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FINITE ELEMENT RESISTIVITY MODELLING USING SPECIALIZED MESH STRUCTURE

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PREFACE

As we know giving a geological model to calculate the anomaly is called a geophysical forward problem. The forward problem is an important one because we want to know the correspondance (or dependence) between the anomalies and the causative bodies.

Usually the forward problems are easy to solve in geophysical prospecting, for example, in magnetics, gravity and seismics, but for a long time it has been difficult for geoelectric prospecting. For most of the applied electric surveying problems, except for few simple geoelectric conditions, we cannot calculate the electrical potential and the apparent resistivity analytically. So people hope to get a numerical approximate solution with computer modelling. These methods include the integral equation method, the finite difference method, the finite element method and the boundary element method.

For the two-dimensional geoelectric condition with a point source of current the finite element method was first applied to electric survey by J.H. Coggon (1971). He derived the equivalent equation of the variation and realized the computation. In 1977 L. Rijo improved the method with reasonable application in his Ph. D. thesis.

For the two-dimensional problem we prefer the finite element method. We have done the finite element resistivity modelling work for years and got some progress. In this book we would like to introduce the theory, the technique and the applications of the finite element resistivity modelling in the two dimensional geoelectric conditions with point sources of the steady current.

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1.1 Introduction of fundamental relations

In this book we will only treat the electric field of steady current. $\overline{E}$, U, $\overline{j}$, ρ and σ represent the electric field intensity, the electric potential, the electric current density, the resistivity and the conductivity of rocks respectively. According to the knowledge of electric field theories, we have the following relations.
The intensity vector is the negative gradient of the potential:

$$\overline{E}=-\overline{\nabla}U \qquad (1.1.1)$$

The differential form of Ohm's low is

$$\overline{j}=\sigma\overline{E} \qquad (1.1.2)$$

or

$$\overline{j}=-\sigma\overline{\nabla}U \qquad (1.1.3)$$

From the conservation of electric charge, the continuity equation of the current density j and the charge density q we obtain

$$\overline{\nabla}\cdot\overline{j}=-\frac{\partial q}{\partial t} \qquad (1.1.4)$$

Then we obtain the equation of the potential and the charge density by joining 1.1.3 and 1.1.4

$$\overline{\nabla}\cdot(\sigma\overline{\nabla}U)=\frac{\partial q}{\partial t} \qquad (1.1.5)$$

In order to produce the differential equation of the potential and the current intensity we need to introduce the Dirac δ-function. For example, if we assume an electric charge distribution e along the x-axix, the charge density is

$$q=\frac{de}{dx}=\lim_{\Delta x\to 0}\frac{\Delta e}{\Delta x}$$

Let's put a point-charge at x_0 then the charge density q equals to zero everywhere except at the point x_0 where q becomes infinite. Assuming

$$\delta(x-x_0)=0 \text{ when } x\neq x_0 \text{ but } \infty \text{ when } x=x_0$$

and

$$\int_{-\infty}^{\infty}\delta(x-x_0)dx=1$$

then the charge density at each point along the x-axis can be written as

$$q=e\delta(x-x_0)$$

In three dimensional problem, we put the point charge e at (x_0, y_0, z_0), then

$$q=e(t)\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)$$

In term of the definition of the electric current intensity

$$I=-\frac{\partial e}{\partial t}$$

so

$$\frac{\partial q}{\partial t}=-I\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)$$

Formula 1.1.5 can be written as

$$\bar{\nabla}\cdot(\sigma\bar{\nabla}\phi)=-I\delta(x-x_0)\delta(y-y_0)\delta(z-z_0) \qquad (1.1.6)$$

where (x_0,y_0,z_0) are the coordinates of the point source. This is the fundamental differential equation of the resistivity method.
For homogeneous medium, σ is a constant and then 1.1.6 simplifies to Poisson's equation

$$\nabla^2 U=-\frac{I}{\sigma}\delta(x-x_0)\delta(y-y_0)\delta(z-z_0) \qquad (1.1.7)$$

With no source, 1.1.7 becomes Laplac's equation

$$\nabla^2 U=0 \qquad (1.1.8)$$

At the interface of two media with conductivity σ_1, and σ_2, respectively, the potential and the normal component of the current density follow certain boundary conditions. In terms of the continuity of the electric potential we have

$$U_1=U_2 \qquad (1.1.9)$$

at the interface. And from the continuity of the current we have

$$\sigma_1\frac{\partial U_1}{\partial n}=\sigma_2\frac{\partial U_2}{\partial n} \qquad (1.1.10)$$

where $\bar{n}$ is the normal vector to the interface.
If we want to determine specific distribution of the electric field, i.e. to determine uniquely the solutions of the differential equation, the boundary conditions given at the boundary of the research region are needed. There are three kinds of boundary conditions.
Consider a boundary which is far away from the current source, we can assume

$$U|_\Gamma=0 \qquad (1.1.11)$$

or

$$U|_\Gamma=f_0 \qquad (1.1.12)$$

where f_0 is known. This is called the first boundary condition or Dirichlet's condition.
The ground is an insulating surface, so we have

$$\frac{\partial U}{\partial n}=0$$

which is called the second boundary condition or Neumann's condition.
In calculating practice of the finite difference and the finite element method, we find that further away than some distance from the point-source, the potential value calculated by condition 1.1.11 is smaller, but the values by condition 1.1.12 is larger than the corresponding analytical ones. Therefore we may choose a mixed boundary condition. In fact, at a position which is far away from the source we always have

$$U(x,y,z)=\frac{C}{\sqrt{(x^2+y^2+z^2)}}=\frac{C}{r}$$

where C is a constant, then

$$\frac{\partial U}{\partial n}=-\frac{C}{r^3}\bar{r}\cdot\bar{n}=-\frac{U}{r}\cos\theta$$

where θ is the angle between the radial vector $\bar{r}$ and the normal vector $\bar{n}$, and

$$\frac{\partial U}{\partial n}+\gamma\frac{U}{r}=0 \qquad (1.1.13)$$

where $\gamma=\cos\theta$. In practice this mixed condition is better than the first and the second one.
As to the two-dimensional problem with line source, 1.1.6 can be simplified as

$$\frac{\partial}{\partial x}(\sigma\frac{\partial U}{\partial x})+\frac{\partial}{\partial z}(\sigma\frac{\partial U}{\partial z})=-I\delta(x-x_0)\delta(z-z_0)$$

Here we assume that y is parallel to the line-source.

1.2 The problem of two-dimensional geoelectric structure with point-source

Suppose the y-axis is parallel to the geological strike, thus there is no change of geoelectric parameters in the y-direction. That is

$$\frac{\partial}{\partial y}[\sigma(x,y,z)]=0$$

then formula 1.1.6 becomes

$$\bar{\nabla}\cdot(\sigma(x,z)\bar{\nabla}U(x,y,z)]=-I\delta(x-x_0)\delta(y-y_0)\delta(z-z_0) \qquad (1.2.1)$$

In order to simplify the calculation we use the cosine Fourier transform

$$\phi(x,k,z)=\int_0^\infty U(x,y,z)\cos ky\,dy \qquad (1.2.2)$$

where k is the wavenumber. It transforms the three-dimensional potential in space (x,y,z) into two-dimensional transformed potential in space (x,k,z).
We transform both sides of 1.1.5 by 1.2.2. Consider the left of 1.1.5, we have

$$\bar{\nabla}\cdot(\sigma\bar{\nabla}U)=\bar{\nabla}\sigma\bar{\nabla}U+\sigma\nabla^2U$$

$$=\frac{\partial\sigma}{\partial x}\frac{\partial U}{\partial x}+\frac{\partial\sigma}{\partial z}\frac{\partial U}{\partial z}+\sigma\left(\frac{\partial^2U}{\partial x^2}+\frac{\partial^2U}{\partial z^2}\right)+\sigma\frac{\partial^2U}{\partial y^2}$$

Transforming this expression, we can interchange the sequence of integration and differentation in the first three terms because they are done to different variables. And we use the differential theorem of Fourier transforms for the last term. Arranging the expression we obtain

$$\bar{\nabla}\sigma\bar{\nabla}\phi+\sigma\nabla^2\phi-k^2\sigma\phi$$

Transforming the right hand side of 1.1.5 and using

$$\int_0^\infty\delta(y)e^{-iky}\,dy=\frac{1}{2}$$

we obtain

$$\bar{\nabla}\cdot(\sigma\bar{\nabla}\phi)-k^2\sigma\phi=-\frac{I}{2}\delta(x-x_0)\delta(z-z_0) \qquad (1.2.3)$$

or

$$\bar{\nabla}\cdot[\sigma(x,z)\bar{\nabla}\phi(x,k,z)]-k^2\sigma(x,z)\phi(x,k,y)$$

$$=-\frac{I}{2}\delta(x-x_0)\delta(z-z_0).$$

This is the fundamental differential equation that the transformed potential ϕ follows under the condition of two-dimensional geoelectric structure with a point source. It is independent of y. Also ϕ is only a function of x and z therefore the original three-dimensional problem described by 1.1.1 is simplified as the two-dimensional problem described by 1.2.3. In equation 1.2.3 k appears only as a parameter. We choose several k values for calculating ϕ. Then we get several ϕ values at each measured point. Each time in the calculation k is a constant. And finally we obtain the values of the potential U(x,y,z) through inverse Fourier transform which is done to ϕ gained by the different k's.

$$U(x,y,z)=\frac{2}{\pi}\int_0^\infty\phi(x,k,z)\cos ky\,dk \qquad (1.2.4)$$

As to the boundary conditions we prefer the mixed boundary condition. A form which is similar to 1.2.3 can be deduced. We know that in space (x,y,z) the electric potential of the point source and the secondary field are inversely proportional to the distance, that is

$$U(x,y,z)=\frac{C}{r}$$

where $r=\sqrt{(x^2+y^2+z^2)}$ is the distance from the point source to the surveying point. C is a constant. Transforming both sides of the expression by 1.2.2 we obtain

$$\phi(x,k,z)=CK_0(kr) \qquad (1.2.5)$$

Where K_0 is Bessel function of order zero,

$$r=\sqrt{(x^2+z^2)}$$

and

$$\frac{\partial \phi}{\partial n} = -CkK_1(kr)\cos\theta \qquad (1.2.6)$$

where θ is the angle between the radial vector $\bar{r}$ and the normal vector $\bar{n}$. K_1 is Bessel function of order one.
From 1.2.5 we get

$$C = \frac{\phi}{K_0(kr)},$$

substituting for C is 1.2.6 we obtain

$$\frac{\partial \phi}{\partial n} + k\frac{K_1(kr)}{K_0(kr)}\cos\theta = 0$$

or simply as

$$\frac{\partial \phi}{\partial n} + \gamma\phi = 0$$

where

$$\gamma = k\frac{K_1(kr)}{K_0(kr)}\cos\theta$$

We can also obtain the second boundary condition when r equals to zero and the first one by using 1.2.5.

2. VARIATIONAL METHOD

The mathematical basis of the finite element method is the variational principle. So we introduce first the variational method.

2.1 Functional and variation

The variational method is a method for studying functional extremum. In fact the functional is a extension of the function concept and the variation is a extension of the differential concept. In mathematics the concept of function is well known. Assuming x stands for the independent variable and y stands for the dependent variable, the function can be expressed as

$$y=y(x)$$

There is a corresponding relation between x and y, that is, every value of x corresponds to some value of y. So we call that y is a function of x.
If J is a function of y, i.e. there is a corresponding value of J for every value of function y. In that case we call J a functional of y(x), expressed as

$$J=J[y]=J[y(x)]$$

It can be seen that functional is different from a general function. Its independent variable is a function, i.e. the functional is a function's function. And one must notice that here the dependent variable J is a real value.
In order to have a concrete understanding of a functional, we give a simple example. Assuming there are two points A and B on a plane, we want to find the curve joining A and B with the shortest distance (apparently it is the straight line joining A and B). This problem can be expressed mathematically as follow. Making a wilful curve y=y(x), which join the points A and B, the differential of the curve is (see fig. 2.1.1)

$$ds=\sqrt{((dx)^2+(dy)^2)}.$$

So the length of the curve is

$$J=\int_A^B ds=\int_A^B \sqrt{((dx)^2+(dy)^2)}=\int_{x_1}^{x_2}\sqrt{(1+(y')^2)}dx \quad (2.2.1)$$

Where y is a function of x, but the length of the curve J is a function of y, so call J[y(x)] a functional. The problem of finding the shortest curve is finding a function y=y(x) with the boundary conditions

$$y_1=y(x_1) \text{ and } y_2=y(x_2) \quad (2.1.2)$$

such that the functional 2.1.1 is minimized.
Now we discuss the concept of the variation of the functional argument and the variation of the functional. Suppose the independent variable y(x) to be a function $y_1(x)$, the increment variable y(x) on $y_1(x)$ would be the difference between the two functions y(x) and $y_1(x)$:

$$\delta y(x)=y(x)-y_1(x).$$

The increment $\delta y(x)$ of the argument $y_1(x)$ is called the variation of argument y(x).
If the increment of the argument of the functional J[y(x)] is $\delta y(x)$, then

$$\Delta J=J[y(x)+\delta y(x)]-J[y(x)]$$

is the increment of the functional. ΔJ can be expressed as

$$\Delta J=L[y(x),\delta y(x)]+\alpha \quad (2.1.3)$$

where $L[y(x),\delta y(x)]$ is homogeneous linear as to $\delta y(x)$, that is

$$L[y(x,\delta y_1(x)+\delta y_2(x)]=L[y(x),\delta y_1(x)]-L[y(x)+\delta y_2(x)],$$

and

$$L[y(x),\lambda\delta y(x)]=\lambda L[y(x),\delta y(x)]$$

is a higher order infinitesimal when $\delta y(x)$ is an infinitesimal. Then we call $L[y(x),\delta y(x)]$ the variation of $J[y(x)]$ at $y(x)$ and write it as $\delta J[y(x)]$. So we have

$$\Delta J=\delta J+\alpha$$

Therefore the variation of a functional is the linear main part of the functional increment. This is a definition of a functional variation. Now we introduce another definition as follow.

Given a functional $J=J[y(x)]$, we consider the value $J[y(x)+t\delta y(x)]$ which the functional takes at $y(x)+t\delta y(x)$. In terms of the previous definition, if the functional has variation in the sense that the main part of functional increment is linear, we have

$$\Delta J=J[y(x)+t\delta y(x)]-J[y(x)]=L[y(x),t\delta y(x)]+\alpha$$

Thus

$$\frac{\partial}{\partial t}\{J[y(x)+t\delta y(x)]\}=\lim_{t\to 0}\frac{\Delta J}{t} \qquad (2.1.4)$$

where $L[y(x),t\delta y(x)]$ is linear in $\delta y(x)$, so

$$L[y(x),t\delta y(x)]=tL[y(x),\delta y(x)]$$

and

$$\lim_{t\to 0}\frac{\alpha}{t}=0.$$

So the variation of a functional $J[y(x)]$ at $y=y(x)$ equals the derivative of the functional $J[y(x)+t\delta y(x)]$ with respect to t when $t=0$. That is

$$\delta J[y(x)]=\frac{d}{dt}\{J[y(x),t\delta y(x)]\}\Big|_{t=0} \qquad (2.1.5)$$

Now we discuss the extremum of a functional. We have mentioned above that the variational method is a method to study the extremum of a functional. As to the functional

$J[y(x)]$, if we have $y_0(x)$, making all $y(x)$ in the region where y exist satisfy the condition.

$$J[y_0(x)]\leq J[y(x)] \text{ or } J[y_0(x)]\geq J[y(x)]$$

We say that $J[y(x)]$ takes minimum (or maximum) at $y_0(x)$. If only y's close to y_0 satisfy the condition, we say that $J[y]$ takes local minimum (or maximum) at y_0.
If the functional $J=J[y(x)]$ has variation and takes minimum or maximum at $y_0(x)$, the first order variation at $y_0(x)$ is

$$\delta J=\delta J[y_0(x)]=0 \qquad (2.1.6)$$

The relation is the necessary condition for $J[y(x)]$ to take extremum at y_0. The function $y=y_0(x)$ is called the extremum function (or extremum curve).
Practically if $J[y(x)]$ takes extremum at y_0 and

$$y(x)-y_0(x)=\delta y(x)$$

we obtain

$$J[y(x)]=J[y_0(x)+t\delta y(x)]$$

for any t. When $y_0(x)$ and $\delta y(x)$ are fixed,

$$J[y_0(x)+t\delta y(x)]=\phi(t)$$

is a function of t. As J takes extremum at y_0, $\phi(t)$ takes an extremum when t=0, i.e.

$$\frac{\partial}{\partial t}J[y_0(x)+t\delta y(x)]\Big|_{t=0}$$

So we have

$$\delta J=L[y_0(x),\delta y(x)]=0$$

The extremum problem of the functional

$$J[y(x)]=\int_{x_1}^{x_2}F(x,y,y')dx \qquad (2.1.7)$$

under the boundary conditions

$$y_1=y(x_1) \quad \text{and} \quad y_2=y(x_2)$$

is discussed as follow. In terms of the definition above the increment of the functional 2.1.7 is

$$\begin{aligned}\Delta J(y)&=\int_{x_1}^{x_2}F(x,y+\delta y,y'+\delta y')dx-\int_{x_1}^{x_2}F(x,y,y')dx\\&=\int_{x_1}^{x_2}[F(x,y+\delta y,y'+\delta y')-F(x,y,y')]dx.\end{aligned}$$

By Taylor expansion

$$F(x,y+\delta y,y'+\delta y')-F(x,y,y')=\frac{\partial F}{\partial y}\delta y+\frac{\partial F}{\partial y'}\delta y'+\alpha'$$

where α' is a higher order infinitesimal, we get

$$\Delta J(y)=\int_{x_1}^{x_2}[\frac{\partial F}{\partial y}\delta y+\frac{\partial F}{\partial y'}\delta y']dx+\int_{x_1}^{x_2}\alpha'dx$$

The first term of the expression is homogeneous linear in δy, the second one is a higher infinitesimal. So the variation of $J(y)$ is

$$\Delta J(y)=\int_{x_1}^{x_2}[\frac{\partial F}{\partial y}\delta y+\frac{\partial F}{\partial y'}\delta y']dx \qquad (2.1.8)$$

It should be pointed out that the variation of a functional is caused by δy. And δy is given by

$$\delta y=y(x)-y_0(x)$$

for any certain x. Thus we can regard x as a constant in the variational calculation of the expression 2.1.8, and $y_0(x)$ is an extremum curve, $y(x)$ is a curve very close to $y_0(x)$.
According to the formula of integration by parts

$$\int_{x_1}^{x_2}uv'dx=uv\Big|_{x_1}^{x_2}-\int_{x_1}^{x_2}vu'dx$$

the second term on the right hand side of 2.1.8 can be written as

$$\int_{x_1}^{x_2}\frac{\partial F}{\partial y'}\delta y'\,dx=\frac{\partial F}{\partial y'}\delta y(x)\Big|_{x_1}^{x_2}-\int_{x_1}^{x_2}\delta y\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)dx$$

using the boundary conditions we have

$$\delta y(x_1)=0 \text{ and } \delta y(x_2)=0$$

Thus the first term on the right hand side equals to zero. So 2.1.8 can be written as

$$\Delta J(y)=\int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y}-\frac{d}{dx}\frac{\partial F}{\partial y'}\right)\delta y\,dx.$$

For any δy to satisfy the necessary condition of extremum

$$\Delta J(y)=0$$

we get

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\frac{\partial F}{\partial y'}=0 \qquad (2.1.9)$$

This expression is called Euler's equation. It is the necessary condition that the functional 2.1.7 takes an extremum.

By solving the differential equation 2.1.9, we obtain infinite extremum curves. Substituting boundary conditions into it, we can obtain the unique extremum curve at last. So the extremum problem of the functional can be boiled down to a solution of corresponding differential equation. For example, in the functional 2.1.1

$$F(x,y,y')=\sqrt{(1+(y')^2)}$$

So the corresponding differential equation is

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\frac{\partial F}{\partial y'}=-\frac{d}{dx}\frac{y'}{\sqrt{(1+(y')^2)}}=0$$

Of course, it is very easy to solve this differential equation. Integrating according to x, we get $y'=C_1\sqrt{(1+(y')^2)}$. Simplifying it we get the equation $y'=C_2$. Integrating again we obtain $y=C_2x+C_3$, where C_1, C_2 and C_3 are constants. Substituting the boundary condition 2.1.2 into the expression, we obtain

$$C_2=\frac{y_2-y_1}{x_2-x_1}x_1$$

and

$$C_3=y_1-\frac{y_2-y_1}{x_2-x_1}x_1$$

hence

$$y=\frac{y_2-y_1}{y_2-y_1}(x-x_1)+y_1$$

This is the equation for a straight line. That is a straight line is the shortest distance between A and B.
In mathematics, we can prove the opposite that the solution of a differential equation can be boiled down to find a extremum of a corresponding functional. Thus the extremum of a functional and the solution of the corresponding differential equation are equivalent.

2.2 One-dimensional finite element technique

All geophysical problems are of course two-dimensional or three-dimensional problems. But for showing the fundamental method of the finite element technique, we will first introduce the one-dimensional finite element method.
Generally classical variational solution is to boil down the extremum problem of a functional to the solution of the corresponding differential equation. But the finite element technique transforms the solution of a differential equation into the extremum problem of the corresponding functional. This is because the solutions of some differential equations are very difficult or impossible to obtain

analytically, so one has to establish first the corresponding functional, then through discretization and interpolation get an approximate minimum solution of the functional which satisfies given boundary conditions with numerical methods.
For example, finding the solution of the ordinary differential equation

$$y''=1 \qquad (2.2.1)$$

under the boundary conditions

$$y(0)=0 \text{ and } y(1)=1$$

The analytical solutions of this equation can be obtained easily. Integrating 2.2.1 twice and substituting for the boundary conditions, we get

$$y=\frac{1}{2}x^2+\frac{1}{2}x \qquad (2.2.2)$$

Here we will use the finite element technique to find an approximate solution of the equation 2.2.1. First, find the functional expression which corresponds to the differential equation. Assuming

$$F(x,y,y')=\frac{1}{2}(y')^2+y$$

substituting it into Euler's equation 2.1.9, we have

$$\frac{\partial F}{\partial y}=\frac{d}{dx}\frac{\partial F}{\partial y'}=1-y''=0$$

This is just the differential equation 2.2.1. Thus the corresponding functional is

$$J(y)=\int_0^1 (\frac{1}{2}(y')^2+y)dx \qquad (2.2.3)$$

Now, we will use the finite element technique to find the extremum of the functional which satisfies the boundary conditions.
The first step, simply using equal dividing points

$$x_0=0,\ x_1.....,x_i..,x_{i+1},....,x_n$$

the interval is divided into n subintervals. These points are called nodes and every subinterval is called an element. The length of the i-th element is

$$x_i-x_{i-1}=h=\frac{1}{n}$$

The function values y(x) at the ends x_0 and x_n of the interval [0,1] has been given by the boundary conditions. But the function values at the nodes inside the interval are to be found. So we transform finding the solution of a continuous function y=y(x) into finding the values of the function at the nodes. This is called the descretization treatment.

The second step, assuming the function y=y(x) is linear within every element (see fig. 2.2.1). The smaller the element, the assumption is closer to the true. Inside the i-th element the function $y_i(x)$ and its first derivative $y_i'(x)$ are

$$y_i(x)=\frac{y_i-y_{i-1}}{h}(x-x_{i-1})+y_{i-1}$$

and

$$y_i'(x)=\frac{y_i-y_{i-1}}{h}$$

This is called linear interpolation.

The third step, dividing the integration 2.2.3 into the integrations of each element. The integration of the i-th element is

$$J_i(y)=\int_{x_{i-1}}^{x_i}[\frac{1}{2}(y'(x))^2+y]dx=\int_{x_{i-1}}^{x_i}\{\frac{1}{2}[y_i'(x)]^2+y_i(x)\}dx$$

We substitute the expressions of $y_i(x)$ and $y_i'(x)$ into the functional. After integrating and arranging it we obtain

$$J_i(y)=\frac{1}{2h}(y_i-y_{i-1})^2+\frac{h}{2}(y_i+y_{i-1}) \qquad (2.2.4)$$

Thus it can be seen that $J_i(y)$ only have to do with the values of the function at the ends of element (i.e. the nodes x_i and x_{i-1}). It can be written as

$$J_i(y)=J_i(y_i,y_{i-1})$$

Making the sum of the integrations over all elements we obtain from the integration 2.2.3

$$J(y)=\int_0^1(y'^2+y)dx=\sum_{i=0}^{n}J_i(y)=\sum_{i=0}^{n}J_i(y_i,y_{i-1})$$

Thus it is shown that $J(y)$ is a function of the values y_i at nodes x_i $(i=1,2,\ldots,n-1)$ inside the interval. We write it as

$$J(y)=J(y_1,y_2,\ldots,y_{n-1})$$

We can regard $J(y)$ as a multivariable function of variables $y_1, y_2, \ldots, y_{n-1}$.

The forth step, writing out the linear equation which the variables $y_1, y_2, \ldots, y_{n-1}$ satisfy. That the functional $J(y)$ takes extremum is equivalent to that the multivariable function $J(y_1,y_2\ldots,y_{n-1})$ takes extremum. It is well known that for a multivariable function to take extremum, it has to satisfy the condition

$$\frac{\partial J(y_1,y_2,\ldots,y_{n-1})}{\partial y_i}=0 \quad i=1,2,\ldots,n-1$$

Because y_i only appears in $J_i(y_i,y_{i-1})$ of the i-th element and $J_{i+1}(y_{i+1},y_i)$ of the (i+1)-th elements we have

$$\frac{\partial J(y_1,y_2,\ldots,y_{n-1})}{\partial y_i}=\frac{\partial J_i(y_i,y_{i-1})}{\partial y_i}+\frac{\partial J_{i+1}(y_{i+1},y_i)}{\partial y_i}=0 \qquad (2.2.5)$$

Similar to 2.2.4, we have

$$J_{i+1}(y_{i+1},y_i)=\frac{1}{2h}(y_{i+1}-y_i)^2+\frac{h}{2}(y_{i+1}+y_i)$$

Taking the partial derivitivs of the expression 2.2.4, and substituting them into 2.2.5, we obtain

$$\frac{\partial J}{\partial y_i}=\frac{-y_{i-1}+2y_i-y_{i+1}}{h}+h=0$$

If the interval [0,1] is divided into four elements i.e. $n=4$, $h=1/4$, $x_1=0.25$, $x_2=0.5$, $x_3=0.75$ and with the boundary conditions $y_0=0$, $y_4=1$, we obtain from the expression a set of equations.

$$\frac{\partial J}{\partial y_1}=4(0+2y_1-y_2)+\frac{1}{4}=0$$

$$\frac{\partial J}{\partial y_2}=4(-y_1+2y_2-y_3)+\frac{1}{4}=0$$

$$\frac{\partial J}{\partial y_3}=4(-y_2+2y_3-1)+\frac{1}{4}=0$$

After arranging we get

$$2y_1-y_2 = -\frac{1}{16}$$

$$-y_1+2y_2-y_3 = -\frac{1}{16}$$

$$-y_2+2y_3 = -\frac{15}{16}$$

The fifth step, solving the linear equation system, we obtain

$$y_1=0.15625,\quad y_2=0.375,\quad y_3=0.65625$$

Subsitituting $x=0.25$, 0.5, 0.75 into 2.2.2, we have the correspondig exact solution as

$$y(0.25)=0.15625,\quad y(0.5)=0.375,\quad y(0.75)=0.65625$$

There is no error in this example. We can divide the interval into more subintervals, but then the work load will increase.

In general, when we use the finite element technique to find the solution of one-dimentional differential equation with boundary conditions, we mustfirst establish the functional expression which is equivalent to the differential equation. Then we transform from solving the differential equation into solving the extremum of the functional. Next, we divide the interval into many small elements. Then we do a linear interpolation of the function inside every element as well as integration of the functional. Finally we sum the integrations over all elements such that we transform the functional of a continuous function into the functional for the values of the function at the descrete nodes. In terms of the necessary condition that a functional takes extremum, we obtain the linear equation system which the value of the function at every nodes satisfies. Solving the equation system, we obtain the values of the function at all nodes. Then these discrete values of the function are the approximate solution of the differential equation.

The basic method of the finite element technique to solve two-dimensional or three-dimensional partial differential equation with boundary conditions are the same.

2.3 Two-dimensional variational problem

At present the two-dimensional forward problem of geoelectric survey is a very important one. Its corresponding equation is Helmholtz's equation (it is easy to reduce it to Poisson's or Laplace's equation) under the first, the second or the third boundary condition. In mathematics, these are all elliptical partial differential equations. Thus here we discuss the variational problem in touch with two-dimensional elliptical equation. First, we discuss the equivalence of the boundary problem of elliptical equation and the corresponding variational problem.

In geoelectric surveying, the elliptical differential equation which the target function (e.g. the electric potential and so on) satisfies in the research region D is

$$LU=-\frac{\partial}{\partial x}(\alpha\frac{\partial U}{\partial x})-\frac{\partial}{\partial z}(\alpha\frac{\partial U}{\partial z})+\beta U=f \qquad (2.3.1)$$

And on the boundary Γ of the region D (Γ is a sealing curve which is smooth segment by segment) the function satisfies one of the conditions listed below:

$$U|_{\Gamma=0} \quad \text{(the first boundary condition)} \qquad (2.3.2)$$

$$\frac{\partial U}{\partial n}|_{\Gamma=0} \quad \text{(the second boundary condition)} \qquad (2.3.3)$$

$$(\frac{\partial U}{\partial n}+\gamma U)|_{\Gamma=0} \quad \text{(the third boundary condition)} \qquad (2.3.4)$$

In the above expressions, $U=U(x,y)$ represents the target function which we want to find, α, β, γ and f are functions of x and z, demanding $\Sigma 0$, $\beta\geq 0$, $\gamma\geq 0$. Equation 2.3.1 is also called Helmholtz's equation.
In mathematics it can be proved that if a function $\overline{U}=\overline{U}(x,y)$ is the solution of the equation LU=f (i.e. 2.3.1) under the boundary condition 2.3.4 (or 2.3.2 or 2.3.3) the function $\overline{U}$ makes the corresponding functional

$$J(U)=(LU,U)-2(f,U) \qquad (2.3.5)$$

to reach minimum, where the parentheses mean the inner product, it is defined as

$$(\phi,\psi)=\int\int_D \phi\psi dxdz.$$

Vice versa, if $\overline{U}$ makes the functional J[U] to reach minimum. $\overline{U}$ is the solution of the equation LU=f under the corresponding boundary condition. That is to say, the boundary value problem of Helmholtz's equation and the variational problem of the quadratic functional is equivalent.

Here we will not give a stringent provement of this but only from the derivation of formula explain their equivalence. First

$$(LU,U)=\iint_D ULU\,dxdz$$

$$=-\iint_D U[\frac{\partial}{\partial x}(\alpha\frac{\partial}{\partial x})+\frac{\partial}{\partial z}(\alpha\frac{\partial}{\partial z})]dxdz+\iint_D \beta U^2 dxdz$$

As

$$U\frac{\partial}{\partial x}(\alpha\frac{\partial U}{\partial x})=\frac{\partial}{\partial x}(\alpha U\frac{\partial U}{\partial x})-\alpha(\frac{\partial U}{\partial x})^2$$

and

$$U\frac{\partial}{\partial z}(\alpha\frac{\partial U}{\partial z})=\frac{\partial}{\partial z}(\alpha U\frac{\partial U}{\partial z})-\alpha(\frac{\partial U}{\partial z})^2$$

we have

$$(LU,U)=\iint_D\{\alpha[(\frac{\partial U}{\partial x})^2+(\frac{\partial U}{\partial z})^2]-\frac{\partial}{\partial x}(\alpha U\frac{\partial U}{\partial x})-\frac{\partial U}{\partial z}(\alpha U\frac{\partial U}{\partial z})\}dxdz$$

$$+\iint_D \beta U^2 dxdz$$

In terms of Green's theorem

$$\iint_D(\frac{\partial Z}{\partial x}+\frac{\partial X}{\partial z})dxdz=\oint(Zdz-Xdx)$$

the last two terms of the integration can be written as

$$\iint_D[\frac{\partial}{\partial x}(\alpha U\frac{\partial U}{\partial x})+\frac{\partial}{\partial z}(\alpha U\frac{\partial U}{\partial z})]dxdz$$

$$=\oint_\Gamma[\alpha U\frac{\partial U}{\partial x}dz-\alpha U\frac{\partial U}{\partial z}dx]$$

From Fig. 2.3.1, it can be seen that

$$\alpha U\frac{\partial U}{\partial x}dz+\alpha U\frac{\partial U}{\partial z}dx=\alpha U\left(\frac{\partial U}{\partial x}\frac{dz}{ds}-\frac{\partial U}{\partial z}\frac{dx}{ds}\right)ds$$

$$\alpha U\left[\frac{\partial U}{\partial x}\cos(n,x)+\frac{\partial U}{\partial z}\cos(n,z)\right]ds=\alpha U\frac{\partial U}{\partial n}ds$$

Hence

$$\iint_D\left[\frac{\partial}{\partial x}\left(\alpha U\frac{\partial U}{\partial x}\right)+\frac{\partial}{\partial z}\left(\alpha U\frac{\partial U}{\partial z}\right)\right]dxdz=\oint_\Gamma \alpha U\frac{\partial U}{\partial n}ds$$

Substituting for the third boundary condition, we obtain

$$\iint_D\left[\frac{\partial}{\partial x}\left(\alpha U\frac{\partial U}{\partial x}\right)+\frac{\partial}{\partial z}\left(\alpha U\frac{\partial U}{\partial z}\right)\right]dxdz=-\oint_\Gamma \alpha\gamma U^2ds$$

where ds is a small segment on Γ, $\bar{n}$ is the direction of the outer normal of this segment. Therefore

$$J(U)=(LU,U)-2(f,U)$$

$$=\iint_D\left\{\alpha\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial U}{\partial z}\right)^2\right]+\beta U^2-2fU\right\}dxdy+\oint_\Gamma \alpha\gamma U^2ds \quad (2.3.6)$$

In fact, because J[U] is a quadratic functional, we can find its first order and second order variation. Assuming $\delta U(x,z)$ is a increment of $u(x,z)$, we have

$$J[U+\delta U]=\iint_D\left\{\alpha\left[\left(\frac{\partial(U+\delta U)}{\delta x}\right)^2+\left(\frac{\partial(U+\delta U)}{\delta z}\right)^2\right]+\beta(U+\delta U)^2\right.$$

$$\left.-2f(U+\delta U)\right\}dxdz+\oint_\Gamma \alpha\gamma(U+\delta U)^2ds$$

$$=\iint_D\left\{\alpha\left[\left(\frac{\partial U}{\partial x}\right)^2+2\left(\frac{\partial U}{\partial x}\right)\left(\frac{\partial\delta U}{\partial x}\right)+\left(\frac{\partial\delta U}{\partial x}\right)^2+\left(\frac{\partial U}{\partial z}\right)^2+2\left(\frac{\partial U}{\partial z}\right)\left(\frac{\partial\delta U}{\partial z}\right)+\left(\frac{\partial\delta U}{\partial z}\right)^2\right]\right.$$

$$\left.-2fU-2f\delta U\right\}dxdz+\oint_\Gamma\left[\alpha\gamma 2U\delta U+\alpha\gamma(\delta U)^2\right]ds=J[U]+\delta J+1/2\delta^2J$$

where

$$\delta J=2\iint_D[\alpha(\frac{\partial U}{\partial x})(\frac{\partial \delta U}{\partial x})+\alpha(\frac{\partial U}{\partial z})(\frac{\partial \delta U}{\partial z})+\beta U\delta U-f\delta U]dxdz$$

$$+2\oint_\Gamma \alpha\gamma U\delta U dUds \qquad (2.3.7)$$

$$\delta^2 J=2\iint_D\{\alpha[(\frac{\partial \delta U}{\partial x})^2+(\frac{\partial \delta U}{\partial z})]+\beta(\delta U)^2\}dxdz+2\oint_\Gamma \alpha\gamma(\delta U)^2$$

The necessary condition for the functional 2.3.6 to take extremum is

$$\delta J=0,$$

but $\Sigma 0$, $\beta\geq 0$, $\gamma\geq 0$ and $\delta U\neq 0$ so $\delta^2 J>0$. Thus the sufficient and necessary condition that the functional 2.2.6 reach minimum is $\delta J=0$ and $\delta^2 J>0$.
Now we rewrite the first order variation δJ using

$$\alpha\frac{\partial U}{\partial x}\frac{\partial \delta U}{\partial x}=\frac{\partial}{\partial x}(\alpha\delta U\frac{\partial U}{\partial x})-\delta U\frac{\partial}{\partial x}(\alpha\frac{\partial U}{\partial x})$$

and

$$\alpha\frac{\partial U}{\partial z}\frac{\partial \delta U}{\partial z}=\frac{\partial}{\partial z}(\alpha\delta U\frac{\partial U}{\partial z})-\delta U\frac{\partial}{\partial z}(\alpha\frac{\partial U}{\partial z})$$

so

$$\delta J=2\iint_D\{[-\frac{\partial}{\partial x}(\alpha\frac{\partial U}{\partial x})-\frac{\partial}{\partial z}(\alpha\frac{\partial U}{\partial z})]\delta U+\frac{\partial}{\partial x}(\alpha\delta U\frac{\partial U}{\partial x})+\frac{\partial}{\partial z}(\alpha\delta U\frac{\partial U}{\partial z})$$

$$+2\oint_\Gamma \alpha\gamma U\partial U ds$$

$$=2\iint_D(LU-f)\delta U dxdz+2\oint_\Gamma \alpha\delta U\frac{\partial U}{\partial n}ds+2\oint_\Gamma \alpha\gamma\delta U ds$$

$$=2\iint_D(LU-f)\delta U dxdz+2\oint_\Gamma \alpha(\frac{\partial U}{\partial n}+\gamma U)\delta U ds$$

As δU is wilful and $\alpha>0$, so from $\delta J=0$ we can derive

$$LU-f=0 \quad \text{and} \quad \left(\frac{\partial U}{\partial n}+\gamma U\right)\Big|_{\Gamma=0} \tag{2.3.9}$$

This is the solution of the third boundary value problem of the equation 2.3.1.
When J=0, the second integral term vanishes. It can be seen easily that $\delta J=0$ corresponds to find a solution of the differential equation under the second boundary condition, i.e.

$$LU-f=0 \quad \text{and} \quad \frac{\partial U}{\partial n}\Big|_{\Gamma=0} \tag{2.3.10}$$

As to the first boundary value problem, as there is no line integral term in the functional 2.3.6, and we consider that $\delta U|_{\Gamma=0}$. So we have

$$\delta J=2\iint_D (LU-f)\delta U dxdz$$

The minimum function U(x,y) corresponding to $\delta J=0$ is the solution of the equation under the first boundary condition.

$$LU-f=0 \quad \text{and} \quad U|_{\Gamma=0} \tag{2.3.11}$$

In the above discussion the case is that when we find a function which causes the functional 2.3.6 to reach minimum, the boundary condition is satisfied automatically by the minimum function and need not to be listed as a condition for determining the solution. Such boundary conditions are called spontaneous boundary condition.
But the first boundary condition, in both the variational problem and the boundary value problem of the differential equation, must be listed as the condition for determining the solution. That is to say, the extremum solution has to be found in the functions which satisfies this boundary condition. This kind of boundary condition is called forced boundary condition.
The discussion above fits with homogeneous medium, so there does not appear physical property papameters of the medium in the formula. When the medium is inhomogeneous we must consider the influence of the separating surface between two media and there will appear physical property para-

meters of the media in the formula. Now we take the electric field as an example.

In terms of the electro-magnetic field theory, on the separating interface between two conductive media, the electric field must satisfy two conditions:

1) the electric potential is continuous, i.e. $U_1=U_2$

2) the normal component of the current density is continuous, i.e. $j_{1n}=j_{2n}$ or

$$\frac{1}{\rho_1}\frac{\partial U_1}{\partial n}=\frac{1}{\rho_2}\frac{\partial U_2}{\partial n}$$

where ρ_1, ρ_2 are the resistivities of the media, $\bar{n}$ is the normal to the interface, pointing from medium 1 to medium 2.

Now there is given a distribution of the two media in Fig. 2.3.2. As to the Laplace equation of the first boundary condition and considering $\alpha=\sigma=1/\rho$, we write the corresponding functional

$$J[U]=\iint_D \frac{1}{\rho}\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial U}{\partial z}\right)^2\right]dxdz$$

$$=\iint_{D_1} \frac{1}{\rho_1}\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial U}{\partial z}\right)^2\right]dxdz+\iint_{D_2} \frac{1}{\rho_2}\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial U}{\partial z}\right)^2\right]dxdz$$

where the region D is made of the regions D_1 and D_2, and $\rho=\rho_1$ in D_1, $\rho=\rho_2$ in D_2. The variation of the expression is written as

$$J[U]=\iint_{D_1} \frac{1}{\rho_1}\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial z^2}\right)\delta U dxdz-\iint_{D_2} \frac{1}{\rho_2}\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial z^2}\right)\delta U dxdz$$

$$+\int_{\Gamma_1} \delta U\frac{1}{\rho_1}\frac{\partial U}{\partial n}ds+\int_{\Gamma_2} \delta U\frac{1}{\rho_2}\frac{\partial U}{\partial n}ds+\int_{\Gamma_{12}^1} \partial U\frac{1}{\rho_1}\frac{\partial U}{\partial n}ds+\int_{\Gamma_{12}^2} \delta U\frac{1}{\rho_2}\frac{\partial U}{\partial n}ds$$

The normal of the separating interface points to the medium 2 from the medium 1, so the last term of the expression is negative. On interface Γ_{12}, because the potential is continuous, the variation of the potential in medium 1 must equal to the variation of it in medium 2, i.e.

$$\delta U|_{\Gamma_{12}^1}=\delta U|_{\Gamma_{12}^2}$$

And because the normal component of the current density is continuous, the last two integral term offset each other. So we obtain the variation of the corresponding functional

$$\delta J[U]=-\iint_D \frac{1}{\rho_1}\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial z^2}\right)\delta U dxdz-\iint_D \frac{1}{\rho_2}\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial z^2}\right)\partial U dxdz$$

$$+\int_{\Gamma_1}\partial U\frac{1}{\rho_1}\frac{\partial U}{\partial n}ds+\int_{\Gamma_2}\partial U\frac{1}{\rho_2}\frac{\partial U}{\partial n}ds$$

$$=-\iint_D \frac{1}{\rho}\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial z^2}\right)\delta U dxdz+\oint_\Gamma \partial U\frac{1}{\rho}\frac{\partial U}{\partial n}ds$$

where $\Gamma=\Gamma_1+\Gamma_2$ only involve the outer boundary, it does not involve the separating interface of the media.

Through the above analysis we know that if the functional expression involves physical property parameters, the variation of the functional has nothing to do with the separating interface and only has to do with the outer boundary. In the process of the functional taking extremum, the boundary condition on the interface will be satisfied automatically. So this condition belongs to the spontaneous one.

3 THE FINITE ELEMENT TECHNIQUE FOR HELMHOLTZ'S EQUATION

As mentioned before, the transformed potential from a point source in the two-dimensional resistivity problem can be described by the two-dimensional Helmholtz's equation 1.2.3. Therefore it is of great importance to apply the corresponding finite element method in the digital modelling of the geoelectric survey.
We know, that the solution of the two-dimensional Helmholtz's equation

$$LU=-\frac{\partial}{\partial x}(\alpha\frac{\partial U}{\partial x})-\frac{\partial}{\partial z}(\alpha\frac{\partial U}{\partial z})+\beta U=f \qquad (3.0.1)$$

with the third boundary condition

$$(\frac{\partial U}{\partial n}+\gamma U|)_{\Gamma=0} \qquad (3.0.2)$$

corresponds to the function which minimizes the functional

$$J(U)=\iint_D\{\alpha[(\frac{\partial U}{\partial x})^2+(\frac{\partial U}{\partial z})^2]+\beta U^2-2fU\}dxdz+\oint_\Gamma\alpha\gamma U^2ds \qquad (3.0.3)$$

In the formula above, U is the target function, D is the research region and Γ is the boundary curve of D.

3.1 The deduction of the coefficient matrix

When we use the finite element technique to resolve the minimum of a functional, the procedure is the same as that of the one-dimensional finite element technique which has been studied in the former section. In domain D, a simple triangular subdivision is used. At the boundary we use an edge of the triangular element to fit the curve approximately. It is not allowed for the edge of a element to cross the interface of two media and and also to a corner of any triangle appear at an edge of another element. It should be avoided to use a very sharp triangle. It is needed to use more elements in the region where the target function U have big variation.

The corners of the elements are called nodes. The mesh consists of all elements and nodes. The number of each element and node is given by a certain order.
Suppose the numbers for the corners of a triangular element are i, j, and m in the anti-clockwise direction (see Fig. 3.1.1). The relevant coordinates are (x_i,z_i), (x_j,z_j) and $x_m,z_m)$, and the values of the function U are Ui, Uj, and U_m. We want to find the values of the function U at all nodes in domain D, except the nodes at the boundary where the first boundary condition is used. Therefore the problem to find the solution of the continuous function U(x,z) is reduced to the problem to find the function values at the nodes.
Suppose the elements are sufficient small and the function is linear in the elements. We have

$$U=\beta_1+\beta_2 x+\beta_3 z \qquad (3.1.1)$$

At the three corners of the element.

$$U_i=\beta_1+\beta_2 x_i+\beta_3 z_i$$

$$U_j=\beta_1+\beta_2 x_j+\beta_3 z_j$$

$$U_m=\beta_1+\beta_2 x_m+\beta_3 z_m$$

We solve the linear equation system by Cramer's rule and get

$$\beta_1=\frac{D_1}{D}=\frac{1}{2\Delta}(a_iU_i+a_jU_j+a_mU_m)$$

$$\beta_2=\frac{D_2}{D}=\frac{1}{2\Delta}(b_iU_i+b_jU_j+b_mU_m)$$

$$\beta_3=\frac{D_3}{D}=\frac{1}{2\Delta}(c_iU_i+c_jU_j+c_mU_m)$$

where

$$D=\begin{vmatrix}1 & x_1 & z_i\\ 1 & x_j & z_j\\ 1 & x_m & z_m\end{vmatrix} \qquad D_1=\begin{vmatrix}U_i & x_i & z_i\\ U_j & x_j & z_j\\ U_m & x_m & z_m\end{vmatrix}$$

$$D_2=\begin{vmatrix}1 & U_i & z_i\\ 1 & U_j & z_j\\ 1 & U_m & z_m\end{vmatrix} \qquad D_3=\begin{vmatrix}1 & x_i & U_i\\ 1 & x_j & U_j\\ 1 & x_m & U_m\end{vmatrix}$$

$$a_i=x_jz_m-x_mz_j, \quad b_i=z_j-z_m, \quad c_i=x_m-x_j$$

$$a_j=x_mz_i-x_iz_m, \quad b_j=z_m-z_i, \quad c_j=x_i-x_m$$

$$a_m=x_iz_j-x_jz_i, \quad b_m=z_i-z_j, \quad c_m=x_j-x_i$$

and

$$\Delta=\frac{D}{2}=\frac{1}{2}(b_ic_j-b_jc_i)$$ is the area of the element

substituting β_1, β_2, β_3 into 3.1.1, we have

$$U(x,z)=\frac{1}{2\Delta}[(a_i+b_ix+c_iz)U_i+(a_j+b_jx+c_jz)U_j+(a_m+b_mx+c_mz)U_m]$$

or

$$U(x,z)=N_i(x,z)U_i+N_j(x,z)U_j+N_m(x,z)U_m \qquad (3.1.2)$$

where

$$N_i=\frac{1}{2\Delta}(a_i+b_ix+c_iz)$$

$$N_j=\frac{1}{2\Delta}(a_j+b_jx+c_jz)$$

$$N_m=\frac{1}{2\Delta}(am+bmx+cmz)$$

which are called the shape functions.
In order to calculate the value of the functional in the triangle, it is necessary to calculate the first derivative of the function U, from 2.1.2, we have

$$\frac{\partial U}{\partial x}=\frac{\partial N_i}{\partial x}U_i+\frac{\partial N_j}{\partial x}U_j\frac{\partial N_m}{\partial x}U_m$$

$$\frac{\partial U}{\partial z}=\frac{\partial N_i}{\partial z}U_i+\frac{\partial N_j}{\partial z}U_j\frac{\partial N_m}{\partial z}U_m$$

where

$$\frac{\partial N_i}{\partial x}=\frac{bi}{2\Delta}\qquad\frac{\partial N_j}{\partial x}=\frac{bj}{2\Delta}\qquad\frac{\partial N_m}{\partial x}=\frac{bm}{2\Delta}$$

$$\frac{\partial N_i}{\partial z}=\frac{ci}{2\Delta}\qquad\frac{\partial N_j}{\partial z}=\frac{cj}{2\Delta}\qquad\frac{\partial N_m}{\partial z}=\frac{cm}{2\Delta}$$

Regard the domain D of the integration for 3.0.2 as the sum of every divided triangle, that is

$$J(U)=\sum_e J_e$$

where J_e is the functional in a triangle element e:

$$J_e=\iint_\Delta\{\alpha[(\frac{\partial U}{dx})^2+(\frac{\partial U}{dz})^2]+\beta U^2-2fU\}dxdz+\oint\Gamma\alpha\gamma U^2ds \quad (3.1.3)$$

Assuming that ν β and γ are all constants, so they can be moved outside the integral. If the element e is in the domain D, the second line integration term equals to zero. If an edge of the triangle element e is at the boundary, the line integration must be calculated. For making our research general there is no harm in assuming that an edge of the element e is at the boundary of the domain D.
Deriving the target-function U is our object. If the functional J(U) is minimized, according to the necessary condition for a minimum, we have

$$\frac{\partial J}{\partial U_l}=0, \quad l=1,2,\ldots,l_o) \tag{3.1.4}$$

where l_o is the sum of the nodes in D. In order to derive the concrete form of the formula, we differentiate formula 3.1.3 with respect to U_l, and assume l=i first.

$$\frac{\partial J_e}{\partial U_i}=\iint_\Delta \{2\alpha[\frac{\partial U}{\partial x}\frac{\partial}{\partial U_i}(\frac{\partial U}{\partial x})+\frac{\partial U}{\partial z}\frac{\partial}{\partial U_i}(\frac{\partial U}{\partial z})]+2\beta U\frac{\partial U}{\partial U_i}-2f\frac{\partial U}{\partial U_i}\}dxdz$$

$$+2\alpha\gamma\int_\Gamma U\frac{\partial U}{\partial U_i}ds \tag{3.1.5}$$

From the expression of $\partial U/\partial x$ and $\partial U/\partial z$ which have been derived before, we obtain

$$\frac{\partial}{\partial U_i}(\frac{\partial U}{\partial x})=\frac{\partial N_i}{\partial x} \qquad \frac{\partial}{\partial U_i}(\frac{\partial U}{\partial z})=\frac{\partial N_i}{\partial z}$$

Hence

$$\frac{\partial U}{\partial x}\frac{\partial}{\partial U_i}(\frac{\partial U}{\partial x})=\frac{\partial U}{\partial x}\frac{\partial N_i}{\partial x}=(\frac{\partial N_i}{\partial x}U_i+\frac{\partial N_j}{\partial x}U_j+\frac{\partial N_m}{\partial x}U_m)\frac{\partial N_i}{\partial x}$$

$$=\frac{b_i}{2\Delta}(\frac{b_i}{2\Delta}U_i+\frac{b_j}{2\Delta}U_j+\frac{b_m}{2\Delta}U_m)=\frac{1}{4\Delta^2}(b_i^2U_i+b_ib_jU_j+b_ib_mU_m)$$

Similarly

$$\frac{\partial U}{\partial z}\frac{\partial}{\partial U_i}(\frac{\partial U}{\partial z})=\frac{1}{4\Delta^2}(c_i^2U_i+c_ic_jU_j+c_ic_mU_m)$$

From 2.1.2, we get

$$\frac{\partial U}{\partial U_i}=N_i \quad \text{and} \quad U\frac{\partial U}{\partial U_i}=(N_iU_i+N_jU_j+N_mU_m)N_i$$

Putting these terms into formula 3.1.5 and considering that

$$\iint_\Delta dxdz=\Delta$$

then after the function in brackets of formula 3.1.5 has been integrated, we get the first integration term of 3.1.5

$$\frac{\alpha}{2\Delta}[(b_i^2Ui+bibjUj+bibmUm)+(c_i^2Ui+cicjUj+cicmUm)]$$

$$=\frac{\alpha}{2\Delta}[(b_i^2+c_i^2)Ui+(bibj+cicj)Uj+(bibm+cicm)Um]$$

The second integration term of 3.1.5 is

$$\int\int_{\Delta}2\beta[N_i^2Ui+NiNjUj+NiNmUm]dxdz$$

Using the following expression (as proved in appendix of the paragraph)

$$\int\int_{\Delta}N_i(x,z)Nj(x,z)dxdz=\frac{\Delta}{12}(1+\delta ij)$$

where

$$\delta_{ij}=0 \quad \text{when } i\neq j$$

$$\delta_{ij}=1 \quad \text{when } i=j$$

we obtain

$$\int\int_{\Delta}2\beta U\frac{\partial U}{\partial U_i}=2\beta(\frac{2\Delta}{12}U_i+\frac{\Delta}{12}Uj+\frac{\Delta}{12}Um)=\frac{\beta\Delta}{6}(2Ui+Uj+Um)$$

The third integration term is a field source term. We know that f has only two possible forms. That is f=0 when there is no field source in the domain, otherwise $f=I/2\delta(x)\delta(z)$ in our problem. Therefore, the term can be written as

$$\int\int_{\Delta}2f\frac{\partial U}{\partial U_i}dxdz=\int\int_{\Delta}I\delta(x-x_o)\delta(z-z_o)N_idxdz$$

We consider two state. First, point i is a current point, that is $x_o=x_i$, $z_o=z_i$. Applying the defination of the Dirac δ-function and the property of the N function (the proof is shown in appendix of this paragraph).

$$N_i(x_i,z_j)=1$$

Put this relation in the integration term, it gives

$$\iint_{\Delta_i} 2f\frac{\partial U}{\delta U_i}dxdz=I$$

where Δ_i is the sum of the elements which are around the point i. Second, the point i is not a current source, hence this term must be zero.
A boundary element has a line integration

$$2\alpha\gamma\int_{\Gamma} U\frac{\partial U}{\partial U_i}ds$$

where i can represent any node. But in boundary element, the boundary nodes are usually represented by j and m. In order to get rid of the confusion, we rewrite i as l in the term, that is

$$2\alpha\gamma\int_{\Gamma} U\frac{\partial U}{\partial U_l}ds \qquad l=j,m$$

we have mentioned that jm is a boundary line segment. To calculate the term we simply assume that U is linear variable in jm as shown in Fig. 3.1.2.
Hence

$$U=(1-t)U_j+tU_m \qquad 0\leq t\leq 1$$

the length of jm is

$$s_i=\sqrt{[(x_j-x_m)^2+(z_i-z_m)^2]}$$

Then we have

$$t=s/s_i.$$

The line integral term for the point j can be written as

$$2\alpha\gamma\int_{jm}U\frac{\partial U}{\partial U_j}ds=2\alpha\gamma s_i\int_0^1[(1-t)U_j+tU_m](1-t)dt$$

$$=2\alpha\gamma s_i[U_j(\int_0^1 dt+\int_0^1 t^2dt-2\int_0^1 tdt)+U_m(\int_0^1 tdt-\int_0^1 t^2dt)]$$

$$=2\alpha\gamma s_i(\frac{1}{3}U_j+\frac{1}{6}U_m)$$

Similarly for the point m.

$$2\alpha\gamma\int_{jm}U\frac{\partial U}{\partial U_m}ds=2\alpha\gamma s_i(\frac{1}{6}U_j+\frac{1}{3}U_m)$$

We have discussed the formula for calculating

$$\frac{\partial J_e}{\partial U_i} \qquad \text{(i.e. where } l=i\text{)}.$$

When $l=j$, $l=m$, we can get the corresponding formula by similar method. Now the discussion can be summed as to any triangle element, the computation of

$$\frac{\partial J_e}{\partial U_j} \qquad \text{(i.e. where } l=i,j,m\text{)}.$$

has the form of the following matrix

$$\begin{matrix}\frac{\partial J_e}{\partial U_i} \\ \\ \frac{\partial J_e}{\partial U_l} \\ \\ \frac{\partial J_e}{\partial U_m}\end{matrix}\quad\begin{matrix}k_{ii} & k_{ij} & k_{im} \\ \\ k_{ji} & k_{jj} & k_{jm} \\ \\ k_{mi} & k_{mj} & k_{mm}\end{matrix}\quad=\quad\begin{matrix}U_i & I \\ \\ U_j & 0 \\ \\ U_m & 0\end{matrix} \qquad (3.1.6)$$

where the elements of the matrix are

$$k_{ii}=\frac{\alpha}{2\Delta}(b_i^2+c_i^2)+\frac{1}{3}\beta\Delta$$

$$k_{jj}=\frac{\alpha}{2\Delta}(b_j^2+c_j^2)+\frac{1}{3}\beta\Delta+\frac{2}{3}\alpha\gamma_{si}$$

$$k_{mm}=\frac{\alpha}{2\Delta}(b_m^2+c_m^2)+\frac{1}{3}\beta\Delta+\frac{2}{3}\alpha\gamma_{2i} \qquad (3.1.7)$$

$$k_{ij}=\frac{\alpha}{2\Delta}(b_ib_j+c_ic_j)+\frac{1}{6}\beta\Delta=k_{ji}$$

$$k_{im}=\frac{\alpha}{2\Delta}(b_ib_m+c_ic_m)+\frac{1}{6}\beta\Delta=k_{mi}$$

$$k_{jm}=\frac{\alpha}{2\Delta}(b_jb_m+c_jc_m)+\frac{1}{6}\beta\Delta+\frac{1}{3}\alpha\gamma_{si}=k_{mj}$$

In the formula, it is only necessary to calculate the third term (including γ_{si}) for the boundary element. Here the surface is not involved, because $\gamma=0$ on the surface, it is not necessary to compute the surface boundary term. 3.1.6 is called the element coefficient matrix.
After calculating the element matrix for all elements in the domain D, we sum up all element matrics for each node. Then we get the total coefficient matrix. The equation system which corresponds with 3.1.4 is

$$\begin{array}{cccccc}
k_{11} & k_{12}\cdots\cdots k_{1l_0} & U_1 & 0 & & \\
k_{21} & k_{22}\cdots\cdots k_{2l_0} & U_1 & 0 & & \\
\cdot & \cdot \quad \cdot & \cdot & \cdot & (3.1.8) & \\
\cdot & \cdot \quad \cdot & \cdot & \cdot & & \\
\cdot & \cdot \quad \cdot & \cdot & I & \text{the supply point} & \\
\cdot & \cdot \quad \cdot & \cdot & 0 & & \\
\cdot & \cdot \quad \cdot & \cdot & \cdot & & \\
k_{l_01} & k_{l_02} \quad k_{l_0l_0} & U_1 & 0 & &
\end{array}$$

where we remove all terms with the current I to the right hand side. Solving the equation system 3.1.8, we can get the values of the target function at all nodes U_l ($l=1,2,\ldots l_0$).

APPENDIX <u>The geometric meaning of N_i, N_j, N_m and their integration</u>

We know that the area of a triangle which has the three corners (x_i,z_i), (x_j,z_j) and (x_m,z_m) is

$$\Delta=\frac{1}{2}\begin{vmatrix}1 & x_i & z_i\\ 1 & x_j & z_j\\ 1 & x_m & z_m\end{vmatrix}$$

Assuming $p(x,z)$ is any point in Δ, then

$$a_i+b_ix+c_iz=\begin{vmatrix}x_j & z_j\\ x_m & z_m\end{vmatrix}-x\begin{vmatrix}1 & z_j\\ 1 & z_m\end{vmatrix}+z\begin{vmatrix}1 & x_j\\ 1 & x_m\end{vmatrix}$$

$$=\begin{vmatrix}1 & x & z\\ 1 & x_j & z_j\\ 1 & x_m & z_m\end{vmatrix}$$

is twice of the area of the triangle pjm (see Fig.3.1.3) so

$$N_i(x,z)=\frac{(a_i+b_ix+c_iz)}{2\Delta}=\frac{2\Delta pjm}{2\Delta}=\frac{\Delta pjm}{\Delta}$$

From the expression we have following properties of N_i, N_j and N_m:

$$N_i(x,z)+N_j(x,z)+N_m(x,z)=1$$

$$N_i(x_i,z_i)=1,\quad N_i(x_j,z_j)=N_i(x_m,z_m)=0$$

$$N_j(x_j,z_j)=1,\quad N_j(x_i,z_i)=N_j(x_m,z_m)=0$$

$$N_m(x_m,z_m)=1,\quad N_m(x_i,z_i)=N_m(x_j,z_j)=0$$

According to 3.1.2

$$N_i=\frac{1}{2\Delta}(a_i+b_ix+c_iz) \qquad N_j=\frac{1}{2\Delta}(a_j+b_jx+c_jz)$$

We regard the formula as a transform from (x,z) to (N_i,N_j). Depending on the properties of N_i and N_j, they transform the points i, j, m in the plane (N_i, N_j). (See Fig. 3.1.4) The transform expression for the area of the elements is

$$dN_i dN_j = \begin{vmatrix} \frac{\partial N_i}{\partial x} & \frac{\partial N_i}{\partial z} \\ \frac{\partial N_j}{\partial x} & \frac{\partial N_j}{\partial z} \end{vmatrix} dxdz$$

$$= \frac{1}{4\Delta^2} \begin{vmatrix} b_i & c_i \\ b_j & c_j \end{vmatrix} dxdz = \frac{1}{2\Delta} dxdz$$

where

$$\begin{vmatrix} b_i & c_i \\ b_j & c_j \end{vmatrix} = \begin{vmatrix} 1 & x_i & z_i \\ 1 & x_j & z_j \\ 1 & x_m & z_m \end{vmatrix} = 2\Delta$$

or

$$dxdz = 2\Delta dN_i dN_j$$

then we obtain

$$\iint_\Delta N_i^2 dxdz = 2\Delta \iint_\Delta N_i^2 dN_i dN_j = 2\Delta \int_0^1 N_i^2 dN_i \int_0^{1-N_i} dN_j$$

$$= 2\Delta \int_0^1 N_i^2 (1-N_i) dN_i = 2\Delta \left(\frac{N_i^3}{3} \Big|_0^1 - \frac{N_i^4}{4} \Big|_0^1 \right) = 2\Delta \left(\frac{1}{3} - \frac{1}{4} \right) = \frac{\Delta}{6}$$

and

$$\iint_\Delta N_i N_j dxdz = 2\Delta \iint_\Delta N_i N_j dN_i dN_j = \Delta \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) = \frac{\Delta}{12}$$

3.2 Mesh and boundary condition

Usually in our case, the research domain D is divided into many rectangular element by a set of lines which are parallel to the x-axis and a set of lines which are parallel to the z-axis. The distances between these parallel lines are arbitrary given depending on the distribution of the geological structures and the topography. Each rectangle is divided into four triangle elements (see Fig. 3.2.2). We will prove later that the points in the center of the rectangles do not need to be stored. So we only have to consider that the crossover points of the two set of lines. By using the triangle element, the mesh can be used easily to follow the shape of any inhomogeneous structure and the topography profile curve.

Of course, using larger mesh and/or more nodes, we can get higher accuracy of the computation. But then more storage and more computing time are needed. So the mesh is usually with the elements finer in the center of its upper part and gets gradually more and more gross outwards as shown in Fig. 3.2.1.

As mentioned before, there are three kinds of boundary conditions. If we use the first condition, we must give the potential values at the boundary of the mesh. If we utilize the zero values at the boundary, a rather large mesh is needed in order to reduce the influence of the inhomogeneous medium.

When the potential values at a part of the boundary are given, we need to rewrite the equation system 3.1.8. We know that the total number of the nodes is l_0, and suppose the number of the nodes at which we want to find the potential is l_1. So the number of the boundary nodes is l_0-l_1. Assuming the boundary nodes are the last nodes (we do not need to do so in the computing practise), then 3.1.8 is written as

$$
\begin{array}{ccccc}
a_{1,1} & \cdots\cdots a_{1,l_1} & \cdots\cdots a_{1,l_0} \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
a_{l_1,1} & \cdots\cdots a_{l_1,l_1} & \cdots a_{l_1,l_0} \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
a_{l_0,1} & \cdots\cdots a_{l_0,l_1} & \cdots a_{l_0,l_0}
\end{array}
\quad
\begin{array}{c}
U_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ U_{l_1} \\ \cdot \\ \cdot \\ U^{\circ}_{l_0}
\end{array}
\quad
\begin{array}{c}
\\ \\ \\ \\ = \\ \\ \\ \\ \\ \\
\end{array}
\quad
\begin{array}{c}
0 \\ \cdot \\ 0 \\ I \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0
\end{array}
$$

Where U° is the given value. In fact, the number of the unknown potential values are l_1. Removing the terms of the known values in the equation system to the right hand side of these equations, we have

$$
\begin{array}{ccccc}
a_{1,1} & \cdots\cdots & a_{1,l} & 0 \cdots\cdots & 0 \\
\cdot & & \cdot & \cdot & \cdot \\
\cdot & & \cdot & \cdot & \cdot \\
\cdot & & \cdot & \cdot & \cdot \\
\cdot & & \cdot & \cdot & \cdot \\
a_{l_1,1} & \cdots\cdots & a_{l_1,l_1} & 0 \cdots\cdots & 0 \\
0 & \cdots\cdots & 0 & 1 \cdots\cdots & 0 \\
\cdot & & \cdot & 0 \; 1 \cdots & \cdot \\
\vdots & & \vdots & \vdots \; 0 & \vdots \; 1
\end{array}
\quad
\begin{array}{c}
U_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ U_{l_1} \\ U_{l_1+1} \\ U_{l_1+2} \\ \vdots \; U_{l_0}
\end{array}
\quad
\begin{array}{c}
-\sum_{t=l_1+1}^{l_0} a_{1,t} U^{\circ}_t \\ \cdot \\ \cdot \\ \cdot \\ I-\sum_{t=l_1+1}^{l_0} a_{i,t} U^{\circ}_t \\ -\sum_{t=l_1+1}^{l_0} a_{l_1,t} U^{\circ}_t \\ U^{\circ}_{l_1+1} \\ U^{\circ}_{l_1+2} \\ \vdots \; U^{\circ}_{l_0}
\end{array}
$$

In the practical compution we do not need to arrange the known potential equations to the last part of the system, we just need to calculate the terms on the right hand side of the system to put 1 at the diagonal position in the matrix corresponding to the node at which the potential value is known, and to put 0 at all corresponding elements in the row and column of the matrix.

A. Dey and H.F. Morrison used the mixed boundary condition in the finite difference method in 1979. Using this condition we do not need to give values at the boundary of the mesh. The third condition 3.2 is only suitable to the homogeneous medium. In inhomogeneous medium it should be better to put a constant λ in the condition, that is

$$(\lambda\frac{\partial U}{\partial n}+\gamma U)\Big|_{\Gamma=0} \qquad (3.2.3)$$

The solution of the Helmholtz's equation 3.0.1 under the condition 3.2.3 corresponds to the function which minimize the functional

$$J(U)=\int\int\{\alpha[(\frac{\partial U}{\partial x})^2+(\frac{\partial U}{\partial z})^2]+\beta U^2-2fU\}dxdz+\frac{1}{\lambda}\oint_\Gamma \alpha\gamma U^2 ds \qquad (3.2.4)$$

The corresponding element coefficient matrix is

$$J_{ii}=\frac{^0}{2\Delta}[b_i^2+c_i^2]+\frac{^0}{3}k^2\Delta$$

$$J_{ij}=J_{ji}=\frac{^0}{2\Delta}[b_ib_j+c_ic_j]+\frac{^0}{6}k^2\Delta$$

$$J_{im}=J_{mi}=\frac{^0}{2\Delta}[b_ib_m+c_ic_m]+\frac{^0}{6}k^2\Delta$$

$$J_{jj}=\frac{^0}{2\Delta}[b_j^2+c_j^2]+\frac{^0}{3}k^2\Delta+\frac{^0}{3}s_i\alpha_i$$

$$J_{jm}=J_{mj}=\frac{^0}{2\Delta}[b_jb_m+c_jc_m]+\frac{^0}{6}\sigma k^2\Delta+\frac{^0}{6}s_i\alpha_i$$

$$J_{mm}=\frac{^0}{2\Delta}[b_m^2+c_m^2]+\frac{^0}{3}k^2\Delta+\frac{^0}{3}s_i\alpha_i$$

Now we prove that it is not necessary to store the points in the center of the rectangles by transforming the calculating formula to eliminate the potential at these points. Taking an arbitrary rectangle as shown in Fig 3.2.2. Suppose the number of the four nodes are g, h, q,

p, the center crossover point is a, the length of the rectangle are Dx in x direction and Dz in z direction, the tringle gap is Δ_1, gah is Δ_2, haq is Δ_3 and paq is Δ_4. The area of the four triangle is the same.

$$\Delta_1=\Delta_2=\Delta_3=\Delta_4=\Delta=\frac{1}{4}DxDz$$

Their conductivity are σ_1, σ_2, σ_3 and σ_4 respectively. According to the formula 3.2.3 and let $\lambda=2$ the contributions of the four triangular element to the rectangular element ceofficient matrix are as follow.
In Δ_3

$$k_{h,h}^{3}=\frac{\sigma_3}{2\Delta}[(\frac{Dz}{2})^2+(\frac{Dx}{2})^2]+\frac{1}{3}\sigma_3 k^2\Delta=\sigma_3(\frac{Dz}{2Dx}+\frac{Dx}{2Dz})+\sigma_3\frac{1}{12}DxDzk^2$$

Similarly, in Δ_2

$$k_{h,h}^{2}=\sigma_2(\frac{Dz}{2Dx}+\frac{Dx}{2Dz})+\sigma_2\frac{1}{12}DxDzk^2$$

From Fig. 3.2.2 we can see that $k_{h,h}$ in the matrix is contributed by the elements Δ_2 and Δ_3. Therefore $k_{h,h}$ is the sum of the two expression above.

$$k_{h,h}=(\sigma_2+\sigma_3)[\frac{Dz}{2Dx}+\frac{Dx}{2Dz}+\frac{1}{12}DxDzk^2]$$

Similarly, the other diagonal elements can be written as

$$k_{g,g}=(\sigma_1+\sigma_2)[\frac{Dz}{2Dx}+\frac{Dx}{2Dz}+\frac{1}{12}DxDzk^2]$$

$$k_{p,p}=(\sigma_1+\sigma_4)[\frac{Dz}{2Dx}+\frac{Dz}{2Dz}+\frac{1}{12}DxDzk^2]$$

$$k_{q,q}=(\sigma_3+\sigma_4)[\frac{Dz}{2Dx}+\frac{Dz}{2Dz}+\frac{1}{12}DxDzk^2]$$

the four triangle will make contribution to $k_{a,a}$

$$k_{a^1,a}=2\sigma_1\frac{Dz}{Dx}+\frac{1}{12}\sigma_1 DxDzk^2$$

$$k_{a^2,a}=2\sigma_2\frac{Dx}{Dz}+\frac{1}{12}\sigma_2 DxDzk^2$$

$$k_{a^3,a}=2\sigma_3\frac{Dz}{Dx}+\frac{1}{12}\sigma_3 DxDzk^2$$

$$k_{a^4,a}=2\sigma_4\frac{Dx}{Dz}+\frac{1}{12}\sigma_4 DxDzk^2$$

so

$$k_{a,a}=2(\sigma_2+\alpha_4)\frac{Dx}{Dz}+(\sigma_1+\sigma_3)\frac{Dz}{Dx}+\frac{1}{12}(\sigma_1+\sigma_2+\sigma_3+\sigma_4)DxDzk^2$$

The non-diagonal elements in the matrix are as follow.

$$k_{h,g}=\frac{\sigma_2}{2\Delta}[(\frac{Dz}{2})^2-(\frac{Dx}{2})^2]+\frac{1}{6}\sigma_2 k^2\Delta=\sigma_2[\frac{Dz}{2Dx}-\frac{Dx}{2Dz}+\frac{1}{24}DxDzk^2]$$

$$k_{h,q}=\sigma_3[\frac{Dx}{2Dz}-\frac{Dz}{2Dx}+\frac{1}{24}DxDzk^2]$$

$$k_{p,q}=\sigma_4[\frac{Dz}{2Dx}-\frac{Dx}{2Dz}+\frac{1}{24}DxDzk^2]$$

$$k_{g,p}=\sigma_1[\frac{Dx}{2Dz}-\frac{Dz}{2Dx}+\frac{1}{24}DxDzk^2]$$

and

$$k_{a^1,g}=k_{a^1,p}=-\sigma_1\frac{Dx}{Dz}+\frac{1}{24}\sigma_1 DxDzk^2$$

$$k_{a^2,g}=k_{a^2,h}=-\sigma_2\frac{Dz}{Dx}+\frac{1}{24}\sigma_2 DxDzk^2$$

$$k_{a^3,p}=k_{a^3,p}=-\sigma_3\frac{Dx}{Dz}+\frac{1}{24}\sigma_3 DxDzk^2$$

$$k_{a,q}=k_{a,p}=-\sigma_4\frac{Dz}{Dx}+\frac{1}{24}\sigma_4 DxDzk^2$$

so

$$k_{a,g}=k_{a,g}^1+k_{a,g}^2=\sigma_1\frac{Dx}{Dz}-\sigma_2\frac{Dz}{Dx}+\frac{1}{24}(\alpha_1+\sigma_2)DxDzk^2$$

$$k_{a,h}=k_{a,h}^2+k_{a,h}^3=\sigma_2\frac{Dz}{Dx}-\sigma_3\frac{Dx}{Dz}+\frac{1}{24}(\alpha_2+\sigma_3)DxDzk^2$$

$$k_{a,p}=k_{a,p}^1+k_{a,p}^4=\sigma_1\frac{Dx}{Dz}-\sigma_4\frac{Dz}{Dx}+\frac{1}{24}(\alpha_1+\sigma_4)DxDzk^2$$

$$k_{a,q}=k_{a,q}^3+k_{a,q}^4=\sigma_3\frac{Dx}{Dz}-\sigma_4\frac{Dz}{Dx}+\frac{1}{24}(\alpha_3+\sigma_4)DxDzk^2$$

Now we can write the linear equation system for these five nodes.

$$k_{a,a}U_a+k_{a,g}U_g+k_{a,h}U_h+k_{a,p}U_p+k_{a,q}U_q=b_a$$

where U_g, U_h, U_p, U_q and U_a are potential values at the nodes g, h, p, q and a. b_a, b_g, b_h, b_p and b_q are the corresponding right hand terms of the system. We use the Gauss elimination to eliminate U_a in the last four equations and we get a new equation system:

$$\begin{aligned}
&k_{a,a}U_a+k_{a,g}U_g+k_{a,h}U_h+k_{a,p}U_p+k_{a,q}U_q=b_a\\
&k'_{g,g}U_g+k'_{g,h}U_h+k'_{g,p}U_p+k'_{g,q}U_q=b'_g\\
&k'_{h,g}U_g+k'_{h,h}U_h+k'_{h,p}U_p+k'_{h,q}U_q=b'_h\\
&k'_{p,g}U_g+k'_{p,h}U_h+k'_{p,p}U_p+k'_{p,q}U_q=b'_p\\
&k'_{q,g}U_g+k'_{q,h}U_h+k'_{q,p}U_p+k'_{q,q}U_q=b'_q
\end{aligned}$$

(3.2.5)

Taking the first equation away we obtain four equations including four unknown values, U_g, U_h, U_p and U_q. Then there is no term including a in the coefficient matrix and

there is no need to store a. But we need to calculate the elements k' of the matrix according to a set of new formulae which are simmilar to:

$$k'_{g,g}=k_{g,g}-(k_{a,g})^2/k_{a,a} \qquad (3.2.6)$$

the rest is very similar and will not be printed here.
We need to sum up the boundary term for the nodes at the boundary of the mesh.
For the left hand boundary:

$$k'_{g,g}=k'_{g,g}+1/3\cdot YDzo_2$$

$$k'_{h,h}=k'_{h,h}+1/3\cdot YDzo_2$$

$$k'_{g,h}=k'_{g,h}+1/6\cdot YDzo_2$$

For the right hand boundary:

$$k'_{p,p}=k'_{p,p}+1/3\cdot YDzo_4$$

$$k'_{g,g}=k'_{g,g}+1/3\cdot YDzo_4 \qquad (3.2.7)$$

$$k'_{g,g}=k'_{g,g}+1/6\cdot YDzo_4$$

For the bottom boundary:

$$k'_{h,h}=k'_{h,h}+1/3\cdot YDxo_3$$

$$k'_{q,q}=k'_{q,q}+1/3\cdot YDxo_3$$

$$k'_{h,q}=k'_{h,q}+1/3\cdot YDxo_3$$

From here we know the total coefficient matrix is a symmetric positive definite and band one. It is shown in Fig. 3.2.3 (the number of the nodes in z direction is 8)

3.3 LLT decomposition for solving linear equation system

Because the coefficient matrix is a symmetric, positive definite band matrix, we will solve the equation system by LLT decomposition. The method for a band matrix comes from the following theorem.
If A is a band, symmetric and positive definite

$$a_{ij}=0 \qquad |i-j|>m$$

where m is the width of the band then there is a real non-singular lower triangular matrix with the relation

$$LLT=A \tag{3.3.1}$$

$$l_{ij}=0 \qquad i-j<0$$

The formula (3.3.1) may be written as

$$\begin{matrix} l_{11} & & & \\ l_{21} & l_{22} & & \\ l_{31} & l_{32} & l_{33} & \\ \cdots & \cdots & \cdots & \cdots \\ l_{n1} & l_{n2} & \cdots & lnn \end{matrix} \qquad \begin{matrix} l_{11} & l_{21} & \cdots & l_{n1} \\ & l_{22} & \cdots & l_{n2} \\ & & \cdots & l_{n3} \\ & & \cdots & \cdots \\ & & & lnn \end{matrix} \quad =$$

$$\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & an_2 & \cdots & ann \end{matrix}$$

The corresponding elements of the two matrices above LLT and A are equal. We can obtain

$$\sum_{k=i-m}^{i} l_{ik}l_{jk}=a_{ij}$$

or

$$l_{ij}=(a_{ij}-\sum_{k=i-m}^{i-1} l_{ik}l_{jk})/l_{ij} \qquad j=i-m,\ldots,i-1 \tag{3.3.2}$$

and

$$\sum_{k=i-m}^{i} l_{ik}^2 = a_{ii}$$

or

$$l_{ii} = \sqrt{(a_{ii} - \sum_{k=i-m}^{i-1} l_{ik}^2)} \qquad (3.3.3)$$

Changing the equation system

$$Ax = b$$

into

$$Ly = b \qquad (3.3.4)$$

where

$$LTx = y \qquad (3.3.5)$$

We may solve y first by (3.3.4), then solve (3.3.5) to get x. According to (3.3.4), we have

$$\begin{array}{llll} l_{11} & & y_1 & b_1 \\ l_{21}\ l_{22} & & y_2 & b_2 \\ l_{31}\ l_{32}\ l_{33} & & y_3 = & b_3 \\ \cdots\cdots\cdots\cdots & & \cdots & \cdots \\ l_{n1}\ l_{n2}\ \cdots\cdots\ l_{nn} & & yn & bn \end{array}$$

or

$$\begin{array}{ll} l_{11}y_1 & = b_1 \\ l_{21}y_1 + l_{22}y_2 & = b_2 \\ l_{31}y_1 + l_{32}y_2 + l_{33}y_3 & = b_3 \\ \cdots\cdots\cdots\cdots\cdots & = \cdots \\ l_{n1}y_1 + l_{n2}y_2 + l_{n3}y_3 + \cdots l_{nn}yn & = bn \end{array}$$

so

$$y_i = (bi - \sum_{k=i-m}^{i-1} l_{ik})/l_{ii} \qquad (3.3.6)$$

And according to (3.3.5) we have

$$\begin{matrix} l_{11} & l_{21} & \cdots\cdots & l_{n1} \\ & l_{22} & \cdots\cdots & l_{n2} \\ & & \cdots\cdots & l_{n3} \\ & & \cdots\cdots & \\ & & & l_{nn} \end{matrix} \quad \begin{matrix} x_1 \\ x_2 \\ x_3 \\ \cdots \\ xn \end{matrix} \quad = \quad \begin{matrix} y_1 \\ y_2 \\ y_3 \\ \cdots \\ yn \end{matrix}$$

or

$$x_i = (y_i - \sum_{k=i+1}^{i+m} l_{ki} x_i) / l_{ii} \qquad (3.3.7)$$

Because the high number of zero elements in the matrices A and L, it is unnecessary to store all elements. In general, we adopt the packed storage as shown in Fig. 3.3.1.

4 INVERSE FOURIER TRANSFORMATION TECHNIQUE

4.1 Properties of potential ϕ

Solving the linear equation system $A\phi=b$ with a certain k value, we can obtain the transformed potential ϕ in space (x,k,z). The potential U in the space (x,y,z) can be gained from a series of ϕ with respect to different k values in terms of the inverse Fourier transformation. The technique and precision of this transformation are important parts which influence the calculating results. Here we mention the properties of the transformed potential ϕ in the wavenumber domain first.
As we know the potential U in homogeneous medium with a point source of current is

$$U=\frac{c}{r}$$

where r is the distance between the source and the measuring point, c is a constant. Using Fourier transformation to both side of the formula with respect to y, we get

$$\phi(x,k,z)=cK_0(kr) \qquad (4.1.1)$$

where K_0 is the modified Bessel function of order zero. From (4.1.1) we see that the properties of ϕ are the properties of K_0 in homogeneous medium if the constant c is neglected.
The curves of K_0 versus kr, K_0 versus ln(kr) and $\ln K_0$ versus kr are shown in Fig. 4.1.1, Fig. 4.1.2 and Fig. 4.1.3 respectivly.
The curve in Fig. 4.1.1 shows $K_0\to\infty$ when $kr\to 0$, $K_0\to 0$ when $kr\to\infty$ and it is a monotonous declining function when kr increase. There are following approximate formulae in mathematics

$$K_0(kr)=\ln\left(\frac{2}{kr}\right) \qquad \text{when } kr\to 0$$

and

$$K_0(kr)=\sqrt{\left(\frac{\pi}{2kr}\right)}e^{-kr}q_0(kr) \qquad \text{when } kr\to\infty$$

where

$$q_0(kr)=1-\frac{1}{1!8kr}+\frac{1}{2!(8kr)^2}-\cdots$$

From Fig. 4.1.2, we see that the curve of K_0 approximates a straight line in a quite large range at the beginning of the argument kr. Then it becomes a concave when kr is greater than a certain value. The $\ln K_0$ curve in Fig. 4.1.3 in a certain region at the beginning is concave, then it becomes almost a straight line when kr is larger than a certain value. So we can consider ϕ as a negative exponential curve in the calculation of the inverse Fourier transformation when kr grows larger.

It is difficult to consider the properties of ϕ for inhomogeneous medium geoelectric section. Here we just consider two layer model as an example of inhomogeneous medium. In this case it is easy to derive the expression of ϕ in the wave number domain.

In the space (x,y,z) the potential U of a point source in horizontal two-layer medium can be calculated by the following formula

$$U=\frac{I\rho_1}{2\pi}\left[\frac{1}{r}+2\sum_{n=1}^{\infty}\frac{K_{12}}{\sqrt{(r^2+(2nh)^2)}}\right] \qquad (4.1.2)$$

where:

$r=\sqrt{(x^2+y^2)}$ is the distance between the source and the measuring point.

$K_{12}=\dfrac{\rho_2-\rho_1}{\rho_2+\rho_1}$

ρ_1 is the resistivity of the upper layer

ρ_2 is the resistivity of the lower layer

h is the depth to the interface between layer 1 and 2

By Fourier transforming (4.1.2), we have

$$\phi(x,k,z)=\int_0^\infty U(x,y,z)e^{-iky}dy$$

$$=\frac{I\rho_1}{2\pi}\left[\int_0^\infty \frac{1}{\sqrt{(x^2+y^2)}}e^{-iky}dy\right.$$

$$\left.+2\sum_{n=1}^{\infty}K_{12}^{n}\int_0^\infty \frac{1}{\sqrt{(x^2+(2nh)^2+y^2)}}e^{-iky}dy\right]$$

$$=\frac{I\rho_1}{2\pi}\left[K_0(kr_1)+2\sum_{n=1}^{\infty}K_{12}^{n}K_0(kr)\right. \qquad (4.1.3)$$

where $r_1=x$ and $r_2=\sqrt{(x^2+(2nh)^2)}$.
From 4.1.3 we can see that the transformed potential ϕ in the two layer model is also function of $K_0(kr)$, but the difference from homogeneous medium is only that ϕ is the weighting sum of a series of K_0 with different argument. The weight values depend on the geoelectric parameters of the model. The curve of ϕ of (4.1.3) is shown in Fig. 4.1.4. The curves has similar properties with that of homogeneous medium.
We have calculated some ϕ function in different inhomogeneous model by the finite element method. All curves of these ϕ function have similar properties.

4.2 Numerical method for inverse Fourier transform

Usually we use the cosine inverse Fourier transformation as

$$U=\frac{2}{\pi}\int_0^\infty \phi\cos kydk \qquad (4.2.1)$$

Depending on the properties of the function ϕ the following approximate numerical method is adopted for realizing the inverse transformation.
First of all as ϕ values decrease rapidly with the increment kr, the infinite integration interval can be replaced by a finite interval, for instance $[0,k_N]$, without serious error. Then 4.2.1 can be written as

$$U=\frac{2}{\pi}\int_0^{k_n}\phi\cos kydk \qquad (4.2.2)$$

The second, the interval $[0,k_N]$ is divided into many subintervals and the integration in the interval $[0,k_N]$ is replaced by summing up the integration in the subintervals

$$\frac{2}{\pi}\int_0^{k_n}\phi\cos kydk=\frac{2}{\pi}\sum_i\int_{k_i}^{k_{i+1}}\phi\cos kydk \qquad (4.2.3)$$

The third, as we mentioned above, we know from Fig. 4.1.3 that $\ln\phi$ becomes a straight line as kr is greater than a certain value. So the ϕ function is considered as a negative exponential curve approximately in the main part of the integration interval for simplifying the calculation of the integration.

Suppose N discrete wavenumbers k_1, k_2, ..., k_N are chosen within the interval $[0,k_N]$. The integration 4.2.1 is divided into three parts

$$U=\frac{2}{\pi}\int_0^{\infty}\phi\cos kydk$$

$$=\frac{2}{\pi}\int_0^{k_1}\phi\cos kydk+\frac{2}{\pi}\int_{k_1}^{k_N}\phi\cos kydk+\frac{2}{\pi}\int_{k_N}^{\infty}\phi\cos kydk \qquad (4.2.4)$$

$$=\frac{2}{\pi}\int_0^{k_1}\phi\cos kydk+\frac{2}{\pi}\sum_{i=1}^{N-1}\int_{k_i}^{k_{i+1}}\phi\cos kydk+\frac{2}{\pi}\int_{k_N}^{\infty}\phi\cos kydk$$

The second term in 4.2.4 is the main part of the transform. If k_1 is chosen suitable, this part may lie in the straight line of $\ln\phi$ curve, then we choose ϕ as an approximatly negative exponential function.

For the main profile $y=y_0=0$, we have

$$U_{II}=\frac{2}{\pi}\sum_{i=1}^{N-1}\int_{k_i}^{k_{i+1}}\phi\cos kydk=\frac{2}{\pi}\sum_{i=1}^{N-1}\int_{k_i}^{k_{i+1}}\phi dk=\frac{2}{\pi}\sum_{i=1}^{N-1}\int_{k_i}^{k_{i+1}}A_ie^{-aik}dk$$

$$=\frac{2}{\pi}\sum_{i=1}^{N-1}A_i\frac{e^{-a_ik_i}-e^{-a_ik_{i+1}}}{a_i}=\frac{2}{\pi}\sum_{i=1}^{N-1}\frac{\phi_i-\phi_{i+1}}{a_i} \qquad (4.2.5)$$

where a_i can be obtained by solving following equations

$$\phi_i = A_i e^{-a_i k_i}$$

$$\phi_{i+1} = A_i e^{-a_i k_{i+1}}$$

so

$$a_i = \frac{\ln(\phi_i / \phi_{i+1})}{k_{i+1} - k_i} \qquad (4.2.6)$$

For non-main profile $y - y_0 \neq 0$, we can suppose $y_0 = 0$ and have

$$U_{II} = \frac{2}{\pi} \sum_{i=1}^{N-1} \int_{k_i}^{k_{i+1}} A_i e^{-a_i k} \cos ky \, dk$$

$$= \frac{2}{\pi} \sum_{i=1}^{N-1} \frac{A_i e^{-a_i k}}{a_i^2 + y^2} [y \sin ky - a_i \cos ky] \Big|_{k_i}^{k_{i+1}} \qquad (4.2.7)$$

$$= \frac{2}{\pi} \sum_{i=1}^{N-1} \frac{1}{a_i^2 + y^2} [\phi_i (y \sin k_{i+1} y - a_i \cos k_{i+1} y) + \phi_i (y \sin k_i - a_i \cos k_i y]$$

where a_i is the same as 4.2.6.
If k_1 chosen is small enough and k_N is large enough then U_{II} will be the main part of the integral value U.
The third part of U is neglected if k_N is chosen large enough. In that case $\phi(k_N r)$ should be very small. Another approximate method for caculating U_{III} is considered if this is not the case. Using the negative exponential approximation and a_{N-1}, A_{N-1} replace a_N, A_N respectively.
For the main profile, $y = y_0 = 0$

$$U_{III} = \frac{2}{\pi} \int_{k_N}^{\infty} A_{N-1} e^{-a_{N-1} k} \, dk = \frac{2 A_{N-1}}{\pi a_{N-1}} e^{-a_{N-1} k_N}$$

as we have

$$\phi_{N-1} = A_{N-1} e^{-a_{N-1} k_{N-1}}$$

$$\phi_N = A_{N-1} e^{-a_{N-1} k_N}$$

so

$$\phi_N-\phi_{N-1}=A_{N-1}(e^{-a_{N-1}k_N}-e^{-a_{N-1}k_{N-1}})$$

or

$$A_{N-1}=\frac{\phi_N-\phi_{N-1}}{e^{-a_{N-1}k_N}-e^{-a_{N-1}k_{N-1}}}$$

Hence we have

$$U_{III}=\frac{2(\phi_N-\phi_{N-1})}{\pi(e^{-a_{N-1}k_N}-e^{-a_{N-1}k_{N-1}})a_{N-1}}e^{-a_{N-1}k_N}=\frac{2\phi_N}{\pi a_{N-1}} \quad (4.2.8)$$

For non-main profile, $y-y_0=0$. Suppose $y_0=0$, we have

$$U_{III}=\frac{2}{\pi}\int_{k_N}^{\infty}A_{N-1}e^{-a_{N-1}k}\cos ky\,dk$$

$$=-\frac{2\phi_N}{\pi(a_{N-1}^2+y^2)}[y\sin k_Ny-a_{N-1}\cos k_Ny] \quad (4.2.9)$$

Because k_1 is usually chosen rather small, the first part of 4.2.4 has a small value. We simply approximate this integral using the trapezoidal rule, the rectangular rule and the triangular rule.
For the trapezoidal approximation, we use

$$U_I=\frac{[\varphi(k_1r)+C]}{\pi}k_1r \quad (4.2.10)$$

where C is a constant, and determined by test. For the rectangular approximation we have

$$U_I=\frac{2}{\pi}\varphi(k_1r)k_1r \quad (4.2.11)$$

For the triangular approximation we have

$$U_I=\frac{1}{\pi}\varphi(k_1r)k_1r \quad (4.2.12)$$

Of course the U_I values obtained by 4.2.11 and 4.2.12 will be smaller than the integral value. Because k is usually chosen rather small, the first part of the interval $[k_1,k_N]$ is not in the straight line of $\ln\varphi$. So the value U_{II} obtained by 4.2.5 using the negative exponential approximative must be larger than the integral value. Here we use 4.2.11 and 4.2.12 for calculating U_I in order to balance the decrease of U_{II}.

4.3 Selection of wavenumbers

In order to get the inverse Fourier transform, the integral interval is divided into the three parts: $[0,k_1]$, $[k_1,k_N]$ and $[k_N,\infty]$. So it is necessary to choose k_1 and k_N as well as the concrete wavenumber values within the interval $[k_1,k_N]$ reasonably.

The princeple for choosing k_1 and k_N, as mentioned above, is that the integration in the interval $[k_1,k_N]$ is the main part (about 90 - 95%) of the whole integration. Usually k_1 is chosen rather small and k_N is large enough, for example, $k_1r=0.02-0.03$ and $k_Nr=2-3$.

In chosing $k_2,k_3,\ldots,k_{N-1}$, we should consider that the curve φ is satisfied by a negative exponential function approximately. In this case the smaller distance of the interval $[k_i, k_{i+1}]$, the higher accuracy of the calculation is. But on the other side, we need to solve the equation system once for every wavenumber k_i, then the higher the number of the k's is the more the amount of the computation is. So the principle for choosing $k_2,k_3,\ldots,k_{N-1}$ is to choose as low number of k as possible under the condition of a certain calculating error to balance the accuracy and the amount of the calculation. Usually the number of k is chosen 5-9.

The wavenumber values within the interval $[k_1,k_N]$ is considered as uniformly proportional:

$$\frac{k_{i+1}}{k_i}=\text{constant}\quad i=1,2,\ldots,N-1$$

we have just discussed how to choose the argument kr. Here we must notice that different k values should be chosen for different distance r (between the source and the measuring

point). If we have chosen a set of kr values as c_1, c_2, ..., c_N, then the wavenumber are

$$k_1=\frac{c_1}{r}$$

$$k_2=\frac{c_2}{r}$$

.....

$$k_N=\frac{c_N}{r}$$

for a fixed electrode distance configuration. But in practice the electrode distance r is changable, for instance in the geoelectric sounding case. And suppose the shortest electrode distance is rmin, the largest one is rmax. Then we chose

$$k_1=\frac{c_1}{rmax}$$

$$k_N=\frac{c_N}{rmin}$$

5 EXAMPLES

5.1 Some tests of different filters

For testing the program based on the method discribed in this text and the chosen build up of the mesh we made a number of runs on a VAX 11/750 computer at NEA. The runs were made on a homogeneuos earth model. The mesh were made of 161*12 nodes. The filter-coefficients in the test runnings were chosen with different distribution and with different intervals. The main conclusion is that the filter has to be chosen with some care because the same accuracy can be obtained for a considerable less number of coefficients by chosing them carefully. The number of coefficients nessecary for a accuracy better than 3% is 7-9 for a mesh able to deal with the setups of Schlumberger and head-on that are used in Iceland. The calculation-time is approximatly proportional to the number of filter-coefficients and therefore it is important to keep their number down. We will later give some information on the calculation-time compared with a similar calculation with a finite difference program based on Dey 1979. In the tests we used the two arrays mentioned before. The Sclumberger array gives a good knowledge on the change with different armlengths of AB and MN but the head-on gives an estimate of what happends when the center of the array travels through the model. It is difficult to give any visual discribtion of the differences for different filters on a homogeneous model so we will only give two examples bye printing the numerical results of two runnings with 6 and 9 filter coefficients (Appendix of this paragraph). From these numbers it can be concluded that there are some structures in the calculated resistivities which are a function of the mesh and also that the 6 coefficients are hardly enough to give a resonable result. There is also a reason to belive that this is rather a function of the vertical build up of the mesh than the horizontal build up because the results in the head-on profiles are constants for the same armlength. This structure of the results are clearly seen in example 1 in Appendix where for the shorter AB/2 the result is little higher than 1.0 but for AB/2=1200m this turns to be too low.

APPENDIX TO 5.1

Table of calculation of homogenous model with 1 Ohmm using respectively filter with 7 and 9 coefficients.
MN/2 is 25 m.

AB/2	7 coeffi.	9 coeffi.
175	1.0177	0.9962
200	1.0269	1.0066
225	1.0327	1.0134
250	1.0366	1.0181
275	1.0392	1.0211
300	1.0409	1.0229
325	1.0417	1.0236
350	1.0418	1.0233
375	1.0411	1.0221
400	1.0397	1.0202
425	1.0378	1.0176
450	1.0353	1.0145
475	1.0323	1.0109
500	1.0289	1.0070
525	1.0251	1.0027
550	1.0209	0.9981
575	1.0165	0.9933
600	1.0118	0.9883
625	1.0069	0.9831
650	1.0018	0.9777

5.2 Tests of different mesh

The mesh can be chosen in lot of different ways but for the horizontal coordinate it is often necessery to have a symmetrical distribution around the center of the mesh or having a long segment where the nodes are with a constant distance between them. So it is often in the selection of the vertical distribution one has some freedom. It is important to keep the number of the z-nodes down because the bandwidth of the matrix is the number of nodes in the z direction plus one. The calculation time groves very fast with the bandwidth so it is important to remember this when one makes the mesh. In our test it seems that for the kind of setups mentioned in chapter 5.1 we need 10 to 12 nodes in the vertical direction. When one wants to make a very complex resistivity distribution downwards it will cost more nodes and a longer calculation time. The example in Fig. 5.2.1 is a three layer model where there is a theoretical calculation with a one-dimenisional program, a running with a 10 nodes in the z-direction and a running with 12 nodes in the z-direction. One can see that there is not a complete fit, this is though not a serious fault because the theoretical curve is calculated with a gradient method and is therefore giving no net effect from the horizontal boundaries in the model.

5.3 Some theoretical examples

The apparent resistivity curves of the dipole-dipole sounding for two layer model are shown in Fig. 5.3.1 where the real curve is the analytical curve, the dot points are the computing results of the finite element method. We see that the relative error of the calculation is within 1%.
For a model of two medium with vertical interface. Fig. 5.3.2 shows the apparent resistivity ρ_a of the dipole-dipole sounding. Fig. 5.3.3 shows the ρ_a curve of the gradient mapping. Fig. 5.3.4 shows the ρ_a curve of the head on profile which is not perpendicular to the strike of geological body. Fig. 5.3.5 shows the ρ_a curve of the pole-pole profile, in this configuration there is a

distance between the measured points MN and the transmitting point A in the y direction. In these figures the legends is the same as in Fig. 5.3.1.
The ρ_a curve of the pole-dipole profile for the angular topography model is shown in Fig 5.3.6.

5.4 Comparission of FEM and FDM on an example from Urridavatn, Iceland

The example is on head-on profile measured in the Urridavatn area in eastern Iceland in 1982. There is a strong vertical low reistivity contact in the profile. On fig. 5.4.1 ,5.4.2 and 5.4.3 we see the measured profile, the FEM and the FDM results. As can be seen there is a similar result from the two methods and the difference is not a serious one. From this it can be concluded that the two methods are of similar quality according to the structural resolution but there is a reason to belive that FEM is more accurate in the quantities. There is also a difference in the calculation time which makes the FEM-program much more atractive than the FDM. The difference for this model is FEM/FDM<0.3. This a very significant difference and there is also a reason to belive that one can by using FEM get a similar accuracy for less number of filter-coefficients than with FDM so the difference can be even greater than this.

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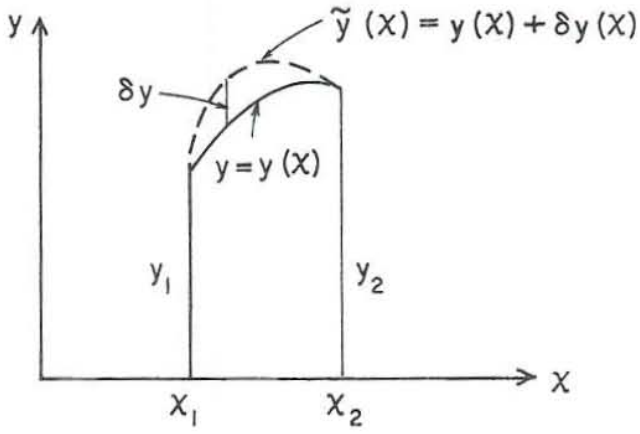


Fig. 2-1-1

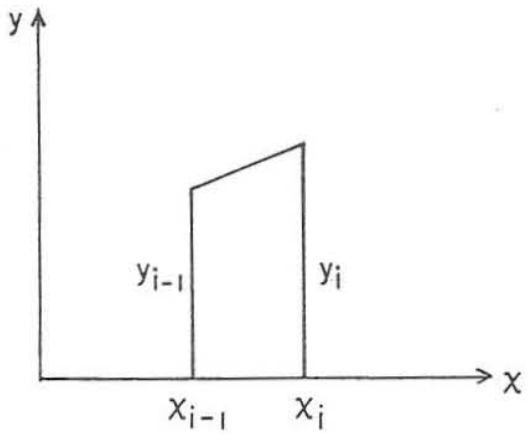


Fig. 2-2-1

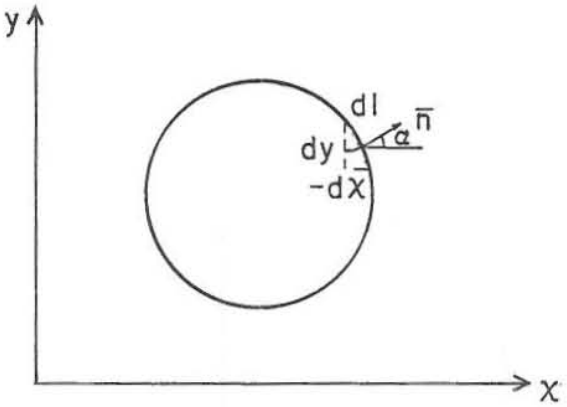


Fig. 2-3-1

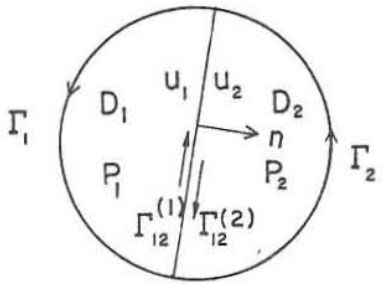


Fig. 2-3-2

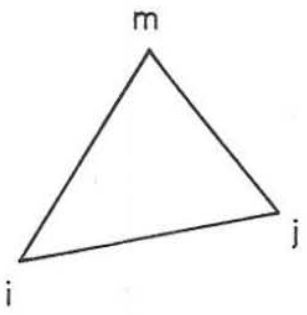


Fig. 3-1-1

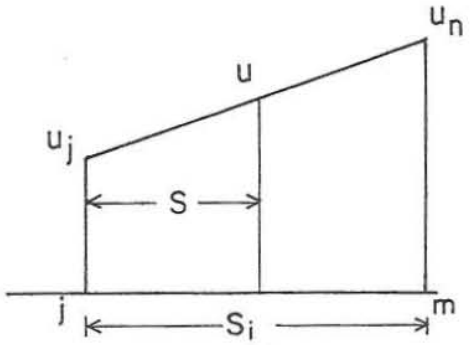


Fig. 3-1-2

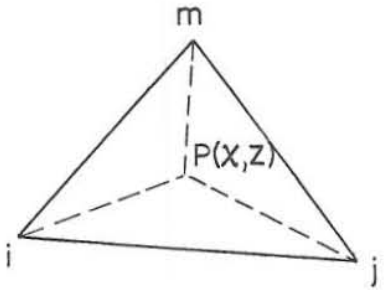


Fig. 3-1-3

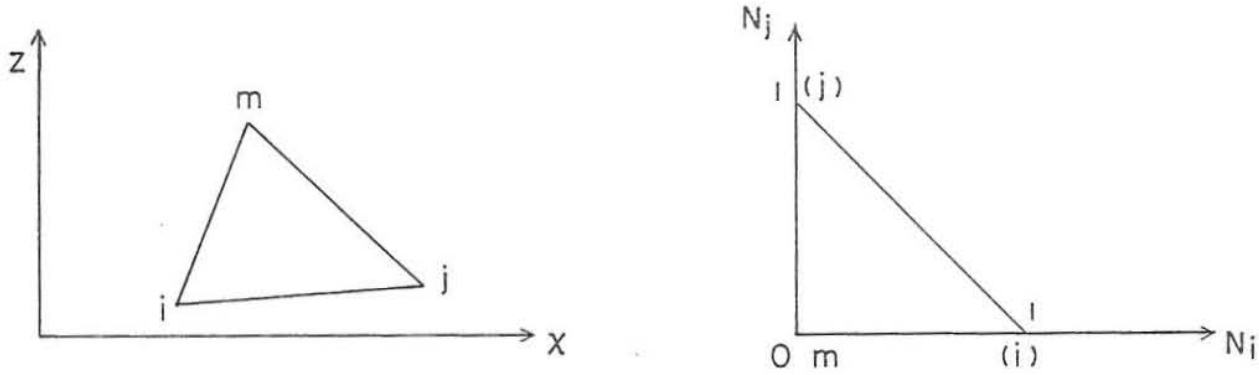


Fig. 3-1-4

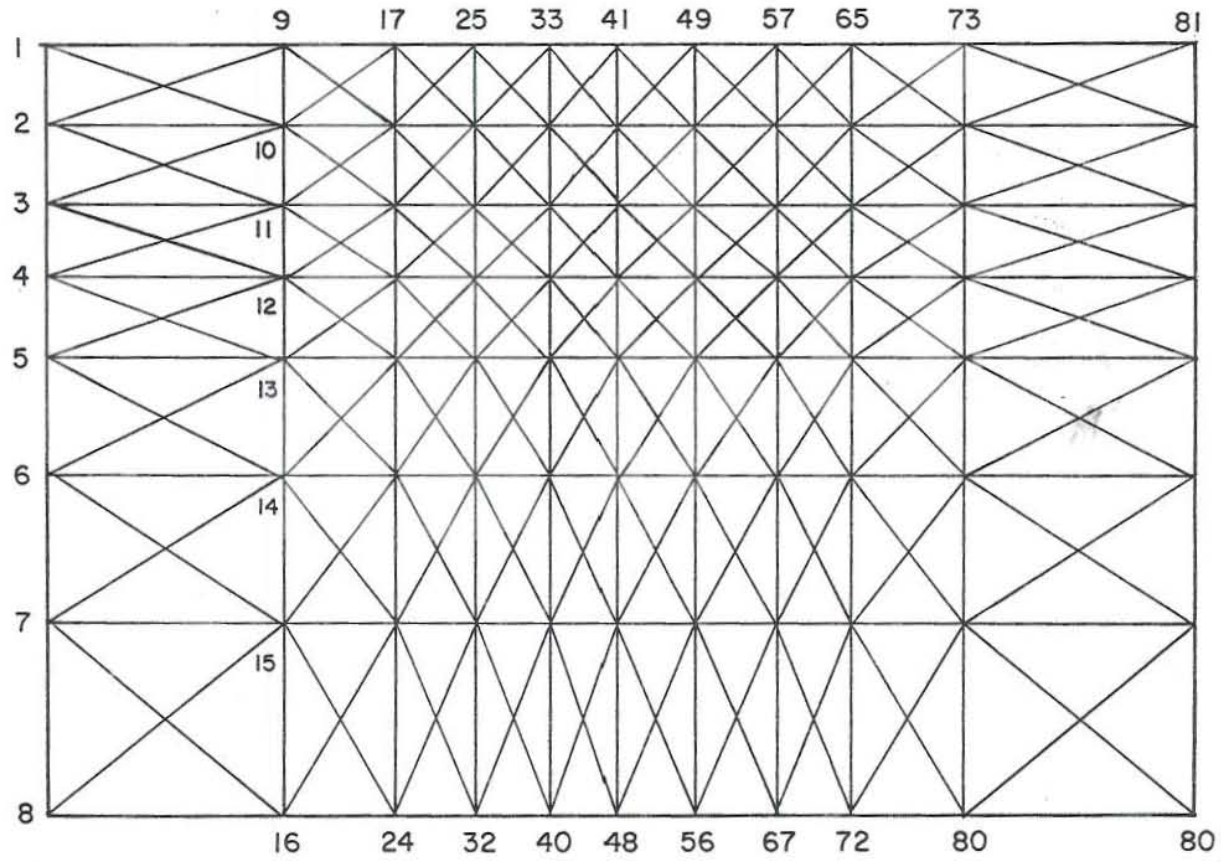


Fig. 3-2-1

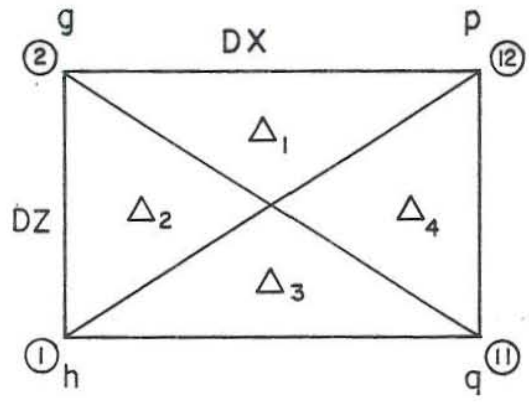


Fig. 3-2-2

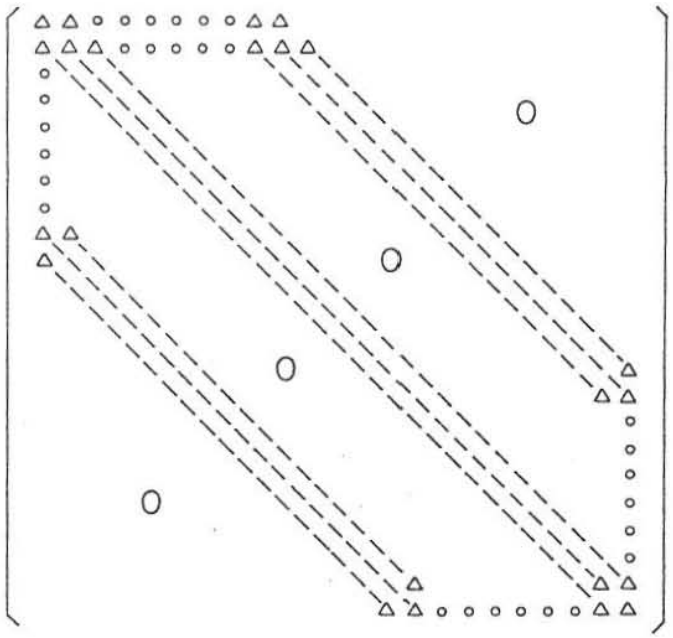


Fig. 3-2-3

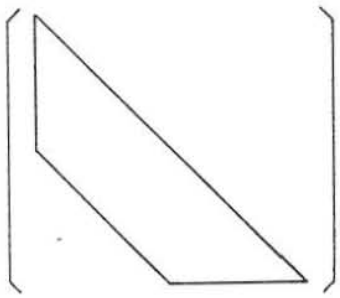
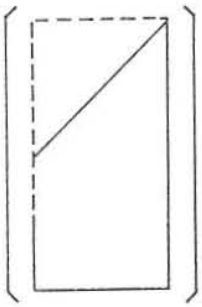


Fig. 3-3-1



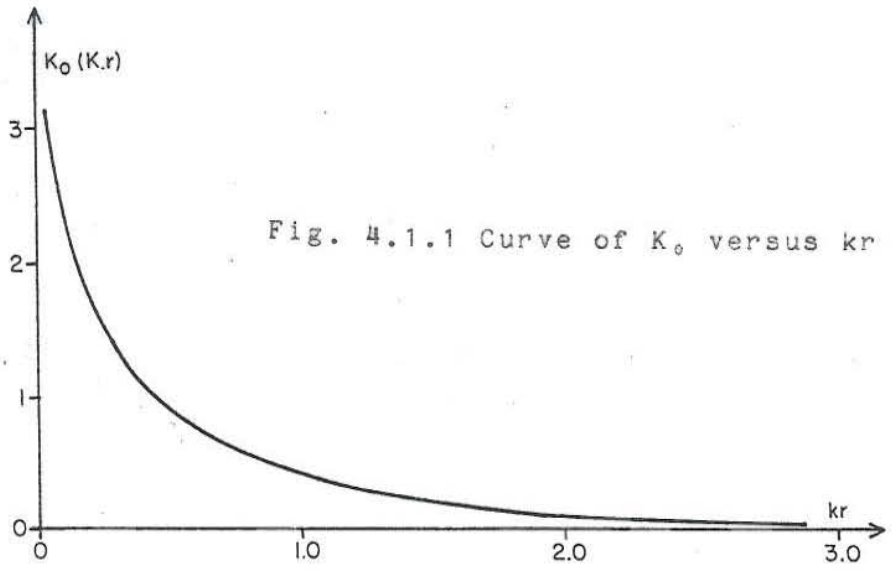


Fig. 4.1.1 Curve of K_0 versus kr

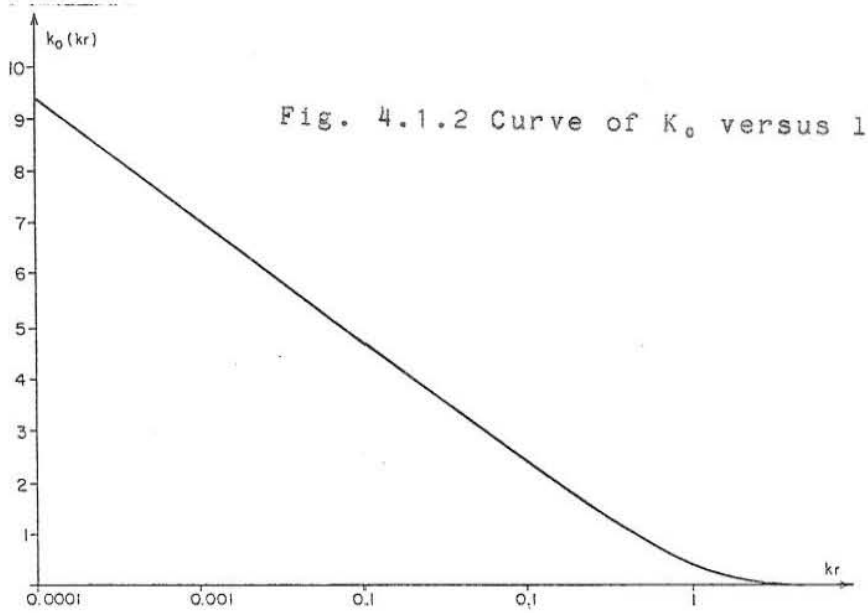


Fig. 4.1.2 Curve of K_0 versus ln(kr)

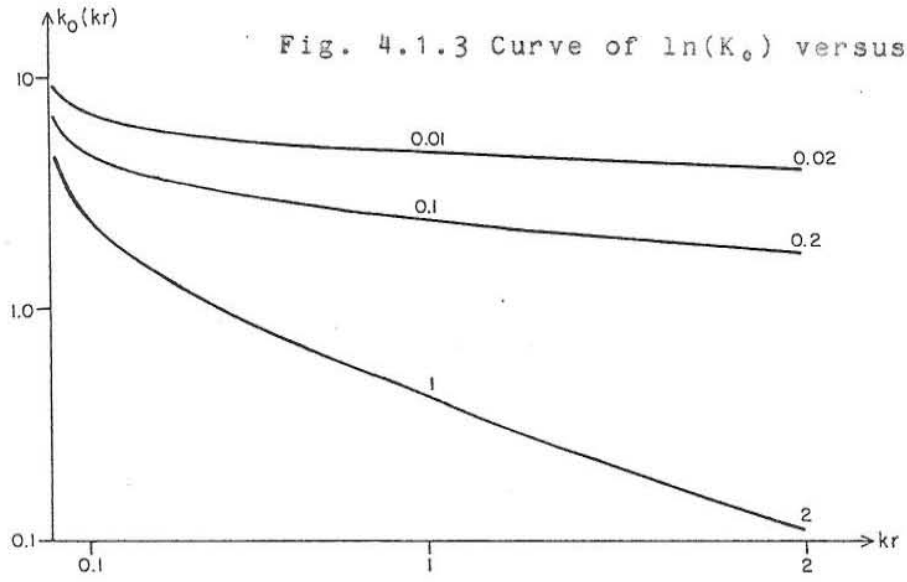


Fig. 4.1.3 Curve of $\ln(K_0)$ versus ln(kr)

Fig. 5.2.1 Three layer model calculated by one dimensional z-direction method and by FEM using 10 and 12 nodes in

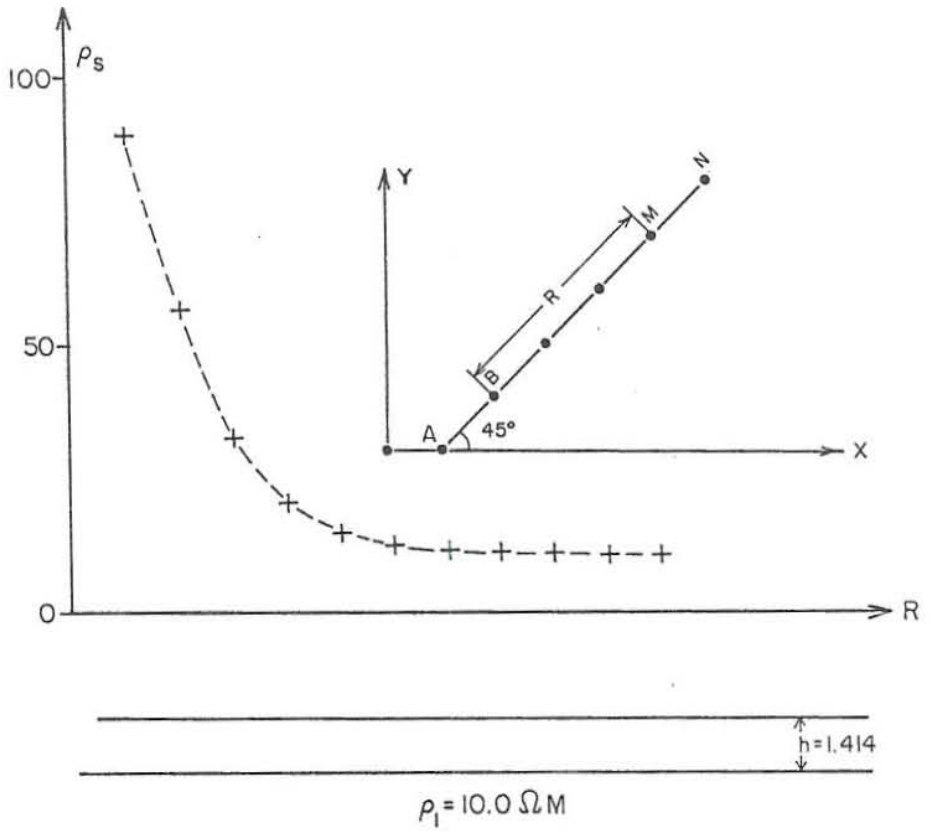


Fig. 5.3.1 Two layer model, dipole-dipole

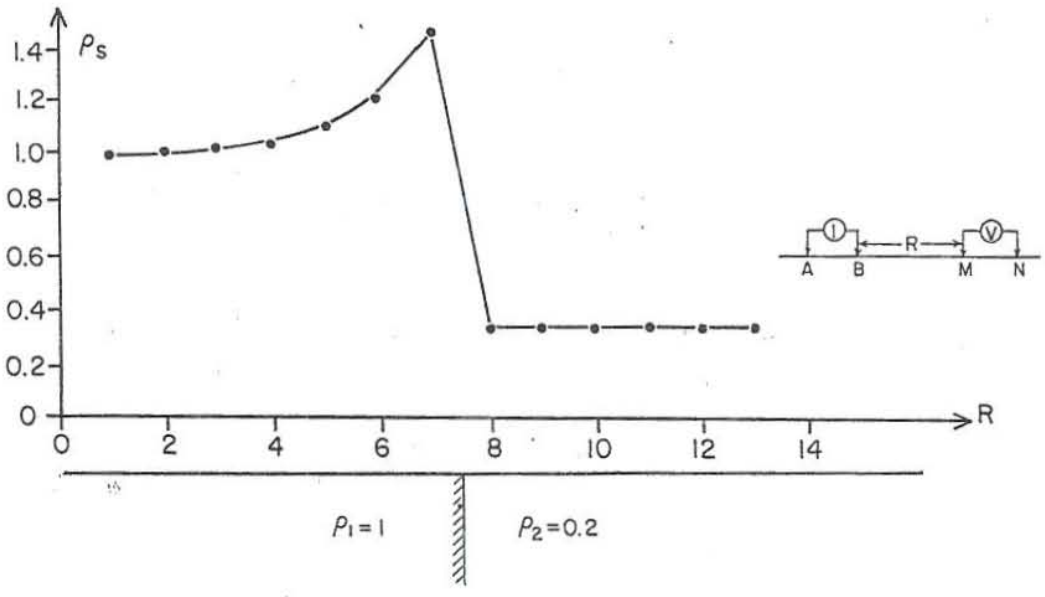


Fig. 5.3.2 Model with vertical interface, dipole-dipole

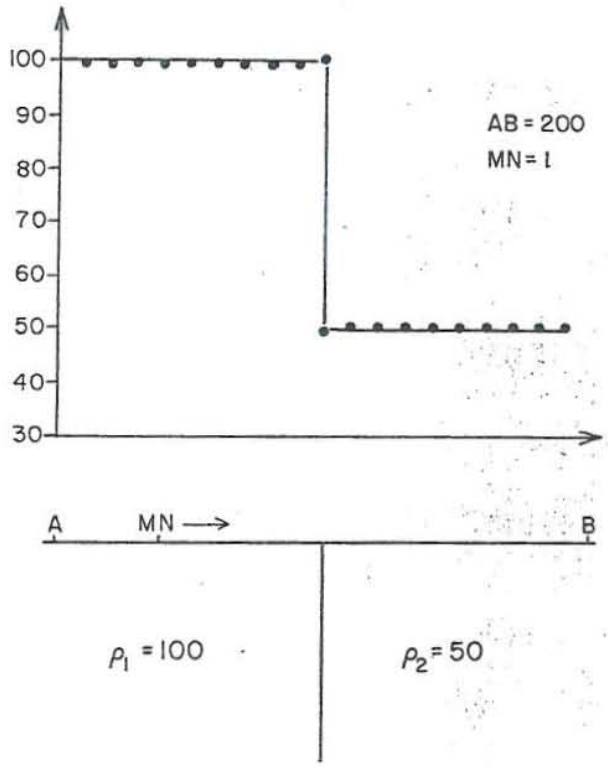


Fig. 5.3.3 Model with vertical interface, Gradient mapping

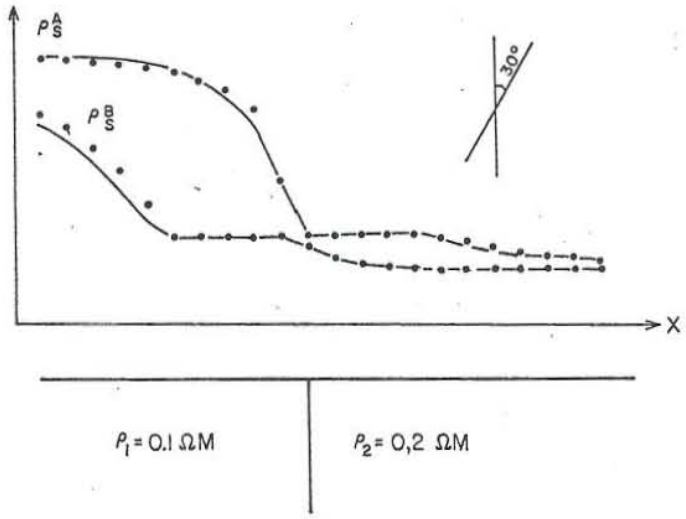


Fig. 5.3.4 Model with vertical interface, head-on profiling

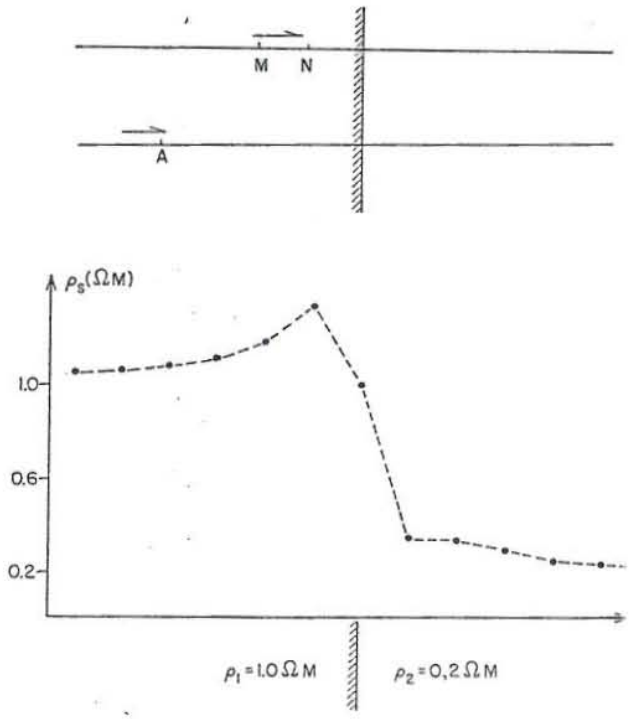


Fig. 5.3.5 Model with vertical interface, pole-pole

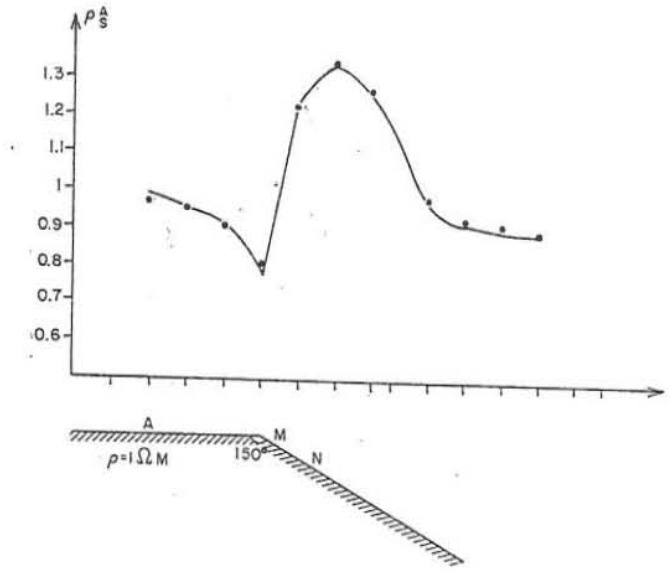


Fig. 5.3.6 Angular topography, pole-dipole

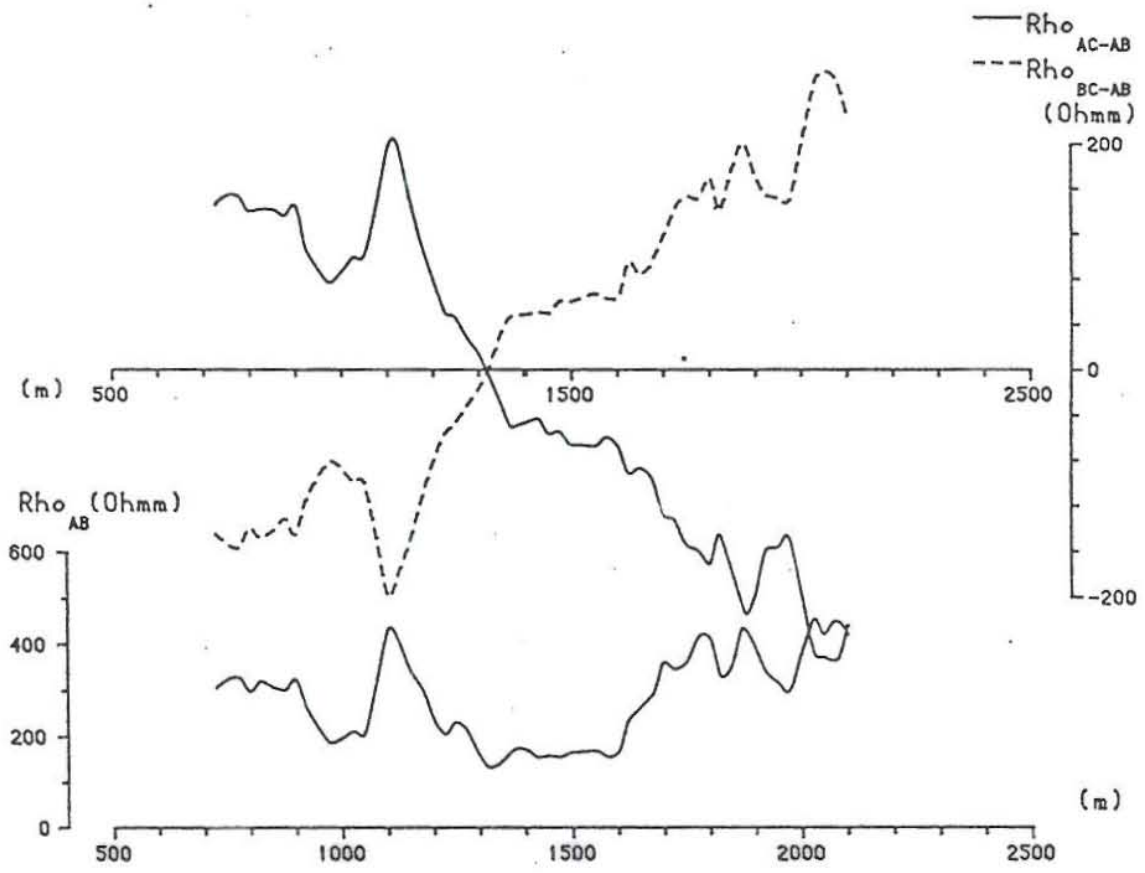


Fig. 5.4.1 Measured data from Urridavatn, Iceland

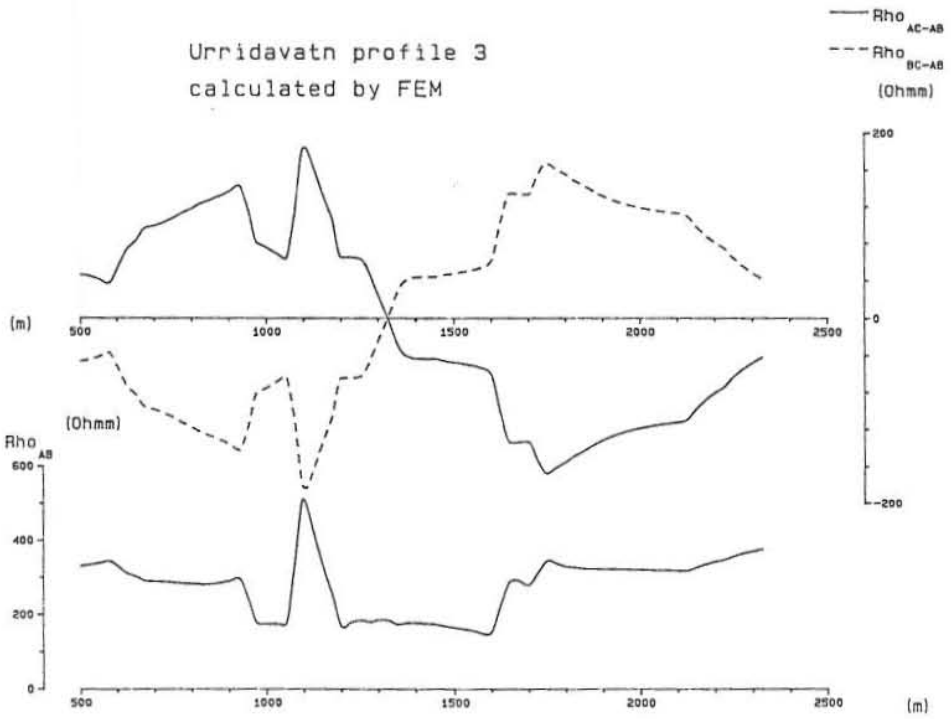


Fig. 5.4.2 FEM calculation

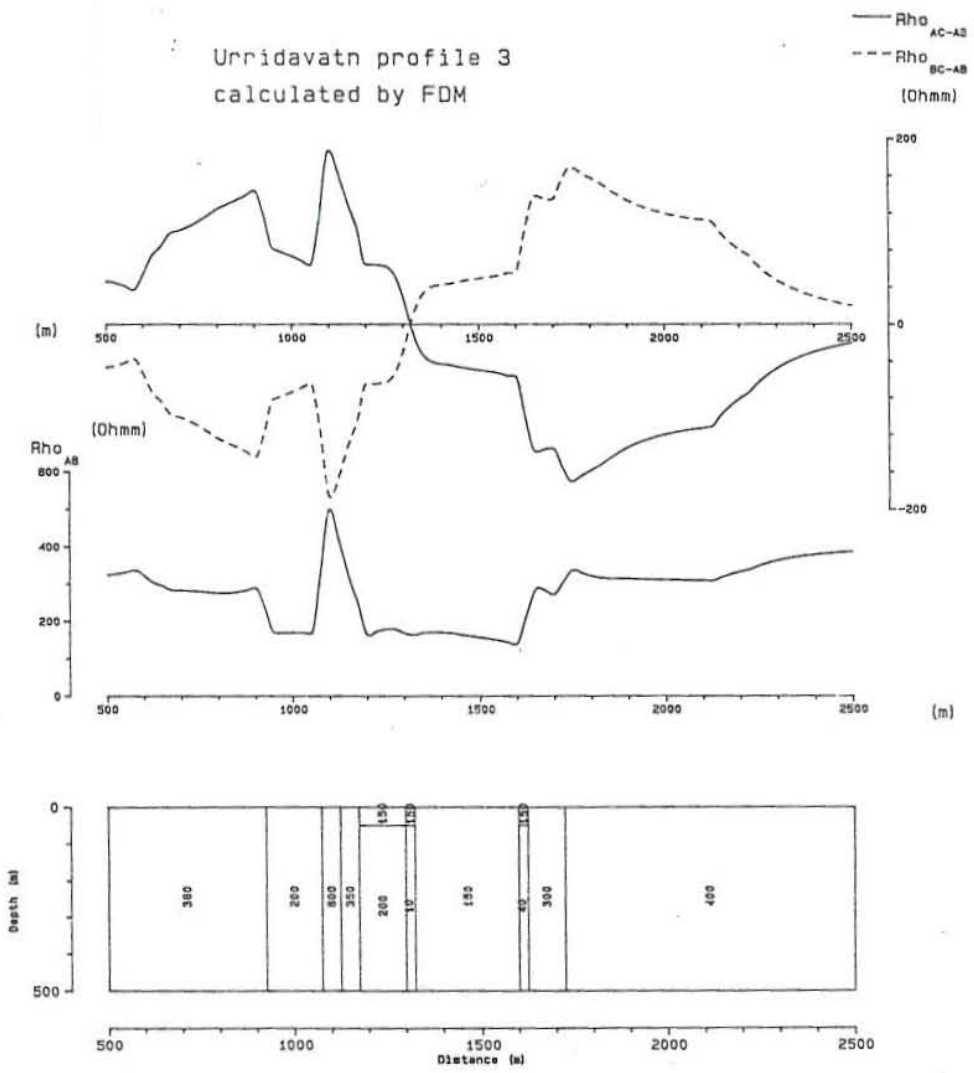


Fig. 5.4.3 FDM calculation