectivity matrix is needed. The element flow origin matrix has the same dimensions as the connectivity matrix. Instead of having two non-zero entries in each column, this matrix has an entry of 1 in the row corresponding to the inflow node into each pipe. The matrix can readily be calculated from the element flow connectivity matrix:

$$\mathbf{E} = \frac{\mathbf{A}_f + \left|\mathbf{A}_f\right|}{2} \tag{47}$$

If a new flow solution is calculated, some flows may have changed direction, and the element flow origin matrix must be recalculated.

5.1 Element types

Three element types are added for the thermal solution. They are:

- *t:* temperature source;
- *q:* heat source; and
- *x:* heat exchanger.

All the element types used in the flow solution are active here, as heat will be transported wit the flowing fluid.

5.2 Pipe heat flow

The heat transported with the flow in a single pipe element is calculated by:

$$q_t = mc_pT_{origin} \tag{48}$$

The origin temperatures for all elements in the network can be found from the nodal temperatures by:

$$\mathbf{T}_{origin} = \mathbf{E}^T \mathbf{T}_n \tag{49}$$

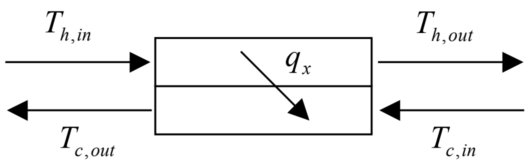
The heat flow for the all the flow elements is then calculated by:

$$\mathbf{q}_t = diag(mc_p)\mathbf{E}^T \mathbf{T}_n \tag{50}$$

5.3 Heat exchangers

Heat exchangers transfer heat from one flowstream to another, without mixing the fluids. They are thus elements with four connection points, as shown in Figure 4.

In order to model the heat exchanger within the network, an equivalent model with two connection points has to be introduced. An equivalent heat transfer coefficient is associated with this simplification. This coefficient is non-linear and dependent on the fluid temperatures, so iteration is necessary for an exact thermal solution. A schematic of the equivalent heat exchanger is shown in Figure 5.



$$\mathbf{q}_x = \frac{(T_{h,in} - T_{c,out}) - (T_{h,out} - T_{c,in})}{\ln \dfrac{T_{h,in} - T_{c,out}}{T_{h,out} - T_{c,in}}} AU$$

FIGURE 4: Schematic of a heat exchanger

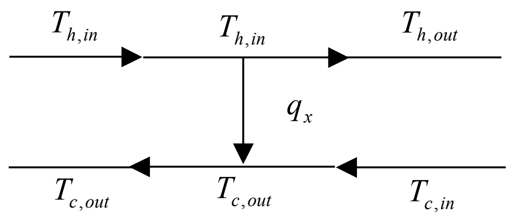


FIGURE 5: Schematic of an equivalent heat exchanger

The heat flow for a heat exchanger elements is then calculated by:

$$q_x = U_{ef}(T_{h,in} - T_{c,out}) \tag{51}$$

In order to relate the heat flow in the heat exchangers to other elements in the network, the heat exchanger connectivity matrix $\mathbf{A}_x$ is defined in the same way as the connectivity matrix. The vector of heat exchanger heat flow is thus:

$$\mathbf{q}_x = \mathbf{U}_{ef} \mathbf{A}_x^T \mathbf{T}_n \tag{52}$$

5.4 Temperature and heat flow elements

These elements are simply treated in the same way as the flow and head elements in the flow solution.

5.5 Steady state thermal solution

The current law of Kirchhoff law for the heat flow in the network is:

$$\begin{bmatrix}\mathbf{A}_f & \mathbf{A}_x & \mathbf{A}_t & \mathbf{A}_q\end{bmatrix}\begin{bmatrix}\mathbf{q}_f\\ \mathbf{q}_x\\ \mathbf{q}_t\\ \mathbf{q}_q\end{bmatrix} = \mathbf{0} \qquad (53)$$

By rearranging the equation so that the known vectors are on the left hand side of the equation:

$$\begin{bmatrix}\mathbf{A}_f & \mathbf{A}_x & \mathbf{A}_t\end{bmatrix}\begin{bmatrix}\mathbf{q}_f\\ \mathbf{q}_x\\ \mathbf{q}_t\end{bmatrix} = -\mathbf{A}_q\mathbf{q}_q \qquad (54)$$

The pipe heat flow and the heat exchanger heat flow can be calculated as:

$$\begin{bmatrix}\mathbf{q}_f\\ \mathbf{q}_x\end{bmatrix} = \begin{bmatrix} diag(\mathbf{m}c_p) & 0\\ 0 & \mathbf{U}_{eq}\end{bmatrix}\begin{bmatrix}\mathbf{E}_T\\ \mathbf{A}_x^T\end{bmatrix} T_n \qquad (55)$$

Equation (55) can now be inserted into equation (54):

$$\begin{bmatrix}\begin{bmatrix}\mathbf{A}_f & \mathbf{A}_x\end{bmatrix}\begin{bmatrix} diag(\mathbf{m}c_p) & 0\\ 0 & \mathbf{U}_{tq}\end{bmatrix}\begin{bmatrix}\mathbf{E}_T\\ \mathbf{A}_x^T\end{bmatrix} & \mathbf{A}_t\end{bmatrix}\begin{bmatrix}T_n\\ \mathbf{q}_t\end{bmatrix} = -\mathbf{A}_q\mathbf{q}_q \qquad (56)$$

This equation has the heat flow in the constant temperature elements as unknowns as well as the nodal temperatures. Information required to find this heat flow is entered by adding an additional row to the equation.

$$\left[\begin{array}{c|c}\begin{bmatrix}\mathbf{A}_f & \mathbf{A}_x\end{bmatrix}\begin{bmatrix} diag(\mathbf{m}c_p) & 0\\ 0 & \mathbf{U}_{tq}\end{bmatrix}\begin{bmatrix}\mathbf{E}_T\\ \mathbf{A}_x^T\end{bmatrix} & \mathbf{A}_t\\ \hline \mathbf{A}_t^T & 0\end{array}\right]\begin{bmatrix}\mathbf{T}_n\\ \mathbf{q}_t\end{bmatrix} = \begin{bmatrix}-\mathbf{A}_q\mathbf{q}_q\\ \mathbf{T}_t\end{bmatrix} \qquad (57)$$

T

his additional row enters information about the value of the temperature of the constant temperature elements (inputs). Expanding the terms in this equation in order to obtain a more readable result:

$$\begin{bmatrix}\mathbf{A}_f diag(mc_p)\mathbf{E}_T + \mathbf{A}_x\mathbf{U}_{eq}\mathbf{A}_x^T & \mathbf{A}_t\\ \mathbf{A}_t^T & 0\end{bmatrix}\begin{bmatrix}T_n\\ \mathbf{q}_t\end{bmatrix} = \begin{bmatrix}-\mathbf{A}_q\mathbf{q}_q\\ T_t\end{bmatrix} \qquad (58)$$

The nodal temperatures and constant temperature heat flow are now found by:

$$\begin{bmatrix}T_n\\ \mathbf{q}_t\end{bmatrix} = \left[\begin{array}{c|c}\mathbf{A}_f diag(mc_p)\mathbf{E}_T + \mathbf{A}_x\mathbf{U}_{eq}\mathbf{A}_x^T & \mathbf{A}_t\\ \hline \mathbf{A}_t^T & \mathbf{0}\end{array}\right]^{-1}\begin{bmatrix}-\mathbf{A}_q\mathbf{q}_q\\ T_t\end{bmatrix} \qquad (59)$$

6. PRACTICAL EXPERIENCE AND CONCLUSION

The microscopic models presented here are just one of many kinds of network calculation models. These models have proven to be powerful, and as they give insight into the mathematics behind the model, the enable a skilled user to do very detailed and accurate analysis. The method is as well very flexible, because the terminology enables the user to adapt these models relatively easily to new fields of application.

6.1 Industrial usage

The Dutch energy company NUON in Arnhem, the Netherlands, has been using these models as their main tool for district heating design and operation since 1995. The main benefit they saw in these models was the flexibility and easy adoption to other systems or for new application. The author has developed the analysis package Pipelab in cooperation with NUON, running under the numerical environment Matlab. This package is not commercial, but is used for research purposes both in academia and industry.

6.2 A sample study from Turkey

Adil Caner Şener, at the Izmir Institute of Technology Geothermal Energy Research Development Test and Education Centre did as well aproject at the United Nations University – Geothermal Training Programme in Reykjavik in 2002. The title of the report was: "Modelling of Balçova geothermal district heating system".

His study analyzed the system, pinpointing various problem areas in the present operation of the system. Optimization of the geothermal supply system was studied, as well as time series methods for load forecasting.

The microscopic models were used for calculation of flow and head loss in the distribution system. Figure 6 is a diagram from the report, showing the head both in the supply and return network as a function of the distance from the supply point.

The thermal solution method was then used to obtain the temperatures in the supply network. Figure 7 is a diagram from the report, showing the temperature as a function of the distance from the supply point.

It is apparent from the diagram, that unacceptable cooling is in a few of the pipes in the network. There were reports from the operation on heating problems by a few of the consumers.

In Figure 8, the distribution system is shown, and the problem areas indicated.

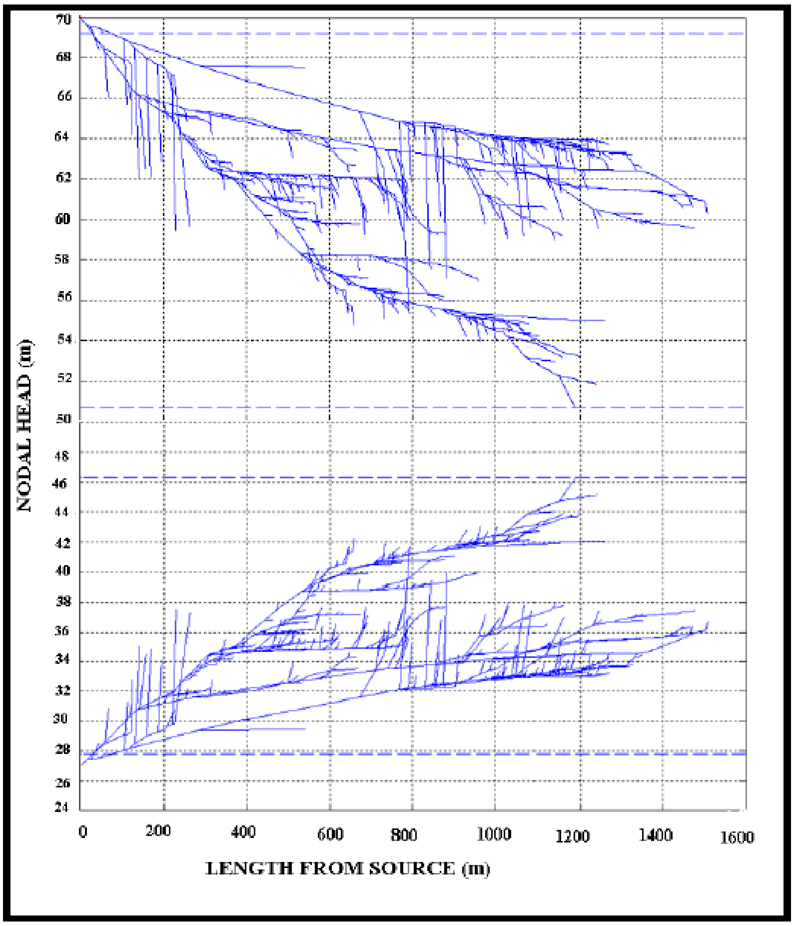


FIGURE 6: Length from source vs. head loss diagram for the Balcova distribution system

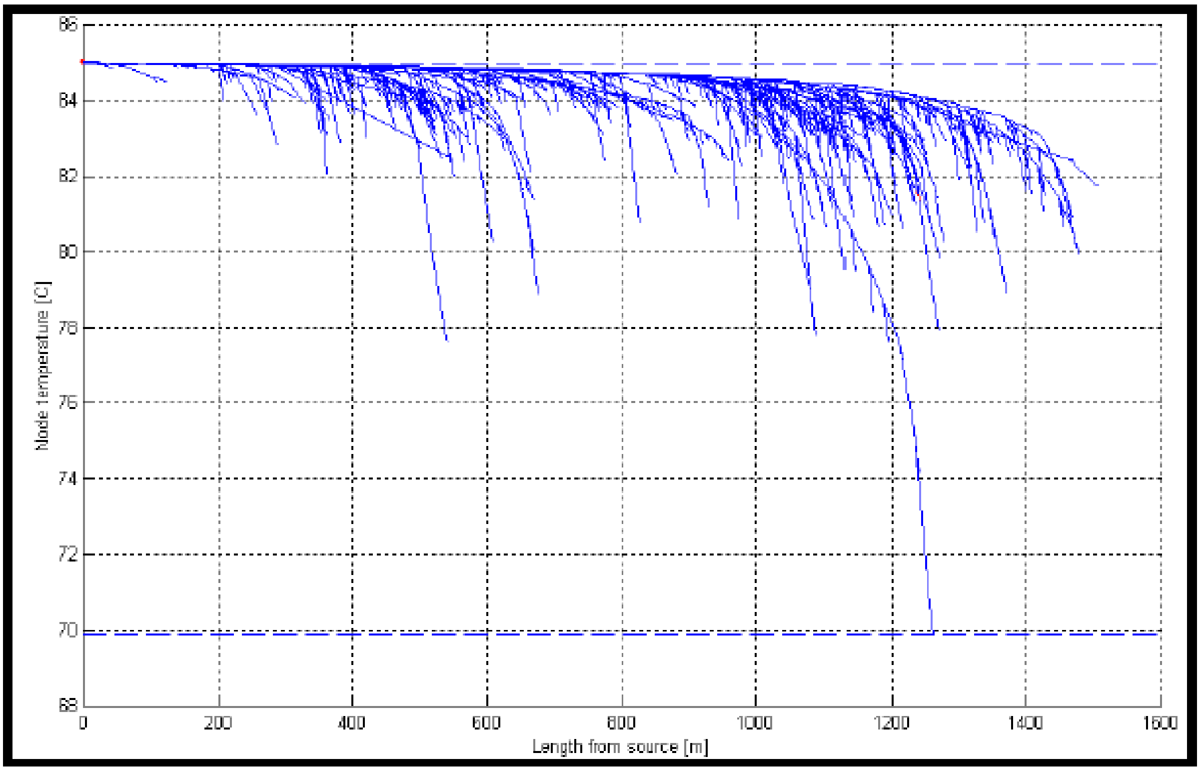


FIGURE 7: Length from source vs. node temperature diagram for the Balcova distribution system

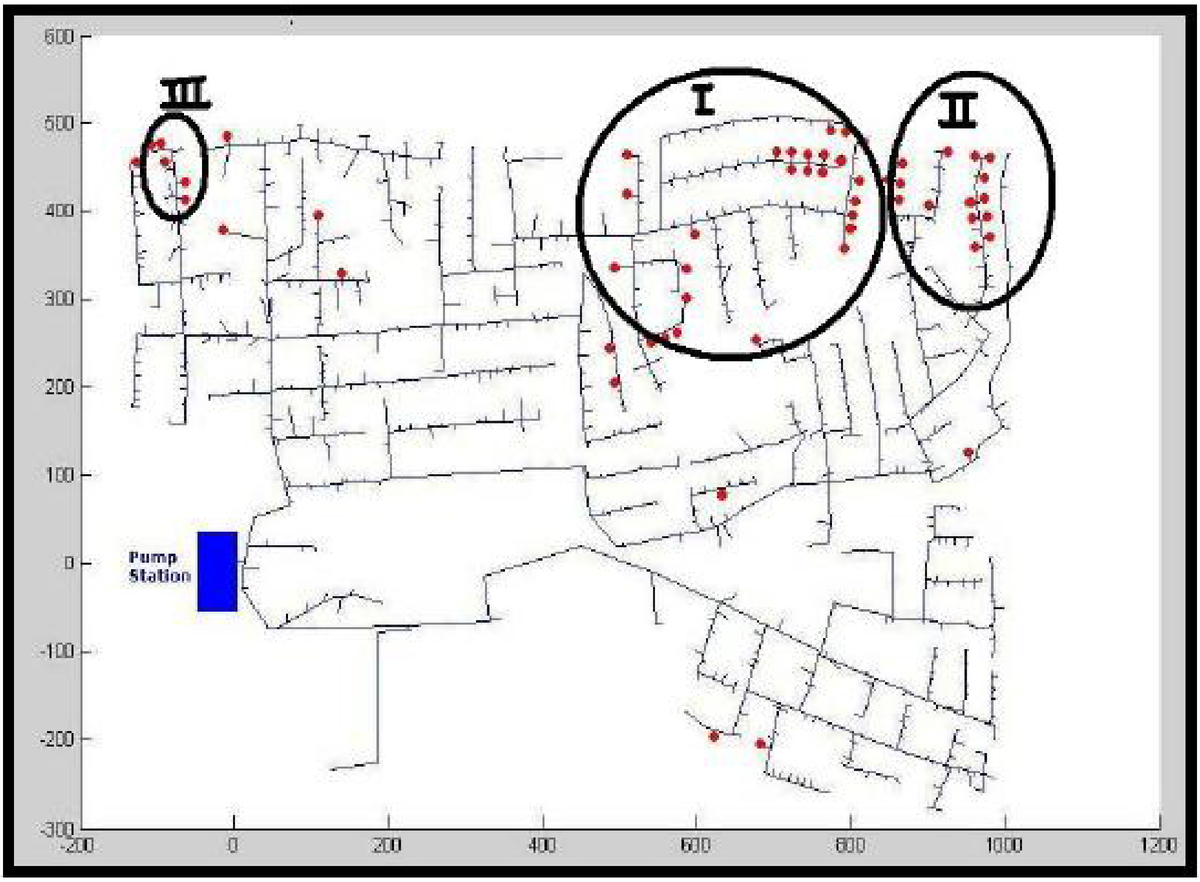


FIGURE 8: Buildings with heating problems

The analysis did show, that the problem areas were related to pipes with abnormally high head loss per unitary length. The problem areas are indicated in Figure 9.

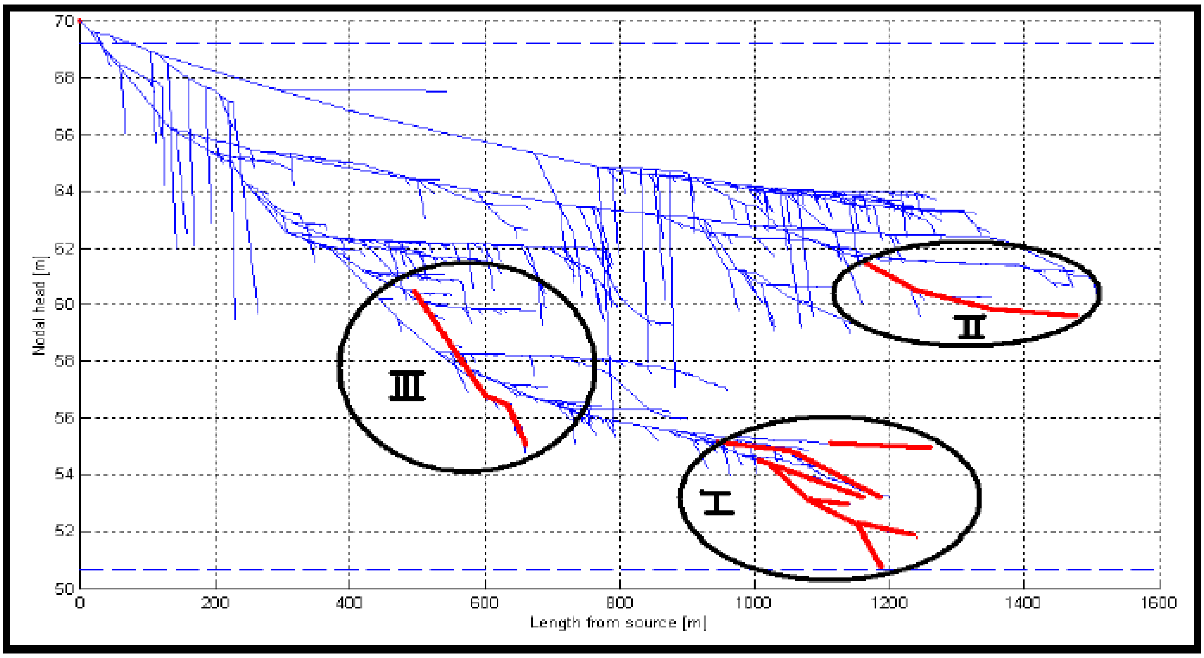


FIGURE 9: Presentation of regions with heating problems in h-l diagram of supply network

Similarly, the area with the high cooling in the supply system was one of the problem areas (Figure 10).

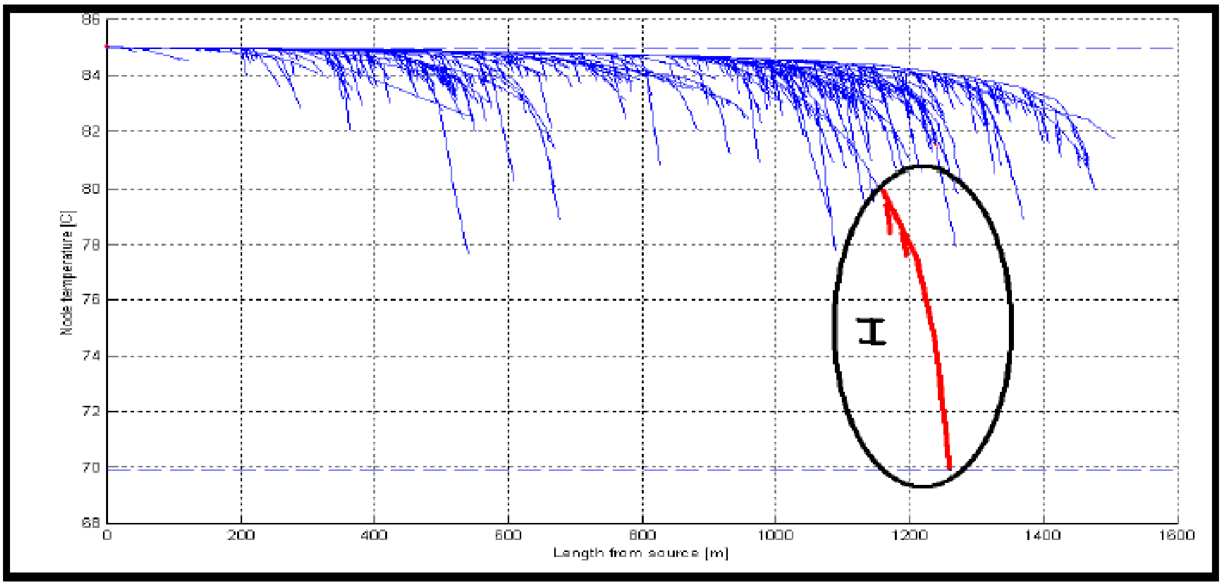


FIGURE 10: Presentation of regions with heating problems in T-l diagram of supply network

The conclusion is, that a microscopic analysis is necessary for safe and good design, operation and troubleshooting of pipe networks.

7. FINAL WORDS

I do sincerely hope that this presentation of the mathematics behind a thorough analysis of district heating networks will give the reader a new insight into this fascinating area of research and design.

NOMENCLATURE

Scalars

A	Heat transfer area (m^2)
a_{ij}	Connectivity matrix entry (-)
C	Heat capacity of house (J/°C)
c_p	Water heat capacity (J/(kg °C))
g	Acceleration due to gravity (m/s^2)
G_n	Graph
G_s	Subgraph
k_i	PI-control parameter (kg/(s^2 °C))
k_l	Building heat loss factor (W/ °C)
k_p	P-control parameter (kg/(s °C))
L	Cotree (the set of links)
m	Water mass flow (kg/s)
m_0	Reference water mass flow (kg/s)
m_{avg}	Average mass flow
n_L	Number of links (-)
n_n	Number of nodes (-)
n_q	Number of constant heat flow elements (-)
n_t	Number of constant temperature elements (-)
n_T	Number of tree branches (-)
q	Heat flow (W)
Q	Heat duty (W)
Q_0	Heat duty at reference conditions (W)
Q_{loss}	Heat loss (W)
Q_{net}	Net heat (W)
q_q	Constant heat flow (W)
Q_{supp}	Heat supply (W)
q_t	Heat flow in constant temperature element (W)
q_x	Heat exchanger duty (W)
R_c	Capacity ratio
T	Temperature (°C)
T	Tree
T_1	Pipe inlet temperature (pumping station) (°C)
T_2	Return temperature at pumping station (°C)
Tc,in	Cold fluid inlet temperature (°C)
Tc,out	Cold fluid outlet temperature (°C)
T_g	Ground temperature (°C)
Th,in	Hot fluid inlet temperature (°C)

Th,out	Hot fluid outlet temperature (°C)
T_i	Indoor temperature (°C)
T_{i0}	Reference indoor temperature (°C)
$T_{i,set}$	Desired indoor temperature in dynamic modelling (°C)
T_o	Outdoor temperature (°C)
T_{o0}	Reference outdoor temperature (°C)
T_r	Return water temperature (primary network) (°C)
T_{r0}	Reference return water temperature (primary network) (°C)
T_{rs}	Return water temperature in secondary network (°C)
T_s	Water supply temperature (primary network) (°C)
T_{s0}	Reference water supply (primary network) (°C)
U	Heat transfer coefficient (W/(m2°C))
Ueq	Equivalent heat transfer coefficient (W/°C)
U_p	Pipe heat loss factor (W/ °C)
y	Variable
z	Variable

Greek symbols

ε	Heat exchanger effectiveness
τ	Pipe transmission effectiveness
τ_0	Pipe transmission effectiveness at reference conditions
ΔT_m	Logarithmic mean temperature difference (°C)
ΔT_{m0}	Logarithmic mean temperature difference at reference conditions (°C)

Vectors and matrices

$\mathbf{A}$	Flow elements connectivity matrix (-)
$\mathbf{A}_f$	Flow connectivity matrix (-)
$\mathbf{A}_L$	Cotree connectivity matrix (-)
$\mathbf{A}_q$	Constant heat flow connectivity matrix (-)
$\mathbf{A}_T$	Tree connectivity matrix (-)
$\mathbf{A}_t$	Constant temperature connectivity matrix (-)
$\mathbf{A}_x$	Heat exchanger connectivity matrix (-)
$\mathbf{D}$	Cutset matrix (-)
$\mathbf{E}$	Element flow origin matrix (-)
$\mathbf{I}_{hT}$	Tree head source identity matrix
$\mathbf{I}_{pT}$	Tree pipe identity matrix
$\mathbf{I}_{pL}$	Cotree pipe identity matrix
$\mathbf{m}$	Flow vector (kg/s)
$\mathbf{m}_{hT}$	Tree head source flow vector (kg/s)
$\mathbf{m}_{mL}$	Link flow source flow vector (kg/s)
$\mathbf{m}_{pL}$	Link pipe flow vector (kg/s)
$\mathbf{m}_{pT}$	Tree pipe flow vector (kg/s)
$\mathbf{F}_{ij}$	Submatrix of the cutset matrix
$\mathbf{q}_f$	Vector of heat flow in flow elements (W)

$\mathbf{q}_q$	Constant heat flow vector (W)
$\mathbf{q}_t$	Vector of heat flow in constant temperature elements (W)
$\mathbf{q}_x$	Heat exchanger duty vector (W)
$\mathbf{T}_n$	Node temperature vector (°C)
$\mathbf{U}_{eq}$	Heat exchanger transfer matrix (W/°C)