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Theoretical and Numerical Models of Groundwater Reservoir Mechanism

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OF

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BY

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J. J. TRYKTEKNIK A/S

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PREFACE

This thesis is one of the requirements for the Degree of Licentiatus Technices according to the Royal Decree of October 16, 1970.

This study was carried out under the supervision of Professor Jónas Elíasson, Faculty of Engineering and Science at the University of Iceland, and Professor Frank Engelund, Institute of Hydrodynamics and Hydraulic Engineering, Technical University of Denmark.

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Reykjavik, december 1975

Snorri Páll Kjaran

Snorri Páll Kjaran.

ABSTRACT

The physical problem dealt with in this article is the unsteady groundwater flow in an isotropic, nonhomogenous aquifer of limited horizontal extent and arbitrary boundary shape. The unsteady groundwater level is described by a parabolic differential equation. Boundary conditions are provided by nature in the form of watertight rocks, rivers, lakes or any kind of constant water level in hydraulic contact with the aquifer, or given boundary inflow or outflow. It is shown that the aquifer system is equivalent to a system of infinitely many parallel linear reservoirs, with a definite time constant. The inflow to each reservoir is given by the corresponding term of an eigenfunction expansion of the infiltration integrated over the area.

The differential equation is solved by using the Galerkin finite element method in the x and y coordinates but keeping the time continuous. Because the time is kept continuous the physical properties of the aquifer are not lost, as they would be in case of using the finite difference method for the time derivative. We solve instead the eigenvalue problem for the matrices involved in the finite element method.

The eigenvalues and the eigenvectors then give us the geometrical and physical properties of the aquifer and need only be solved once and for all for the same aquifer. If the input function changes we can use the already calculated properties to give us the groundwater level. If we had used the finite difference method for the time derivative we would have to solve the whole problem again if the input function changes.

The flexibility of the finite element method gives you the opportunity to use geological and hydrological insight, when constructing the finite element mesh.

The accuracy of the eigenfunction expansion depends upon how well they approximate the input function, the gradients and the boundary conditions. As the gradients in most problems are small, and the input function does not fluctuate very much over the area, one can be satisfied with a very limited number of eigenfunctions.

Finally a brief description of the problems in parameter estimation is given.

Several theoretical examples are given together with a practical example from a site in Iceland.

1. INTRODUCTION

1.1 A little history

Ancient Arabian wells and elaborate Chinese and Egyptian irrigation systems show the ability of early hydrologist, but hydrology as we know it today, is a young science. The development of theoretical groundwaterhydrology started with a French engineer, Darcy, /10/, who started observations on flow of water through sands. Darcy's name is given to the basic linear law of flow,

$$v = K \cdot I \qquad (1.1)$$

where I is the hydraulic gradient and v is the velocity in the direction of the gradient. Darcy verified this law experimentally, but since then several theoretical verifications of Darcy's law have been considered, f.ex. capillary tube models, fissure models, hydraulic radius models, resistiance to flow models, statistical models, averaging the Navier-Stokes equations, Ferrandon's model.

Dupuit,/13/, extended Darcy's linear law into a quadratic law at larger Reynolds numbers, though it bears the name of a later Austrian pioneer, Forchheimer, /28/. Dupuit also used the first approximate solution of flow to wells in unconfined aquifers, where he suggests using hydrostatic pressure distribution. Boussinesq,/ 4/, extended his idea to unsteady flow. Thousands of papers have since been published in agricultural, hydrological, geological and engineering journals on flow of water and other fluids through soils and other porous media. A number of extensive monographs, theses and papers have also been published.

The best known textbooks are in the United States of America by Muskat, /57/, on flow of homogeneous and heterogeneous fluids, mainly for petroleum engineers. More elementary textbooks are by Todd, /82/, Domenico, /12/, and

Harr, /38/. In Russia there is a highly mathematical book by Polubarinova - Kochina, /69/, also available in English translation. There is another book in Russian by Aravin-Numerov, /62/, with an English translation. In Canada Scheidegger, /72/, has written a textbook, mainly for use in petroleum engineering. In Israel, Bear, Zaslavsky and Irmay, /44/, have written a book in English covering the subject and later Bear, /13/, presents an extensive survey of the theory of dynamics of fluids in porous media, as applicable to many disciplines of science and engineering.

The basic ideas of the pioneers mentioned before were in the coming years studied theoretically, and solutions to different flow problems in many areas were formed.

New ideas in groundwater hydrology were then introduced by Meinzer, /53/. He disagreed with the generally assumed idea that confined aquifers were rigid, incompressible bodies, and changes in fluid pressure were not accompanied by changes in pore volume. He supported his theories by observations and measurements, which showed that the fluid compressibility alone could not explain the occuring phenomena. Theis, /79/, derived his famous equation about a decade later, based upon Meinzer's work. Then Jacob, /46/, derived the differential equation for compressible flow in elastic aquifers. Jacob's equation was later derived using deforming coordinate system. The derivation showed that his equation was approximately correct when the vertical coordinate z is taken as the deforming coordinate. In the years to come groundwater hydrologist became concerned with solving the differential equations derived and parameter estimation methods. In most cases the differential equations could not be solved analytically, then numerical procedures were used, and thus loosing sight of the physical properties of the aquifers under study. To begin with, finite difference approximation to the differential equation, were used. See f.ex. Pinder and Bredehoeft, /67/, and Young and Bredehoeft, /90/, and Bredehoeft and Pinder, /5/. Parameters were estimated from simple equations, such as Theis' equation. These methods are based upon observing the aquifer response to some external source and then using some simple relation between the response and the source to calculate the parameters. See f.ex. Pinder, Bredehoeft and Cooper, /68/. As the finite difference method is not very efficient to deal with aquifers having arbitrary boundaries, the finite element method seems more suitable to aquifer analysis. One of the first to use the finite element method in groundwater flow modelling were Pinder and Frind, /63/. They use the finite element method to discretisize in the x and y coordinates, but the finite difference method to discretisize the time derivative. Later, three-dimensional finite elements were used. See /64/ and /40/. See also /59/, /65/, /66/ and /55/ for further use of the finite element method in aquifer modelling.

Parameter estimation methods were also improved. They took into account the more detailed description of the aquifer, such as boundary conditions, and inhomogeneity. Most of the methods have concentrated on finding the value of the unknown transmissivity and leaving the determination of the storage coefficient aside. Venuri and Karplus, /85/, and Vemuri, Dracup, Erdmann, /86/, use an iterative method to minimize a norm with respect to the unknown parameters. Kleinecke, /48/, uses linear programming as an estimation procedure. Later Emsellem and Marsily, /23/, give an automatic solution for the inverse problem by not only minimizing a single error functional, but use multiple objectives, as Neuman, /58/, calls it. He uses the finite element approach to the differential equation and minimizes an error functional with subjective constraints on the unknown parameters. Lovell, Duckstein and Kiesel, /50/, discuss the use of subjective information in estimation of aquifer parameters. Gates and Giesel, /32/, calculate the worth of additional data to a groundwater model by using statistical decision theory. Frind and Pinder, /30/, use the Galerkin finite element method to solve the inverse problem. The inverse problem occurs also in other disciplines of geophysics. The problem of direct interpretation of apparent resistivity curves from horizontally layered models is accomplished by using the generalized linear inverse theory. See /42/ and /43/. Glenn et al., /33/, demonstrate that the generalized linear inverse theory may be applied to vertical magnetic dipole sounding problems.

The present thesis deals with the two problems, solving the differential equation for aquifer flow and parameter estimation. The differential equation is solved by using the Galerkin finite element method to discretizise the x and y coordinates and keeping the time continuous. Because the time is kept continuous the physical properties of the aquifer are not lost, as they would be in case of using f.ex. the Crank Nicholson finite difference method for the time derivative, or some three dimensional finite element methods to solve the differential equation. Instead the eigenvalue problem for the matrices involved in the finite element method is solved.

The eigenvalues and the eigenvectors then give us the geometrical and physical properties of the aquifer and need only be solved once and for all for the same aquifer. If the input function changes we can use the already calculated properties to give us the desired output. If the finite difference method had been used for the time derivative, the whole problem would have to be solved again, if the input function changes. The flexibility of the finite element method gives you the opportunity to use subjective information, geological and hydrological insight, when constructing the finite element mesh.

The accuracy of the eigenfunctions expansion depends upon how well they approximate the input function, the gradients and the boundary conditions. As the gradients in most problems are small, and the input function does not fluctuate very much over the area, one can be satisfied with a very limited number of eigenfunctions.

In parameter estimation it is shown that a weigted average between linear inverse theory and non-linear inverse theory must be considered. When estimating the parameters it is not wise to use the eigenfunctions expansion because the eigenfunctions and eigenvalues dependance on the unknown parameters is very complicated. If geologic information tells us that the aquifer is fairly homogeneous and therefore the use of just one pair of parameters is justified, the eigenfunctions and eigenvalues do not depend upon the parameters. In that case the use of the eigenfunction expansion could be very practical.

1.2 Aquifer properties.

The aquifer is the porous medium domain treated by the groundwater hydrologist. Following is a brief description of these domains and the fluid present in them.

An aquifer is a geologic formation or a stratum, that contains water and allows significant amounts of water to move through it. An aquiclude is on the contrary a formation that may contain water, but is incapable of transmitting significant quantities. The aquiclude may therefore for practical purposes be considered impervious. An aquitard is a semipervious geologic formation transmitting water at a very slow rate as compared to the aquifer. It is often referred to as a leaky formation. An aquifuge is an impervious formation that neither contains nor transmits water.

The term groundwater will be used here to denote water contained in the zone of saturation. See fig. 1.1.

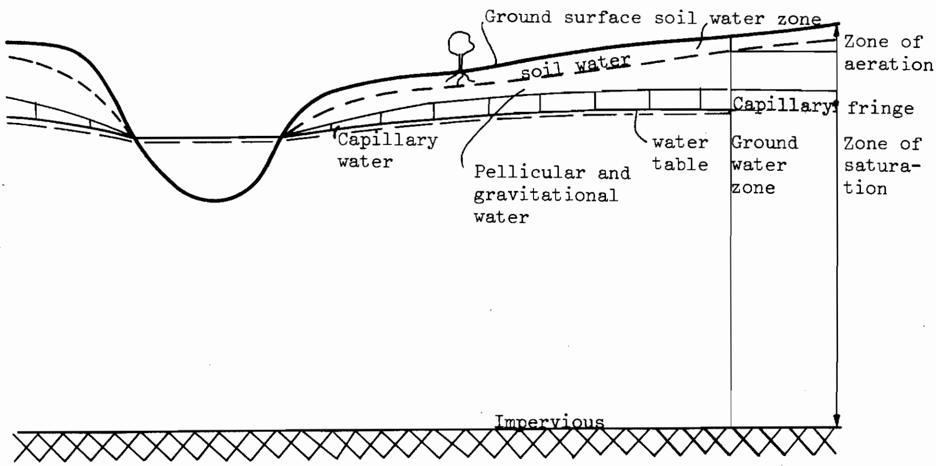

Fig. 1.1.

Aquifers can be classified into two main types. A confined aquifer is bounded above and below by impervious formations. In a well penetrating such an aquifer the water level will rise above the base of the confining formation. The water levels in a number of observation wells define an imaginary surface, the piezometric surface. When the flow in the aquifer is practically horizontal, such that equipotential surfaces are vertical, the depth of the piezometer opening is immaterial. Otherwise, a different piezometric surface is obtained for piezometers that have openings at different elevations. Fortunately, except in the neighbourhood of outlets such as partially penetrating wells or springs, the flow in aquifers is essentially horizontal.

An artesian aquifer is a confined aquifer, where the elevations of the piezometric surface are above ground surface. A well in such an aquifer will flow freely without pumping. Sometimes the term artesian is used to denote a confined aquifer.

The region supplying water to a confined aquifer is called a recharge area. A confined aquifer that has at least one semipervious confining stratum is called a leaky confined aquifer.

A phreatic aquifer or an unconfined aquifer is one with a water table serving as its upper boundary. A phreatic aquifer is recharged from the ground surface above it. A phreatic aquifer that rests on a semipervious layer is a leaky phreatic aquifer.

A special case of a phreatic aquifer is the perched aquifer, that occurs wherever an impervious layer of limited horizontal extent is located between the water table of a phreatic aquifer and the ground surface. Another ground water body is then built above this impervious layer. See fig. 1.2. for further explanation of the above definitions.

The following contains a brief description of the main properties of the fluid and the aquifer.

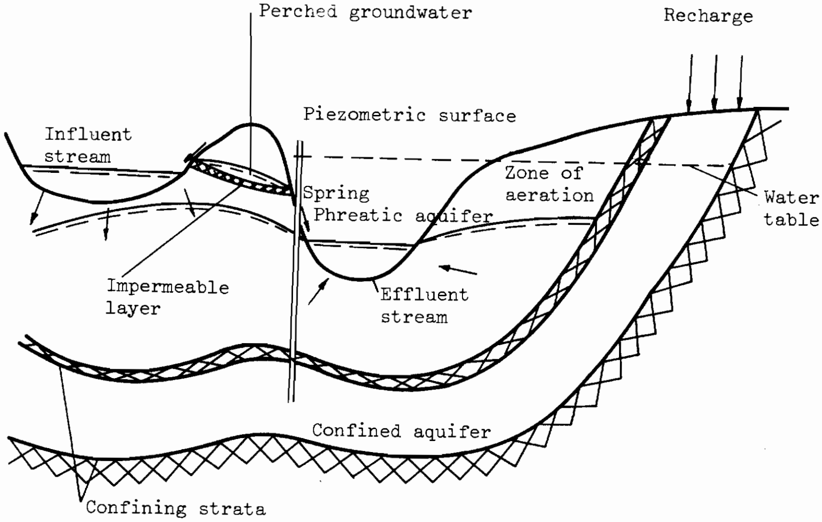

fig. 1.2

Fluid density is defined as the mass of the fluid per unit volume. In general it varies with pressure and temperature according to the equation of state.

$$\rho = \rho(P, T) \tag{1.2}$$

In most cases it is rather complicated. In the following we take the density to be independent of temperature and thus just a function of pressure. We also take the fluid to be homogeneous. A related property is the specific weight defined as the weight of fluid per unit volume.

$$\gamma = \rho \cdot g \tag{1.3}$$

where g is the gravity acceleration.

Fluid viscosity is a measure of the reluctance of the fluid to yield to shear when the fluid is in motion. Newtonian fluids are those fluids having the shear stress proportional to the velocity gradient. The constant of proportionality, μ, is called the dynamic viscosity of the fluid. It is measured in Poise, 1P = 1 g cm^{-1} sec^{-1} in c.g.s. units. Only Newtonian fluids will be considered in the following. We have also the kinematic viscosity $\nu = \mu/\rho$, which is measured in Stokes, 1S = 1 cm^2 sec^{-1} in c.g.s. units. The viscosity is highly temperature dependent, but in the range of temperature we are working in the following, we take it to be independent of temperature. The viscosity is relatively insensitive to pressure until rather high pressures have been obtained. The viscosity is included in our equations only through the coefficient of permeability.

Fluid compressibility is the measure of volume changes when the fluid is subjected to changes in normal pressure. Taking the fluid to be homogeneous and temperatureindependent we have for the fluid compressibility.

$$\beta = \frac{1}{\rho} \frac{\partial \rho}{\partial P} \tag{1.4}$$

In the following we take β to be independent of pressure.

Aquifer porosity is denoted by n and is a macroscopic porous medium property. It is defined as the ratio of volume of the void space (V_v) to the bulk volume (V).

$$n = \frac{V_v}{V} = \frac{(V - V_s)}{V} \tag{1.5}$$

where V_s is the volume of solids. The void ratio, e, is defined as the ratio between the volume of voids and the volume of solids

$$e = \frac{V_v}{V_s} = \frac{n}{1-n} \tag{1.6}$$

Equal balls in tight hexagonal packing have n = 0.26, but in cubic packing n = 0.48. Furthermore, grain size distribution and grain form play a role which is just as important as the packing degree. Sand deposits usually have n = 0.5 with variations to 0.2. Clay is somewhat more porous than sand and so is gravel.

From the standpoint of flow through the porous medium, only interconnected pores are of interest. Hence, the concept of effective porosity, n_e, defined as the ratio of the interconnected (effective) pore volume, V_v^e , to the total volume of the medium, is introduced.

$$n_e = \frac{V_v^e}{V} \tag{1.7}$$

Aquifer compressibility.

Under natural conditions, a porous medium volume at some depth in a groundwater aquifer is subjected to an internal stress or hydrostatic pressure of the fluid saturating the medium, and to an external stress exerted by the formation in which the particular volume is embedded. The effect of overburden load is included in this stress.

The coefficient of bulk-compressibility is defined for a saturated porous medium as the fractional change in the bulk volume with a unit change in total stress.

$$\alpha_b' = -\left(\frac{1}{V}\frac{dV}{d\sigma}\right)_{p=constant} \tag{1.8}$$

In many cases in practice the total stress remains virtually constant, another coefficient of bulk compressibility is often defined with respect to a unit change in P

$$\alpha_b = -\left(\frac{1}{V}\frac{dV}{dP}\right)_{\sigma=constant} \tag{1.9}$$

Using the relation

$$\sigma = \sigma_z + P = constant \tag{1.10}$$

eq. 1.9 can also be written as

$$\alpha = -\left(\frac{1}{V}\frac{dV}{d\sigma_z}\right)_{\sigma=constant} \tag{1.11}$$

where σ_z is the effective stress. In the following we take α to be independent of σ_z .

Aquifer permeability, or the coefficient of permeability, is proportional to Darcy's coefficient K and these two terms are often confused. The coefficient of permeability, k, depends heavily on the porosity. Engelund gives in /24/ the formula

$$k = \frac{n^2}{(1-n)^3}\,\frac{k_o}{2} \tag{1.12}$$

where k_o is the permeability coefficient for the same aquifer when the porosity is 0.5. The relation is valid for homogeneous sand. The factor k_o is sometimes given by the empirical formula:

$$k_o = c \cdot d_{10}^2 \tag{1.13}$$

Here d_{10} is the 10% sieve diameter in mm and the constant equal to 2 gives a mean value. The constant equal to 1 or 5 gives the extremes.

According to this, the permeability depends on the square of d_{10}, so small inhomogeneties in the grading can produce great variation in local permeabilities. This causes no difficulties when dealing with the whole aquifer, because then just global values are used for aquifer parameters. Just if local measurement techniques are used to estimate the parameters, care must be taken. Aquifer parameters from pumping tests are thus local values and cannot be used without further investigation instead of some global values in the analysis of the aquifer as a whole.

Porous media making up aquifers are seldom homogeneous with respect to hydraulic conductivity, K. However, when a rock system is composed of distinct layers, the average conductivity of the system can be determined for some simple flow cases. For flow parallel to layers we have:

$$K_p = \frac{\sum_{i=1}^{N} K_i D_i}{\sum_{i=1}^{N} D_i} = \sum_{i=1}^{N} K_i l_i \tag{1.14}$$

where K_i is the hydraulic conductivity of each layer, D_i is the thickness of each layer and l_i the relative thickness.

For flow perpendicular to the layers we get:

$$\frac{1}{K_N} = \frac{\sum_{i=1}^{N} \left(\frac{D_i}{K_i}\right)}{\sum_{i=1}^{N} D_i} = \sum_{i=1}^{N} \frac{l_i}{K_i} \tag{1.15}$$

From 1.14 and 1.15 we see that the vertical conductivity of stratified aquifers is smaller than the horizontal one. This is also frequently met with in practice as stratification is a very common cause of inhomogeneity. Sediments are stratified due to varying water discharges of the alluvial stream, and stratified lava flows are frequently separated by scoriaceous contact zones, which are again separated by the almost impervious lava cores, and the only vertical water courses are occasional fissures or pseudo-craters.

In the following we assume that we can find such average values in a vertical section but these average values may vary from section to section, and we take the flow to be isotropic with respect to the horizontal plane.

For large Reynold's numbers we have a nonlinear relationship between the velocity and the hydraulic gradient. In the following we assume Darcy's law to be valid.

From analytic derivations of Darcy's law, or from a dimensional analysis, it follows that we may express the coefficient of permeability as

$$k = \frac{\nu K}{g} \tag{1.16}$$

where ν is the kinematic viscosity, K is the hydraulic conductivity or Darcy's coefficient and g is the acceleration of gravity.

Transmissivity is derived from the above basic definitions. It is defined as

$$T = K_p D \tag{1.17}$$

where D is the total thickness of the aquifer.

Storage coefficient is also a derived definition. It is defined to be equal to the volume of water removed from each vertical column of aquifer of height D and unit basal area when the head declines by one unit. It is therefore a ratio of a volume of water to a volume of aquifer, is dimensionless, and is less than one. For a confined aquifer it is:

$$S = D\gamma(\alpha + n\beta) \tag{1.18}$$

and for an unconfined aquifer it is approximately:

$$S \simeq n_e \tag{1.19}$$

2. HORIZONTAL GROUNDWATER FLOW.

2.1 Confined aquifer

If hydraulic conductivity is important to an understanding of the ease with which water moves through an aquifer, compressibility is equally important to an understanding of the manner in which aquifers respond to natural and man-made stresses. The term compressibility refers to the relations at a point between changes in the pressure exerted by the overlying rocks or enclosed fluid and corresponding changes in porosity. These basic concepts are deceptively simple, but their hydrological significance was not discovered at once, but over a period of several years and through the active research of many scientists. Before Meinzer, /53/, it was generally assumed that confined aquifers were rigid incompressible bodies, and changes in fluid pressure were not accompanied by changes in pore volume. Already in 1925, Meinzer disagreed with this principle. His theories were supported by observations and measurements, which showed that the fluid compressibility alone could not explain the occuring phenomena. Several methods were now used to show that aquifers compressed when the fluid pressure was reduced, and expanded when the fluid pressure was increased, such as the response of water levels in wells to ocean tides and passing trains and so on. Based upon Meinzer's work Theis, /79/, derived his famous equation about a decade later. About 1940 Jacob, /46/, derived the differential equation for compressible flow in elastic aquifers. It is interesting to note that Jacob's equation, which leads to Theis' solution for prescribed set of initial and boundary conditions was derived after the solution itself was known. Since then active research has been going on in mathematical hydrology, both seeking to find analytical and numerical solutions to hydrological problems. It is not our intention here to review the recent development in hydrology more than has been done in the introduction, but we will now show how the basic ideas, conductivity and elasticity play an important role in the differential equation for groundwater flow in a confined aquifer. The aquifer is sketched in fig. 2.1.

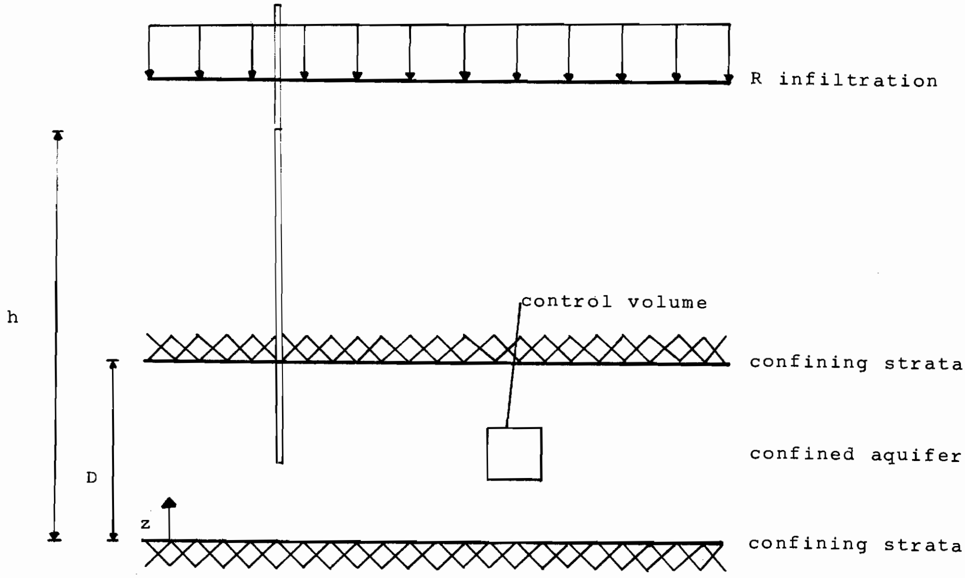

fig. 2.1

We now consider the control volume V of fig. 2.1 and 2.2. We assume that the aquifer is completely saturated. The control volume will be allowed to deform with the medium so that there is no relative velocity between the control surfaces and the grains lying in these surfaces. We assume the solid volume to remain constant and our volume element deforms only vertically.

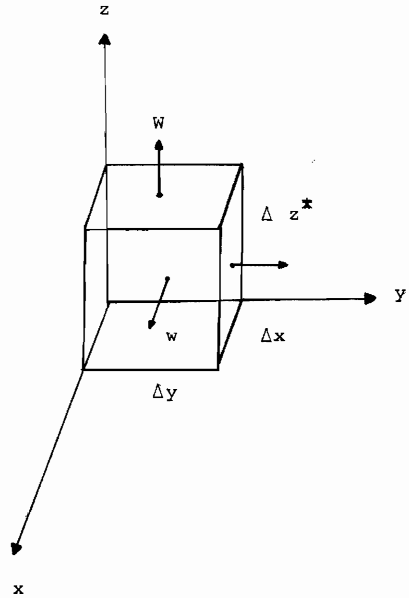

fig. 2.2

We shall use the following definitions:

V = total volume of element

e = void ratio = $\frac{V_v}{V_s}$

V_v = volume of voids = $\frac{e}{1+e}$ y

V_s = Volume of solids = $\frac{1}{1+e}$ V

n = porosity = $\frac{V_v}{V}$ = $\frac{e}{1+e}$

V_w = volume of water

R = infiltration

h = piezometric head

D = aquifer thickness

The mass of fluid, M within the control volume at any time is given by:

$$M = n\rho V = n\rho \, \Delta x \, \Delta y \, \Delta z^{*} \qquad (2.1)$$

Now applying the principle of mass conservation, we get:

$$-\int_A \rho \underline{q} \cdot d\underline{A} = \frac{\partial}{\partial t}\int_V n\rho \, dV \qquad (2.2)$$

where dA is the outward normal to the control surface and $\underline{q}$ = (u, v, w). By using Green's theorem to integrate the left-hand side, and Leibnitz's rule to integrate the right hand side eq. 2.2 becomes:

$$-\nabla \cdot \rho \underline{q} \, \Delta x \, \Delta y \, \Delta z^{*} = \frac{\partial}{\partial t}(n\rho) \, \Delta x \, \Delta y \, \Delta z^{*} + n\rho \frac{\partial V}{\partial t} \qquad (2.3)$$

We now define:

$$\alpha = -\frac{\Delta(\Delta z^{*})/\Delta z^{*}}{\Delta \sigma_z} \qquad (2.4)$$

where σ_z is the normal intergranular stress on a horizontal plane. We find as the solid volume was taken to remain constant,

$$\Delta V = \Delta V_v \qquad (2.5)$$

and 2.4 then becomes:

$$\alpha = -\frac{\Delta e}{(1+e)\Delta \sigma_z}$$

Using 2.4 and 2.5 we can write:

$$\frac{\partial n}{\partial t} = \frac{1}{V}\frac{\partial V_v}{\partial t} - \frac{V_v}{V^2}\frac{\partial V}{\partial t} = -(1-n)\,\alpha\,\frac{\partial \sigma_z}{\partial t} \tag{2.6}$$

The compressibility β of the fluid can be written:

$$\beta = \frac{\Delta\rho/\rho}{\Delta P} \tag{2.7}$$

We thus have:

$$\frac{\partial \rho}{\partial t} = \beta\rho\frac{\partial P}{\partial t} \tag{2.8}$$

Equation 2.3 now becomes:

$$-\nabla\cdot\rho\underline{q} = n\rho\beta\frac{\partial P}{\partial t} - \rho\alpha\frac{\partial \sigma_z}{\partial t} \tag{2.9}$$

This is a general expression for steady or unsteady flow in a fully saturated, elastic medium. For steady flow 2.9 reduces to the well known equation:

$$\nabla\cdot(\rho\underline{q}) = 0 \tag{2.10}$$

and for steady, incompressible flow:

$$\nabla\cdot\underline{q} = 0 \tag{2.11}$$

The stresses σ_z and P are related in some way. We have that the total stress is equal to the pore pressure plus the effective stress, that is:

$$\sigma = \sigma_z + P = constant \tag{2.12}$$

Differentiating 2.12 with respect to the time and inserting in 2.9 we get:

$$-\nabla \cdot \rho \underline{q} = \rho(\alpha + n\beta)\frac{\partial P}{\partial t} \qquad (2.13)$$

We have that the piezometric head is given by:

$$h = z + \frac{1}{g}\int_0^P \frac{dP}{\rho(P)} \qquad (2.14)$$

Differentiating 2.14 we get:

$$\frac{\partial h}{\partial t} = \frac{1}{\gamma}\frac{\partial P}{\partial t} \qquad (2.15)$$

We have at last the continuity equation for isothermal, compressible flow in a saturated, confined, elastic aquifer:

$$-\nabla \cdot \rho \underline{q} = \rho \frac{S}{D}\frac{\partial h}{\partial t} \qquad (2.16)$$

where:

$$S = D\gamma(\alpha + n\beta) \qquad (2.17)$$

S is the storage coefficient, defined as the volume of water an aquifer releases from, or takes into, storage per unit surface area of aquifer per unit change in the component of head normal to that surface. Stated in another way, the storativity is equal to the volume of water removed from each vertical column of aquifer of height D and unit basal area when the head declines by one unit. It is, therefore, a ratio of a volume of water to a volume of aquifer, is dimensionless, and is less than one.

We have now seen how the storage coefficient may be defined for a confined aquifer. We will now not continue to calculate the left hand term, the divergence term, in eq. 2.16. We will instead in view of the definition of the storage coefficient calculate the flow in a confined aquifer with variable depth. Let us look at the control volume in fig. 2.3.

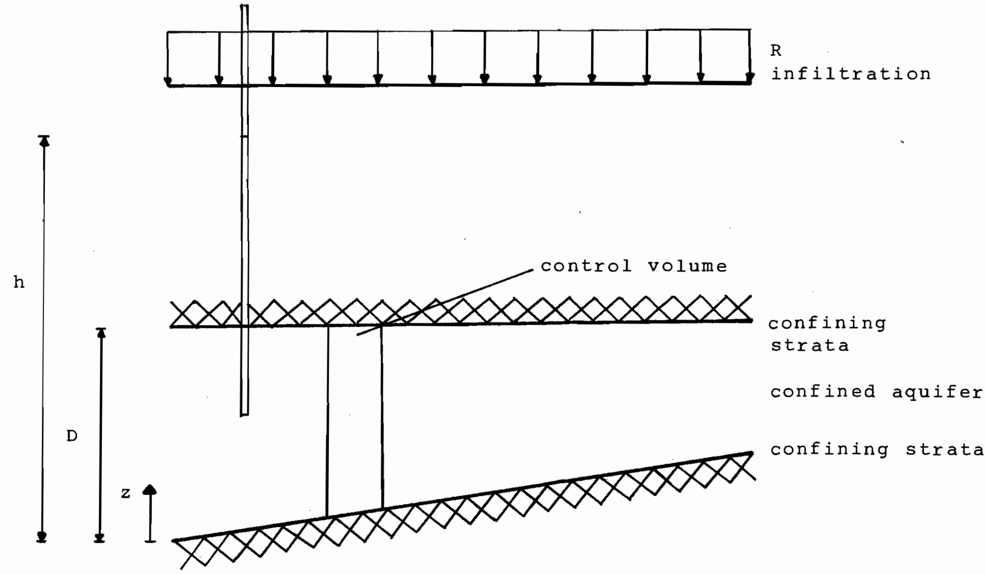

fig. 2.3

The continuity equation for the control volume is:

$$-div(\rho D \underline{q})\,\Delta x \Delta y = \rho S \frac{\partial h}{\partial t}\,\Delta x \Delta y - \rho R\,\Delta x \Delta y \tag{2.18}$$

Let the velocity components be given by Darcy's law as:

$$u = -K\frac{\partial h}{\partial x}$$
$$v = -K\frac{\partial h}{\partial y} \qquad (2.19)$$
$$w = -m\frac{\partial h}{\partial z}$$

We now approximate the flow to be essentially horizontal, which gives W=0. We then have approximately:

$$\frac{\partial(KD\frac{\partial h}{\partial x})}{\partial x} + \frac{\partial(KD\frac{\partial h}{\partial y})}{\partial y} = S\frac{\partial h}{\partial t} - R \qquad (2.20)$$

Defining the transmissivity coefficient:

$$T = KD \qquad (2.21)$$

We then finally have the differential equation:

$$\nabla\cdot(T\nabla h) = S\frac{\partial h}{\partial t} - R \qquad (2.22)$$

which is the governing differential equation for horizontal flow in an confined aquifer with variable aquifer coefficients and infiltration. For constant values of S and T we get the following differential equation:

$$\nabla^2 h = \frac{S}{T}\frac{\partial h}{\partial t} - \frac{R}{T} \qquad (2.23)$$

The coefficient defined as:

$$\mathcal{D} = \frac{T}{S} \qquad (2.24)$$

is sometimes called the aquifer diffusivity.

2.2 Unconfined aquifer.

In an unconfined aquifer eq. 2.9 is still valid but simplification of it through use of 2.12 is no longer permissible. We, of course, still have that the total pressure is equal to the effective pressure plus the porepressure, but it is no longer constant, due to the movement of the free water surface. Eq. 2.16 becomes therefore more complicated by including the movement of the free surface. From a hydraulic point of view, the compressibility of the medium and of the fluid are relatively unimportant in the unconfined case when compared with the changes of fluid volume which accompany changes in elevation of the water table. For that purpose eq. 2.11 is adequate.

To solve this equation we have the boundary conditions in the x,y - plane, the nonlinear boundary condition at the free surface and the boundary condition at the bottom, f.ex. an impermeable bottom. The nonlinearity of the boundary condition at the free surface, together with the fact that the location of this boundary is a priory unknown and is, in fact, part of the required solution, makes an exact analytical solution of a flow problem with such a boundary most difficult, if not practically impossible, in all but a very limited number of cases. A way to circumvent some of the difficulties is to derive analytically approximate solutions based on a linearization of the boundary conditions. Other methods make use of approximations in the continuity equations describing unconfined flows. The Dupuit approximation is one of those, and is among the most powerful tools for treating unconfined flows. Dupuit, /13/, developed a theory based on a number of simplifying assumptions resulting from the observation that in most ground water flows the slope of the phreatic surface is very small. The Dupuit approximations are equivalent to assuming that equipotential surfaces are vertical and the flow essentially horizontal, or to assuming that we have a hydrostatic pressure distribution. The important advantage gained by employing the Dupuit assumptions is that the number of independent variables of the original problem (x,y,z) has been reduced by one; z does not appear as an independent variable. Let us now look at the control volume in fig. 2.4 for deriving the continuity equation for unconfined flow by use of the Dupuit approximation

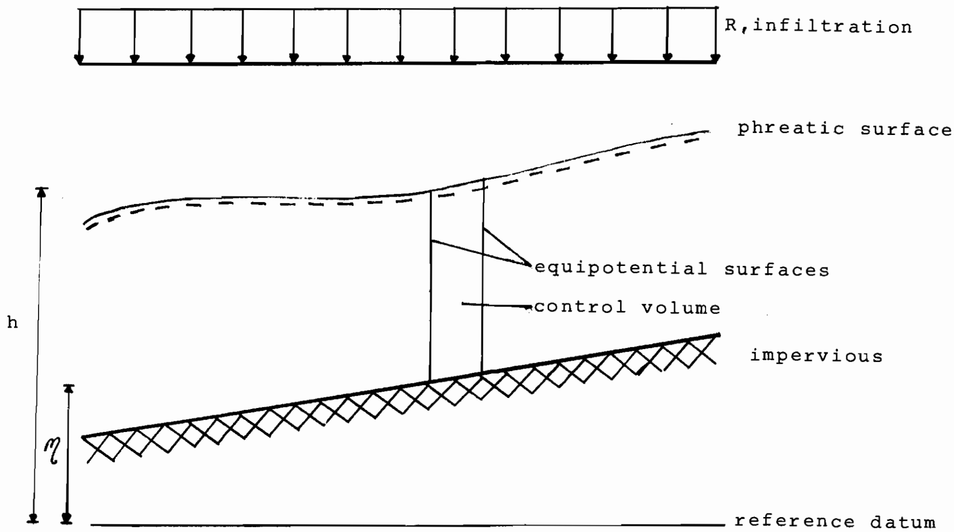

fig. 2.4

We have the continuity equation:

$$-div(\rho(h-\eta)\underline{V})\,\Delta x\,\Delta y + R\rho\,\Delta x\,\Delta y = n_e\,\rho\,\frac{\partial h}{\partial t}\,\Delta x\,\Delta y \qquad (2.25)$$

where $\underline{V}$ is the velocity vector and n_e is the effective porosity.

Now defining the actual thickness of the flow domain:

$$D(x,y,t) = h(x,y,t) - \eta(x,y) \qquad (2.26)$$

Let the horizontal velocity vector, $\underline{V}$, be given by Darcy's law as:

$$\underline{V} = -K\,grad\,h \qquad (2.27)$$

By inserting 2.26 and 2.27 in 2.25 we have:

$$\nabla\cdot(DK\nabla h) = n_e\frac{\partial h}{\partial t} - R \qquad (2.28)$$

We define the storage coefficient:

$$S = n_e \qquad (2.29)$$

and the transmissivity:

$$T = KD \qquad (2.30)$$

and insert in 2.28, getting:

$$\nabla\cdot(T\nabla h) = S\frac{\partial h}{\partial t} - R \qquad (2.31)$$

According to 2.26 and 2.30 the transmissivity varies with time. In the following we approximate it to be constant in time, thus replacing it by some timeaverage value.

3. METHODS OF SOLUTION.

3.1 Timeinvariant boundary conditions.

We now consider a method of solution for eq. 2.23 and 2.31 with timeinvariant boundary conditions. We have the following equations with constant values of S and T.

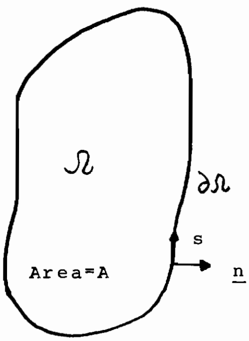

fig. 3.1

Differential equation:

$$\nabla^2 h = \frac{S}{T}\frac{\partial h}{\partial t} - \frac{R}{T} \quad \text{in } \Omega \qquad (3.1)$$

Boundary conditions:

$$h + \alpha \frac{\partial h}{\partial n} = \beta \quad \text{on } \partial\Omega \qquad (3.2)$$

where β is not a function of t.

Let us take the initial condition in $t = -\infty$. That is we assume that the infiltration R has been known for a very long time, thus making the influence of the initial condition negligible.

The groundwater level is now divided into a stationary and a transient part:

$$h(x,y,t) = h_1(x,y,t) + h_0(x,y) \qquad (3.3)$$

Let the stationary level satisfy the following equations:

Differential equation:

$$\nabla^2 h_0 = -\frac{\bar{R}}{T} \qquad in\ \Omega \tag{3.4}$$

Boundary conditions:

$$h_0 + \alpha \frac{\partial h_0}{\partial n} = \beta \qquad on\ \partial\Omega \tag{3.5}$$

where $\bar{R}$ is defined as the mean infiltration level:

$$\bar{R}(x,y) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} R(x,y,t)\,dt \tag{3.6}$$

The transient part must then satisfy:

Differential equation:

$$\nabla^2 h_1 = \frac{S}{T}\frac{\partial h_1}{\partial t} - \left(\frac{R}{T} - \frac{\bar{R}}{T}\right) \qquad in\ \Omega \tag{3.7}$$

Boundary conditions:

$$h_1 + \alpha \frac{\partial h_1}{\partial n} = 0 \qquad on\ \partial\Omega \tag{3.8}$$

We now define the Fourier-transforms:

$$H(x,y,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h_1(x,y,t)\,e^{-i\omega t}\,dt \tag{3.9}$$

$$F(x,y,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{R(x,y,t)}{T} - \frac{\bar{R}(x,y)}{T}\right)e^{-i\omega t}\,dt \tag{3.10}$$

By taking the Fourier-transform of 3.7 and 3.8 and inserting 3.9 and 3.10 we get:

Differential equation:

$$\nabla^2 H = \frac{S}{T} i\omega H - F \quad \text{in } \Omega \tag{3.11}$$

Boundary conditions:

$$H + \alpha \frac{\partial H}{\partial n} = 0 \quad \text{on } \partial\Omega \tag{3.12}$$

Let us define the operator:

$$L = L_1 + L_0 = \left(\nabla^2 - \frac{S}{T} i\omega \right) \tag{3.13}$$

Let us take Ψ for the Green's function for the operator L_1 in Ω with the homogeneous boundary conditions corresponding to 3.2. We have:

$$H(x,y,\omega) = \int_{\Omega} \Psi(x,y,\xi,\eta)\, \Delta H(\xi,\eta)\, d\xi\, d\eta = \Psi * \Delta H$$

From 3.11 and 3.13 we have :

$$L_1 H + L_0 H = - F$$

We then derive:

$$\Psi * L_1 H + \Psi * L_0 H = - \Psi * F$$

and from the above we get :

$$(I + \Psi * L_0) H = - \Psi * F$$

That is:

$$H(x,y,\omega) = -\left(I - i\omega \frac{S}{T} \psi *\right)^{-1} \psi * F(x,y,\omega)$$

From the inverse transform of 3.9 we get the following equation for $h_1(x,y,t)$:

$$h_1(x,y,t) = \int_{-\infty}^{\infty} H(x,y,\omega)\, e^{i\omega t}\, d\omega$$

(3.14)

$$= \int_{-\infty}^{\infty} -e^{i\omega t} \left(I - i\omega \frac{S}{T} \psi *\right)^{-1} \psi * F(x,y,\omega)\, d\omega$$

By introducing 3.10 and defining the integral operator:

$$K = -\frac{S}{T} \psi * \qquad (3.15)$$

we get:

$$h_1(x,y,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} (I + i\omega K)^{-1} K \left[\int_{-\infty}^{\infty} e^{-i\omega\eta} \left(\frac{R(x,y,\eta)}{S} - \frac{\bar{R}(x,y)}{S} \right) d\eta \right] d\omega$$

(3.16)

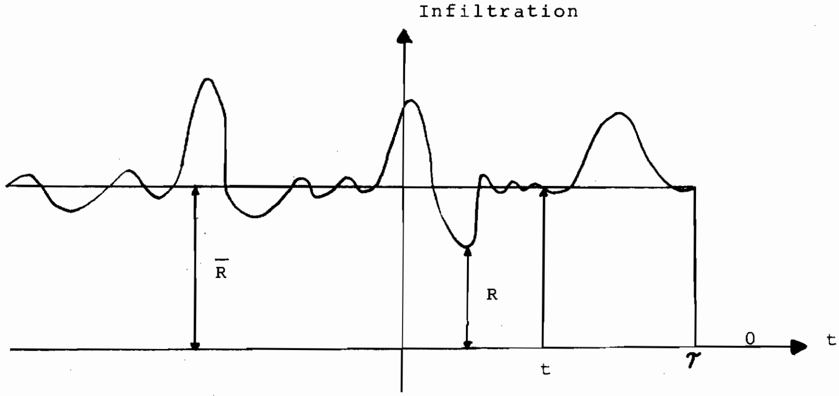

fig. 3.2

We see from fig. 3.2, as the infiltration is zero after time τ we need only integrate in 3.16 up to τ.

$$h_1(x,y,t) = \frac{1}{2\pi}\int_{-\infty}^{\tau}\frac{1}{S}\left[\int_{-\infty}^{\infty} e^{i\omega(t-\eta)}(I+i\omega K)^{-1}d\omega\right]K\left(R(x,y,\eta)-\bar{R}(x,y)\right)d\eta \quad (3.17)$$

The eigenvalues of the operator K are μ_i, that is :

$$Ke_i = \mu_i e_i \quad (3.18)$$

where e_i are the eigenfunctions. We then have:

$$(I+i\omega K)^{-1}e_i = (1+i\omega\mu_i)^{-1}e_i \quad (3.19)$$

This operator has the eigenvalues $1/(1 + i\,\omega\mu_i)$. Let us call the eigenvalues for the Laplacian for λ_i, that is:

$$L_1 e_i = \lambda_i e_i \quad (3.20)$$

We have, $\lambda_i \leq 0$ and $\lambda_i \to -\infty$ when $i \to \infty$. As the operator K is the Green's function integraloperator corresponding to the Laplacian we have:

$$\mu_i = -\frac{S}{T}\frac{1}{\lambda_i} \qquad (3.21)$$

We have $\mu_i \geq 0$ and $\mu_i \to 0$ when $i \to \infty$.
The poles for what is inside the brackets in 3.17 can now be found.

$$\omega_0^i = \frac{i}{\mu_i} = -\frac{T}{S}\, i\lambda_i \geq 0 \qquad (3.22)$$

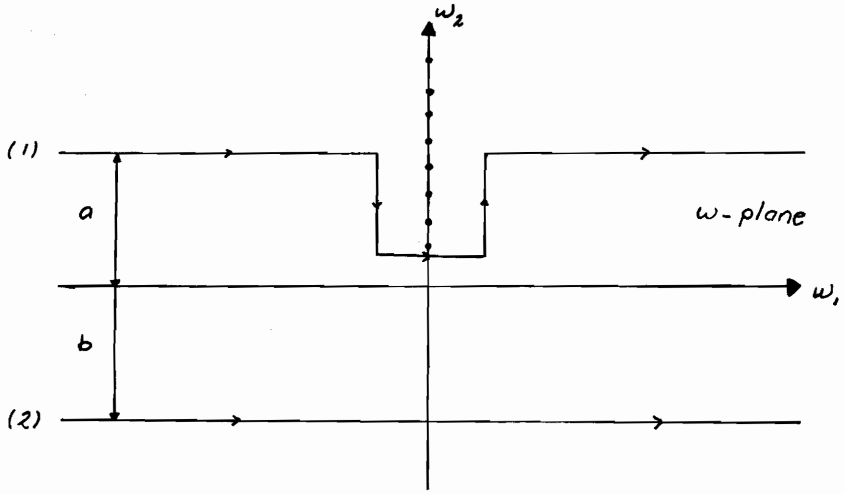

fig. 3.3

By using the integrationpath (2) for great values of b, we get $h_1 = 0$ for $t < \eta$. By using the integrationpath (1) for great values of a, we get $h_1 > 0$ for $t > \eta$. The integrations limit in 3.17 need now only to be taken to t instead of τ. See fig. 3.2. We have thus obtained mathematically the obvious physical result that the groundwater level at time t is independent of the infiltration in the future. 3.17 can then be written as:

$$h_1(x,y,t) = \frac{1}{2\pi}\int_{-\infty}^{t} \frac{1}{S}\left[\int_{-\infty}^{\infty} e^{i\omega(t-\eta)} (I + i\omega K)^{-1} K\,d\omega\right]\left(R(x,y,\eta) - \bar{R}(x,y)\right)d\eta \qquad (3.23)$$

Let us define the mapping:

$$L = \int_{-\infty}^{\infty} e^{i\omega(t-\eta)} (I + i\omega K)^{-1} K\,d\omega \qquad (3.24)$$

we then have:

$$h_1(x,y,t) = \frac{1}{2\pi}\int_{-\infty}^{t} \frac{1}{S}\, L\left\{R(x,y,\eta) - \bar{R}(x,y)\right\}d\eta \qquad (3.25)$$

In order to find the transformation by the operator L, we let it operate on its eigenvectors e_i. We have:

$$L e_i = \int_{-\infty}^{\infty} e^{i\omega(t-\eta)} (I + i\omega K)^{-1} K\, e_i\, d\omega \qquad (3.26)$$

By introducing $\alpha = t - \eta \geq 0$, we now get as the eigenvectors are independent of ω:

$$L e_\vartheta = \int_{-\infty}^{\infty} e^{i\alpha\omega} \frac{\mu_\vartheta}{1 + i\omega\mu_\vartheta} d\omega \, e_\vartheta \qquad (3.27)$$

By taking the integration path along (2) in fig. 3.3 for $\omega_2 = -b$ and introducing $z = 1 + i\omega_1\mu_\vartheta$ we find:

$$L e_\vartheta = \frac{1}{i} \int_{-i\infty + \mu_\vartheta b + 1}^{i\infty + \mu_\vartheta b + 1} \frac{e^{\alpha\left(\frac{z-1}{\mu_\vartheta}\right)}}{z} dz \, e_\vartheta$$

$$= \frac{e^{-\frac{\alpha}{\mu_\vartheta}}}{i} \int_{-i\infty + \mu_\vartheta b + 1}^{i\infty + \mu_\vartheta b + 1} \frac{e^{\frac{\alpha}{\mu_\vartheta} z}}{z} dz \, e_\vartheta \qquad (3.28)$$

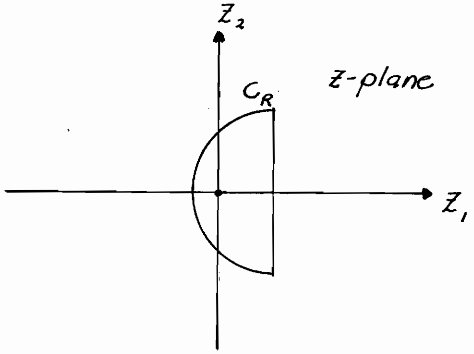

fig. 3.4

We have $\alpha/\mu_\vartheta > 0$ and we now integrate along the integrationpath C_R in fig. 3.4. Bearing in mind that the residue for the function $e^{\frac{\alpha}{\mu_\vartheta} z} / z$ in the pole z = 0 is equal to one, we find:

$$\frac{1}{2\pi i}\int_{-iR+\mu_\nu b+1}^{iR+\mu_\nu b+1} \frac{1}{z} e^{\frac{\alpha}{\mu_\vartheta} z}\, dz + \frac{1}{2\pi i}\int_{C_R} \frac{1}{z} e^{\frac{\alpha}{\mu_\vartheta} z}\, dz = 1$$

In the limit R—> ∞ we get:

$$\frac{1}{i}\int_{-i\infty+\mu_\nu b+1}^{i\infty+\mu_\nu b+1} \frac{1}{z} e^{\frac{\alpha}{\mu_\vartheta} z}\, dz = 2\pi \qquad (3.29)$$

We then have:

$$L e_\vartheta = 2\pi e^{-\frac{\alpha}{\mu_\vartheta}} e_\vartheta \qquad (3.30)$$

Remembering that the μ_ϑ's were the eigenvalues for the K operator we find:

$$L\{R-\bar{R}\} = 2\pi e^{-\alpha K^{-1}}\{R-\bar{R}\} \qquad (3.31)$$

We then have from 3.25:

$$h_1(x,y,t) = \frac{1}{S}\int_{-\infty}^{t} e^{-(t-\eta)K^{-1}}\{R(x,y,\eta) - \bar{R}(x,y)\}\, d\eta \qquad (3.32)$$

We now expand $(R - \bar{R})$ in the series:

$$\left(R(x,y,t) - \bar{R}(x,y)\right) = \sum_{n=1}^{\infty} b_n(t)\, \varphi_n(x,y) \tag{3.33}$$

where ϕ_n are the eigenfunctions for the Laplacian eigen-valueproblem :

$$\begin{aligned} \Delta \varphi_n &= -\frac{\lambda_n}{A} \varphi_n \\ \lambda_n &\geq 0 \end{aligned} \tag{3.34}$$

A is the area of Ω.

We now find from the definition of the K operator that :

$$K^{-1} \varphi_n = \frac{T}{AS} \lambda_n \varphi_n \tag{3.35}$$

3.32 now gives:

$$h_1(x,y,t) = \frac{1}{S} \int_{-\infty}^{t} e^{-(t-\eta)K^{-1}} \sum_{n=1}^{\infty} b_n(\eta)\, \varphi_n(x,y)\, d\eta$$

$$= \frac{1}{S} \sum_{n=1}^{\infty} \int_{-\infty}^{t} e^{-(t-\eta)\frac{T\lambda_n}{AS}} b_n(\eta)\, \varphi_n(x,y)\, d\eta \tag{3.36}$$

Now using the orthogonality relationship for the eigen-functions and supposing that they have been normalized such that:

$$\int_{\Omega} \varphi_n^2\, dx\, dy = 1$$

we get from 3.33 :

$$b_n(t)=\int_{\Omega} \left(R(x,y,t)-\bar{R}(x,y)\right)\varphi_n(x,y)\,dx\,dy \qquad (3.37)$$

Inserting in 3.36 we have finally :

$$h_1(x,y,t)=\frac{1}{S}\sum_{n=1}^{\infty}\varphi_n(x,y)\int_{\Omega}\varphi_n(\xi,\eta)\left\{\int_{-\infty}^{t} e^{-(t-\tau)\frac{T\lambda_n}{AS}}\left(R(\xi,\eta,\tau)-\bar{R}(\xi,\eta)\right)d\tau\right\}d\xi\,d\eta \qquad (3.38)$$

By introducing the timeconstants :

$$K_n=\frac{AS}{\lambda_n T} \qquad (3.39)$$

and changing the integration parameters in 3.38 we finally find for the transient part of the groundwater level :

$$h_1(x,y,t)=\frac{1}{S}\sum_{n=1}^{\infty}\varphi_n(x,y)\int_{\Omega}\varphi_n(\xi,\eta)\left\{\int_{0}^{\infty} e^{-\frac{\tau}{K_n}}\left(R(\xi,\eta,t-\tau)-\bar{R}(\xi,\eta)\right)d\tau\right\}d\xi\,d\eta \qquad (3.40)$$

We finally summarize the results:

$$\nabla^2 h_0=\frac{-\bar{R}}{T} \qquad (3.41)$$

$$h_1(x,y,t) = \frac{1}{S}\sum_{n=1}^{\infty} \varphi_n(x,y)\int_{\Omega} \varphi_n(\xi,\eta)\left\{\int_0^{\infty} e^{-\frac{\tau}{k_n}} \left(R(\xi,\eta,t-\tau) - \bar{R}(\xi,\eta)\right) d\tau\right\} d\xi\, d\eta \tag{3.42}$$

$$h(x,y,t) = h_0(x,y) + h_1(x,y,t) \tag{3.43}$$

3.2 Timedependent boundary conditions.

We now consider a method of solution for eq. 2.23 and 2.31 with timedependent boundary conditions but constant values of aquifer parameters. We now suppose that we have an initial-condition at some time t_o. The equations are:

Differential equation:

$$\nabla^2 h = \frac{S}{T}\frac{\partial h}{\partial t} - \frac{R}{T} \quad \text{in } \Omega \tag{3.44}$$

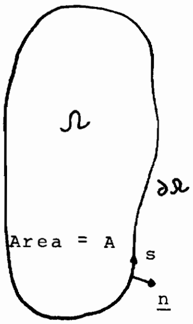

fig. 3.6

Boundary conditions:

$$h(x,y,t) + \alpha \frac{\partial h(x,y,t)}{\partial n} = g(x,y,t) \tag{3.45}$$

$$\text{on } \partial\Omega$$

Initial condition:

$$h(x,y,t_o) = f(x,y) \tag{3.46}$$

By defining the operator:

$$L = \left(\frac{S}{T}\frac{\partial}{\partial t} - \nabla^2\right) \tag{3.47}$$

We have for the differentialequation 3.44 :

$$Lh = \frac{R}{T} \tag{3.48}$$

Let the kernel T have the following properties:

$$
\begin{aligned}
&1)\ LT(x_1, y_1, x_2, y_2, t) = 0 \\
&2)\ T(x_1, y_1, x_2, y_2, t) \in C^{\infty}(\Omega) \\
&3)\ \lim_{t \to 0} \int_{\Omega} f(x_2, y_2) T(x_1, y_1, x_2, y_2, t)\, dx_2\, dy_2 = f(x_1, y_1) \\
&4)\ T(x_1, y_1, x_2, y_2, t) + \alpha \frac{\partial T(x_1, y_1, x_2, y_2, t)}{\partial n} = 0 \\
&\qquad \text{on } \partial\Omega
\end{aligned}
\tag{3.49}
$$

We now multiply the first equation in 3.49 by h(x,y,τ) and integrate over Ω and the timedomain:

$$I = \int_{t_0}^{t} \int_{\Omega} h(x,y,\tau)\, LT(x,y,\xi,\eta,t-\tau)\, dx\, dy\, d\tau$$

$$= \int_{t_0}^{t} \int_{\Omega} h(x,y,\tau) \left\{ \frac{S}{T} \frac{\partial T(x,y,\xi,\eta,t-\tau)}{\partial (t-\tau)} - \nabla^2 T(x,y,\xi,\eta,t-\tau) \right\} dx\, dy\, d\tau$$

$$= 0$$

The differentialoperations on the T-function can now be transferred to the h-function by including the proper boundary integrals. Let us start with the time differentiation:

$$I = \int_{\Omega}\int_{t_o}^{t} \frac{S}{T} \frac{\partial h(x,y,\tau)}{\partial \tau} \mathcal{T}(x,y,\xi,\eta,t-\tau)\, d\tau\, dx\, dy$$

$$-\frac{S}{T}\int_{\Omega}\Big[h(x,y,\tau)\,\mathcal{T}(x,y,\xi,\eta,t-\tau)\Big]_{t_o}^{t} dx\, dy$$

$$-\int_{t_o}^{t}\int_{\Omega} h(x,y,\tau)\, \nabla^2 \mathcal{T}(x,y,\xi,\eta,t-\tau)\, dx\, dy\, d\tau$$

$$= \int_{\Omega}\int_{t_o}^{t} \frac{S}{T} \frac{\partial h(x,y,\tau)}{\partial \tau} \mathcal{T}(x,y,\xi,\eta,t-\tau)\, d\tau\, dx\, dy$$

$$-\frac{S}{T} h(\xi,\eta,t) + \frac{S}{T}\int_{\Omega} f(x,y)\,\mathcal{T}(x,y,\xi,\eta,t-t_o)\, dx\, dy$$

$$-\int_{t_o}^{t}\int_{\Omega} h(x,y,\tau)\, \nabla^2 \mathcal{T}(x,y,\xi,\eta,t-\tau)\, dx\, dy\, d\tau$$

We have now used the initial condition 3.46 and the third property of the kernelfunction. We now turn to the ∇^2-operator writing:

$$I = \int_{\Omega}\int_{t_0}^{t} \frac{S}{T} \frac{\partial h(x,y,\tau)}{\partial \tau} \mathcal{T}(x,y,\xi,\eta,t-\tau)\, d\tau\, dx\, dy$$

$$- \frac{S}{T} h(\xi,\eta,t) + \frac{S}{T}\int_{\Omega} f(x,y)\, \mathcal{T}(x,y,\xi,\eta,t-t_0)\, dx\, dy$$

$$+ \int_{t_0}^{t}\int_{\Omega} \nabla h(x,y,\tau) \cdot \nabla \mathcal{T}(x,y,\xi,\eta,t-\tau)\, dx\, dy\, d\tau$$

$$- \int_{t_0}^{t}\int_{\partial\Omega} h(x,y,\tau)\, \underline{n} \cdot \nabla \mathcal{T}(x,y,\xi,\eta,t-\tau)\, ds\, d\tau$$

$$= \int_{\Omega}\int_{t_0}^{t} \frac{S}{T} \frac{\partial h(x,y,\tau)}{\partial \tau} \mathcal{T}(x,y,\xi,\eta,t-\tau)\, d\tau\, dx\, dy$$

$$- \frac{S}{T} h(\xi,\eta,t) + \frac{S}{T}\int_{\Omega} f(x,y)\, \mathcal{T}(x,y,\xi,\eta,t-t_0)\, dx\, dy$$

$$- \int_{t_0}^{t}\int_{\Omega} \nabla^2 h(x,y,\tau) \cdot \mathcal{T}(x,y,\xi,\eta,t-\tau)\, dx\, dy\, d\tau$$

$$- \int_{t_0}^{t}\int_{\partial\Omega} h(x,y,\tau)\, \underline{n} \cdot \nabla \mathcal{T}(x,y,\xi,\eta,t-\tau)\, ds\, d\tau$$

$$+ \int_{t_0}^{t}\int_{\partial\Omega} \mathcal{T}(x,y,\xi,\eta,t-\tau)\, \underline{n} \cdot \nabla h(x,y,\tau)\, ds\, d\tau$$

We now have:

$$h(\xi,\eta,t) = \frac{T}{S}\int_{\Omega}\int_{t_o}^{t}\left(\frac{S}{T}\frac{\partial h(x,y,\tau)}{\partial \tau} - \nabla^2 h(x,y,\tau)\right) \Gamma(x,y,\xi,\eta,t-\tau)\, dx\, dy\, d\tau$$

$$+\int_{\Omega} f(x,y)\, \Gamma(x,y,\xi,\eta,t-t_o)\, dx\, dy$$

$$+\frac{T}{S}\int_{t_o}^{t}\int_{\partial\Omega}\left(\Gamma(x,y,\xi,\eta,t-\tau)\frac{\partial h(x,y,\tau)}{\partial n} - h(x,y,\tau)\frac{\partial \Gamma(x,y,\xi,\eta,t-\tau)}{\partial n}\right) ds\, d\tau$$

Now, using the definition of the L operator, 3.47, and the boundary conditions for h, 3.45, and the fourth property of the kernel, 3.49, we find:

$$h(\xi,\eta,t) = \int_{\Omega} f(x,y)\, \Gamma(x,y,\xi,\eta,t-t_o)\, dx\, dy$$

$$+\frac{1}{S}\int_{t_o}^{t}\int_{\Omega} R(x,y,\tau)\, \Gamma(x,y,\xi,\eta,t-\tau)\, dx\, dy\, d\tau$$

$$-\frac{T}{S}\int_{t_o}^{t}\int_{\partial\Omega} g(x,y,\tau)\frac{\partial \Gamma(x,y,\xi,\eta,t-\tau)}{\partial n}\, ds\, d\tau \qquad (3.50)$$

We have three terms in the equation for h (ξ,η,t). The first term is from the initial condition, the second from the source function and the third from the boundary conditions.

Now stated without proof the kernel is given by the following equation:

$$\Gamma(x,y,\xi,\eta,t) = \sum_{n=1}^{\infty} \varphi_n(x,y)\varphi_n(\xi,\eta)\, e^{\frac{\lambda_n T}{AS} t} \qquad (3.51)$$

where the λ_n and ϕ_n are the eigenvalues and eigenfunctions respectively of the eigenproblem:

$$\nabla^2 \varphi_n + \frac{\lambda_n}{A} \varphi_n = 0 \qquad \text{in } \Omega$$

$$\varphi_n + \alpha \frac{\partial \varphi_n}{\partial n} = 0 \qquad \text{on } \partial\Omega \tag{3.52}$$

We normalize the eigenfunctions with respect to the norm:

$$\| \varphi_n \|^2 = \int_\Omega \varphi_n^2 \, dx\, dy \tag{3.53}$$

The eigenfunctions are known to be orthogonal with respect to the innerproduct:

$$\langle \varphi_n, \varphi_m \rangle = \int_\Omega \varphi_n \varphi_m \, dx\, dy = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases} \tag{3.54}$$

The groundwaterlevel can now be written as :

$$h(\xi, \eta, t) = \int_\Omega F(x,y) \sum_{n=1}^{\infty} \varphi_n(x,y) \varphi_n(\xi,\eta) e^{-\frac{(t-t_0)}{K_n}} \, dx\, dy$$

$$+ \frac{1}{S} \sum_{n=1}^{\infty} \varphi_n(\xi,\eta) \int_{t_0}^{t} \int_\Omega R(x,y,\tau)\, \varphi_n(x,y)\, e^{-\frac{(t-\tau)}{K_n}} \, dx\, dy\, d\tau$$

$$- \frac{T}{S} \int_{t_0}^{t} \int_{\partial\Omega} g(x,y,\tau) \sum_{n=1}^{\infty} \varphi_n(\xi,\eta) \frac{\partial \varphi_n(x,y)}{\partial n} e^{-\frac{(t-\tau)}{K_n}} \, ds\, d\tau$$

(3.55)

We have introduced K_n from 3.39. We have now found the integral kernel function T for the heat equation 3.44 for a region Ω closed by the boundary $\partial\Omega$. For comparison the kernel for $\Omega = \mathbb{R}^2$ will now be derived. $\mathbb{R}^2$ means the whole x,y plane. We have the same differential equation with the same initial condition.

The double Fouriertransform of h is given by:

$$H(\varkappa,\lambda,t) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y,t)\, e^{-i(\varkappa x+\lambda y)}\,dx\,dy \tag{3.56}$$

and the inverse transform:

$$h(x,y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(\varkappa,\lambda,t)\, e^{i(\varkappa x+\lambda y)}\,d\varkappa\,d\lambda \tag{3.57}$$

Inserting 3.57 in the differential equation 3.44 we get:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[-(\lambda^2+\varkappa^2)\,H - \frac{S}{T}\frac{\partial H}{\partial t} + \frac{F}{T}\right] e^{i(\varkappa x+\lambda y)}\,d\varkappa\,d\lambda$$

where F is the Fouriertransform of the infiltration R given by:

$$F(\varkappa,\lambda,t) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R(x,y,t)\, e^{-i(\varkappa x+\lambda y)}\,dx\,dy \tag{3.58}$$

and the inverse transform:

$$R(x,y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\varkappa,\lambda,t)\, e^{i(\varkappa x+\lambda y)}\,d\varkappa\,d\lambda \tag{3.59}$$

we have:

$$\frac{\partial H}{\partial t} + \frac{T}{S}(\varkappa^2+\lambda^2)\,H = \frac{F}{S} \tag{3.60}$$

The initial condition is given by:

$$H_0 = H(\varkappa,\lambda,0) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-i(\varkappa x+\lambda y)}\, dx\, dy \qquad (3.61)$$

The solution to this initial value problem is given by:

$$H = e^{-\frac{T}{S}(\varkappa^2+\lambda^2)t} H_0 + \frac{1}{S}\int_0^t F(\varkappa,\lambda,\tau)\, e^{-\frac{T}{S}(\varkappa^2+\lambda^2)(t-\tau)}\, d\tau$$

Now using 3.57, 3.58 and 3.61 we have:

$$h(x,y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{ e^{-\frac{T}{S}(\varkappa^2+\lambda^2)t} \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\, e^{-i(\varkappa\xi+\lambda\eta)}\, d\xi\, d\eta\; e^{i(\varkappa x+\lambda y)}\right\} d\varkappa\, d\lambda$$

$$+\frac{1}{S}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_0^t e^{-\frac{T}{S}(\varkappa^2+\lambda^2)(t-\tau)} \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R(\xi,\eta,\tau)\, e^{-i(\varkappa\xi+\lambda\eta)}\, d\xi\, d\eta\, d\tau\right\} e^{i(\varkappa x+\lambda y)}\, d\varkappa\, d\lambda$$

Rearranging the terms we get:

$$h(x,y,t) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\left\{\int_{-\infty}^{\infty} e^{-\frac{T}{S}\varkappa^2 t + i\varkappa(x-\xi)}\, d\varkappa \int_{-\infty}^{\infty} e^{-\frac{T}{S}\lambda^2 t + i\lambda(y-\eta)}\, d\lambda\right\} d\xi\, d\eta$$

$$+\frac{1}{4\pi^2 S}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^t R(\xi,\eta,\tau)\left\{\int_{-\infty}^{\infty} e^{-\frac{T}{S}\varkappa^2(t-\tau) + i\varkappa(x-\xi)}\, d\varkappa \int_{-\infty}^{\infty} e^{-\frac{T}{S}\lambda^2(t-\tau) + i\lambda(y-\eta)}\, d\lambda\right\} d\tau\, d\xi\, d\eta$$

using that:

$$\int_{-\infty}^{\infty} e^{-\frac{T}{S}\mu(t-\tau) + i\mu(x-\xi)}\, d\mu = \sqrt{\frac{\pi}{\frac{T}{S}t}}\; e^{-\frac{(x-\xi)^2}{4\frac{T}{S}t}}$$

we find:

$$h(x,y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\xi,\eta) \frac{S}{4\pi T t} e^{-\frac{\{(x-\xi)^2+(y-\eta)^2\} S}{4Tt}} d\xi d\eta$$

$$+\frac{1}{S}\int_0^t \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R(\xi,\eta,\tau) \frac{S}{4\pi T(t-\tau)} e^{-\frac{\{(x-\xi)^2+(y-\eta)^2\} S}{4T(t-\tau)}} d\xi d\eta d\tau$$

Defining:

$$\mathcal{T}_{\mathbb{R}^2}(x,y,\xi,\eta,t) = \frac{S}{4\pi T t} e^{-\frac{S\{(x-\xi)^2+(y-\eta)^2\}}{4Tt}} \qquad (3.62)$$

we have:

$$h(x,y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\xi,\eta)\, \mathcal{T}_{\mathbb{R}^2}(x,y,\xi,\eta,t)\, d\xi d\eta$$

$$+\frac{1}{S}\int_0^t \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R(\xi,\eta,\tau)\, \mathcal{T}_{\mathbb{R}^2}(x,y,\xi,\eta,t-\tau)\, d\xi d\eta d\tau \qquad (3.63)$$

We see that 3.63 is analog to 3.50 which uses the integral kernel function $\mathcal{T}$ for a bounded region Ω. We state here without proof that :

$$\mathcal{T} \rightarrow \mathcal{T}_{\mathbb{R}^2} \quad \text{when } \Omega \rightarrow \mathbb{R}^2$$

Let us now consider the case of pumping from a single well. See fig. 3.7.

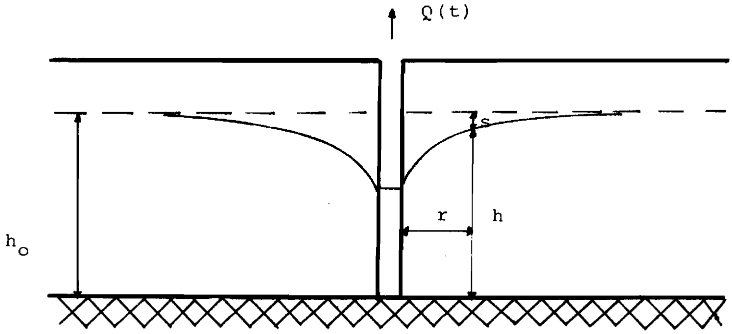

fig. 3.7

We have:

$$f(\xi,\eta) = h_o$$

$$R(\xi,\eta,\tau) = Q(\tau)\,\delta(\xi)\,\delta(\eta) \qquad (3.64)$$

where the δ's are Dirac's delta functions. Inserting 3.64 in 3.63 we get:

$$h(x,y,t) = h_o \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T_{\mathbb{R}^2}(x,y,\xi,\eta,t)\,d\xi\,d\eta$$

$$+ \frac{1}{S}\int_0^t Q(\tau)\,T(x,y,0,0,t-\tau)\,d\tau$$

$$= h_o + \frac{1}{S}\int_0^t \frac{S}{4\pi T(t-\tau)}\,e^{-\frac{S(x^2+y^2)}{4T(t-\tau)}}\,Q(\tau)\,d\tau$$

We now have used the fact that:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T_{R^2}(x, y, \xi, \eta, t)\, d\xi\, d\eta = 1 \qquad (3.65)$$

Using the new integrationvariable:

$$z = \frac{r^2 S}{4T\tau} \qquad (3.66)$$

we find:

$$h(x, y, t) = h_0 + \frac{1}{4\pi T}\int_{\frac{r^2 S}{4Tt}}^{\infty} Q\left(t - \frac{r^2 S}{4Tz}\right)\frac{e^{-z}}{z}\, dz \qquad (3.67)$$

In the case of constant pumping we get:

$$h(x, y, t) = h_0 + \frac{Q}{4\pi T}\int_{\frac{r^2 S}{4Tt}}^{\infty} \frac{e^{-z}}{z}\, dz \qquad (3.68)$$

This is the classical Theis formula, the integral:

$\int_{u}^{\infty} \frac{e^{-z}}{z}\, dz$ is called $W(u)$

but $-E_i(-u)$ in most mathematical tables. It tends to ∞ as $u \rightarrow 0$ and goes rapidly to 0 for increasing values of u.

3.3 Runoff and linear reservoirs.

Let us now consider the boundary outflow. It is given by the equation:

$$q(t) = -\int_{\partial\Omega} T \frac{\partial h}{\partial n} \, ds \qquad (3.69)$$

Now using the Gauss' theorem we get:

$$q(t) = -T\int_{\partial\Omega} \frac{\partial h}{\partial n} \, ds = -T\int_{\Omega} \operatorname{div} \operatorname{grad} h \, dx\,dy$$

$$= -T\int_{\Omega} \nabla^2 h \, dx\,dy$$

By making use of 3.3 and 3.4 we find:

$$q(t) = -T\int_{\Omega} \nabla^2 h_0 \, dx\,dy - T\int_{\Omega} \nabla^2 h_1 \, dx\,dy$$

$$= \int_{\Omega} \bar{R}(x,y) \, dx\,dy - T\int_{\Omega} \nabla^2 h_1 \, dx\,dy$$

Performing the ∇^2-operation on h_1 in 3.40 and inserting, simultaneously changing the summation and integration we have:

$$q(t) = \int_{\Omega} \bar{R}(x,y) \, dx\,dy$$

$$- \frac{T}{S} \sum_{n=1}^{\infty} \int_{\Omega} \nabla^2 \varphi_n(x,y) \, dx\,dy \left[\int_{\Omega} \varphi_n(\xi,\eta) \left\{ \int_0^{\infty} e^{-\frac{\tau}{K_n}} \left(R(\xi,\eta,t-\tau) - \bar{R}(\xi,\eta) \right) d\tau \right\} d\xi \, d\eta \right]$$

Using 3.34 we get:

$$q(t) = \int_{\Omega} \bar{R}(x,y)\,dx\,dy$$

$$+\sum_{n=1}^{\infty} \frac{1}{K_n} \int_{\Omega} \varphi_n(x,y)\,dx\,dy \left[\int_{\Omega} \varphi_n(\xi,\eta) \left\{ \int_0^{\infty} e^{-\frac{\tau}{K_n}} \left(R(\xi,\eta,t-\tau) - \bar{R}(\xi,\eta) \right) d\tau \right\} d\xi\, d\eta \right] \tag{3.70}$$

We define:

$$q(t) = q_0 + \sum_{n=1}^{\infty} q_n(t) \tag{3.71}$$

$$q_0 = \int_{\Omega} \bar{R}(x,y)\,dx\,dy \tag{3.72}$$

$$q_n(t) = \frac{1}{K_n} \int_{\Omega} \varphi_n(x,y)\,dx\,dy \left[\int_{\Omega} \varphi_n(\xi,\eta) \left\{ \int_0^{\infty} e^{-\frac{\tau}{K_n}} \left(R(\xi,\eta,t-\tau) - \bar{R}(\xi,\eta) \right) d\tau \right\} d\xi\, d\eta \right] \tag{3.73}$$

By differentiating 3.73 we find:

$$q_n + K_n \frac{dq_n}{dt} = \int_{\Omega} \varphi_n(x,y)\,dx\,dy \int_{\Omega} \varphi_n(\xi,\eta) \left(R(\xi,\eta,t) - \bar{R}(\xi,\eta) \right) d\xi\, d\eta$$

$$= Z_n \tag{3.74}$$

This is a linear, ordinary differential equation, and is the differential equation for a linear reservoir with time-constant K_n. The boundary outflow due to the transient term of the groundwaterlevel is therefore the outflow from a system of infinitely many parallel simple linear reservoirs as illustrated in fig. 3.8.

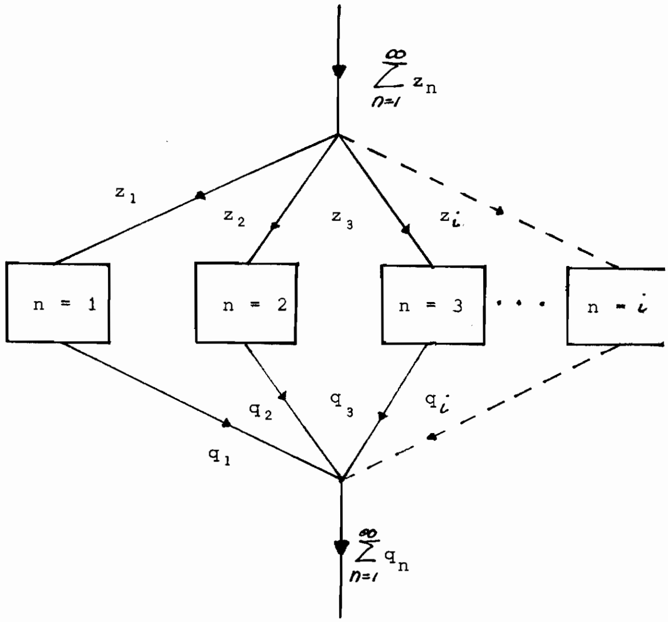

fig. 3.8

The inflow to each reservoir is given by:

$$Z_n = \int_{\Omega} \varphi_n(x,y)\,dx\,dy \int_{\Omega} \varphi_n(\xi,\eta)\left(R(\xi,\eta,t) - \bar{R}(\xi,\eta)\right) d\xi\, d\eta$$

There now remains to check that the total inflow satisfies the necessary continuity equation.

From 3.33 we have:

$$(R-\bar{R}) = \sum_{n=1}^{\infty} \langle (R-\bar{R}), \varphi_n \rangle \varphi_n$$

Where the inner product is defined as before:

$$\langle (R-\bar{R}), \varphi_n \rangle = \int_{\Omega} (R(x,y,t) - \bar{R}(x,y)) \varphi_n (x,y) dx dy$$

$\sum_{n=1}^{\infty} z_n$ can now be written as:

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} \int_{\Omega} \langle (R-\bar{R}), \varphi_n \rangle \varphi_n (x,y) dx dy$$

$$= \int_{\Omega} (R(x,y,t) - \bar{R}(x,y)) dx dy \qquad (3.75)$$

The necessary continuity equation is therefore satisfied as was to be expected. We see that the n'th inflowterm to the linear reservoirs is given by the n'th term in the eigenfunction expansion of the infiltration. The eigenvalues have the following properties:

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots\cdots \leq \lambda_n$$

and $\lambda_n \to \infty$, when $n \to \infty$. We therefore have that $K_n \to 0$, when $n \to \infty$, see fig. 3.9.

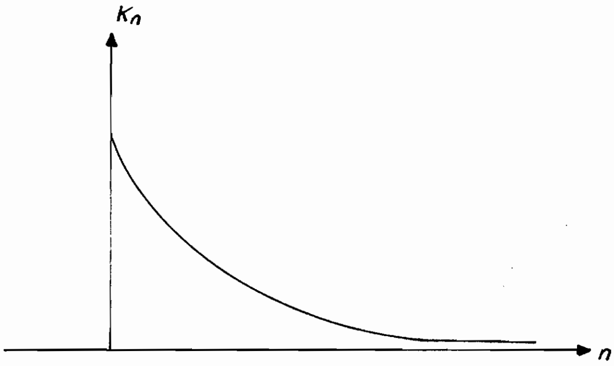

fig. 3.9

The higher order inflow terms are therefore delayed very little in time.

If $(R - \bar{R})$ is not very rich in the higher harmonics, that is to say it does not fluctuate very much over the area, just a few of the inflow terms give any significant value to the total inflow. In that case just the few first terms in the eigenfunction expansion need to be evaluated.

The watermass in storage due to the transient term, V, is related to the respective boundary outflow by the following equation:

$$q_n = \frac{V_n}{K_n} \tag{3.76}$$

$$V = \sum_{n=1}^{\infty} V_n \tag{3.77}$$

which is the characteristic equation for linear reservoirs. That is to say there is a linear relationship between storage and outflow.

We will now determine the IHU (instantaneous unit hydrograph) function of our aquifer.

We have from 3.70:

$$q(t) = \int_{\Omega} \bar{R}(x,y)\,dx\,dy - \sum_{n=1}^{\infty} \int_{\Omega} \varphi_n(x,y)\,dx\,dy \int_{\Omega} \varphi_n(\xi,\eta)\,\bar{R}(\xi,\eta)\,d\xi\,d\eta$$

$$+ \sum_{n=1}^{\infty} \frac{1}{K_n} \int_{\Omega} \varphi_n(x,y)\,dx\,dy \int_{\Omega} \varphi_n(\xi,\eta) \left\{ \int_0^{\infty} e^{-\frac{\tau}{K_n}} R(\xi,\eta,t-\tau)\,d\tau \right\} d\xi\,d\eta$$

Now using that,

$$\bar{R}(x,y) = \sum_{n=1}^{\infty} \varphi_n(x,y) \int_{\Omega} \varphi_n(\xi,\eta)\,\bar{R}(\xi,\eta)\,d\xi\,d\eta$$

we get:

$$q(t) = \sum_{n=1}^{\infty} \frac{1}{K_n} \int_{\Omega} \varphi_n(x,y)\,dx\,dy \int_{\Omega} \varphi_n(\xi,\eta) \left\{ \int_0^{\infty} e^{-\frac{\tau}{K_n}} R(\xi,\eta,t-\tau)\,d\tau \right\} d\xi\,d\eta \tag{3.78}$$

Herefrom we see that the IUH is given by:

$$u(t) = \frac{1}{A} \sum_{n=1}^{\infty} \frac{1}{K_n} \left(\int_{\Omega} \varphi_n(x,y)\,dx\,dy \right)^2 e^{-\frac{t}{K_n}} \tag{3.79}$$

The IUH function is sketched in fig. 3.10

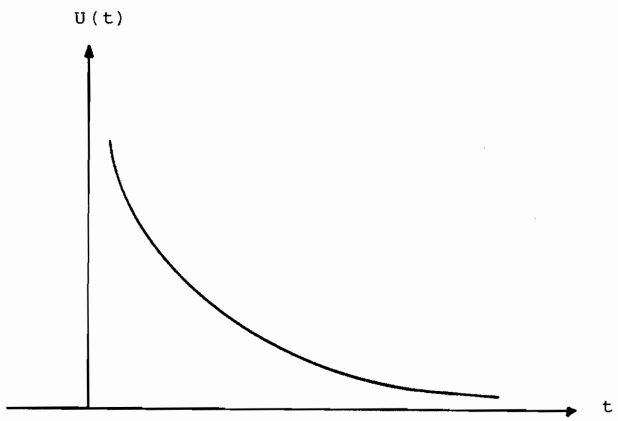

fig. 3.10

We will at last show that the area under the IUH function is equal to one, as it should be. We have:

$$\int_0^\infty U(t)\,dt = \frac{1}{A}\sum_{n=1}^{\infty}\frac{1}{K_n}\left(\int_\Omega \varphi_n(x,y)\,dx\,dy\right)^2\int_0^\infty e^{-\frac{t}{K_n}}\,dt$$

$$= \frac{1}{A}\sum_{n=1}^{\infty}\left(\int_\Omega \varphi_n(x,y)\,dx\,dy\right)^2$$

We define the norm and the inner product in our function space as before that is,

$$\langle \varphi_n, 1\rangle = \int_\Omega \varphi_n(x,y)\,dx\,dy$$

$$\|1\|^2 = \int_\Omega 1^2 \, dx\, dy = A$$

Inserting these definitions we get:

$$\int_0^\infty u(t)\, dt = \frac{1}{A} \sum_{n=1}^{\infty} \langle \varphi_n, 1 \rangle^2$$

The Parseval equality is:

$$\sum_{n=1}^{\infty} \langle \varphi_n, 1 \rangle^2 = \|1\|^2 \qquad (3.80)$$

We finally get:

$$\int_0^\infty u(t)\, dt = \frac{1}{A} \|1\|^2 = 1 \qquad (3.81)$$

We have then obtained the desired result.

3.4 Fourier analysis.

We will now write the infiltration as the Fouriercosine series in the timeinterval $0 \leq t \leq T$:

$$R = A_0 + \sum_{m=1}^{\infty} A_m \cos m\omega t \tag{3.82}$$

where $\omega = \frac{\pi}{T}$.

Inserting this in 3.78 we get :

$$q(t) = \sum_{n=1}^{\infty} \frac{1}{K_n} \int_{\Omega} \varphi_n(x,y)\,dx\,dy \int_{\Omega} \varphi_n(\xi,\eta) \left\{ \int_0^{\infty} e^{-\frac{\tau}{K_n}} \left(A_0 + \sum_{m=1}^{\infty} A_m \cos m\omega(t-\tau) \right) d\tau \right\} d\xi\, d\eta$$

Now defining :

$$R_n = R_{n,0} + \sum_{m=1}^{\infty} R_{n,m} \cos m\omega t$$

$$R_{n,m} = \int_{\Omega} A_m \varphi_n \,dx\,dy$$

and inserting in the equation for q(t) we get:

$$\begin{aligned} q(t) = {} & \sum_{n=1}^{\infty} R_{n,0} \int_{\Omega} \varphi_n(x,y)\,dx\,dy \\ & + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{R_{n,m} \int_{\Omega} \varphi_n(x,y)\,dx\,dy}{(1+(m\omega K_n)^2)^{1/2}} \cos(m\omega t - \gamma_{n,m}) \end{aligned} \tag{3.83}$$

where

$$e^{i\gamma_{n,m}} = \frac{1 + i m\omega K_n}{(1+(m\omega K_n)^2)^{1/2}} \tag{3.84}$$

We must consider three frequency intervals.

Low frequency $\omega m K_n << 1$

High frequency $\omega m K_n >> 1$

and we have frequency between the low and high frequency intervals.

In the low frequency case we have:

$$\gamma_{n,m} \cong m\omega K_n \tag{3.85}$$

Inserting in 3.83 we get:

$$q_{low}(t) = \sum_{n=1}^{\infty} R_{n,0} \int_{\Omega} \varphi_n(x,y)\,dx\,dy$$

$$+ \sum_{m << \frac{1}{\omega K_n}} \sum_{n=1}^{\infty} R_{n,m} \int_{\Omega} \varphi_n(x,y)\,dx\,dy \cos m\omega(t-K_n) \tag{3.86}$$

In the high frequency case we get:

$$\gamma_{n,m} \cong \frac{\pi}{2} \tag{3.87}$$

Inserting in 3.83 we find:

$$q_{high}(t) = \sum_{n=1}^{\infty} R_{n,0} \int_{\Omega} \varphi_n(x,y)\,dx\,dy$$

$$+ \sum_{m >> \frac{1}{\omega K_n}} \sum_{n=1}^{\infty} \frac{R_{n,m} \int_{\Omega} \varphi_n(x,y)\,dx\,dy}{m\omega K_n} \sin m\omega t \tag{3.88}$$

From 3.86 we see that in the low frequency case the soil properties just have influence on the phase difference but not on the amplitude. The amplitude is determined by the geometry of our aquifer. This is completely reversed in the high frequency case.

Here, the phase difference is constant equal to $\pi/2$. The amplitude is greatly damped.

If we analyse the transient groundwaterlevel in the same way as the boundary outflow we find from 3.42:

$$h_1(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{K_n \varphi_n(x,y) R_{n,m}}{S(1+(m\omega K_n)^2)^{1/2}} \cos(m\omega t - \gamma_{n,m}) \quad (3.89)$$

In the low frequency case we get:

$$\gamma_{n,m} \cong m\omega K_n \quad (3.90)$$

$$h_{1,low}(x,y,t) = \sum_{m \ll \frac{1}{\omega K_n}} \sum_{n=1}^{\infty} \frac{K_n \varphi_n(x,y)}{S} R_{n,m} \cos m\omega(t - K_n)$$

In the high frequency case we get:

$$\gamma_{n,m} \cong \frac{\pi}{2}$$

$$h_{1,high}(x,y,t) = \sum_{m \gg \frac{1}{\omega K_n}} \sum_{n=1}^{\infty} \frac{\varphi_n(x,y)}{m\omega S} R_{n,m} \sin m\omega t \quad (3.91)$$

We see that the soil properties have both influence on the amplitude and the phase difference in the low frequency case. This is almost completely reversed in the high frequency case. Here, we have constant phase difference equal to $\pi/2$. The amplitude is just reduced by the storage coefficient, S. The amplitude is greatly damped.

If the infiltration is mainly in the high frequency range, then a river that is mainly fed from a groundwater reservoir must have a very constant water discharge, because of the damping effect. The high frequency fluctuations in the rain are damped by the reservoir mechanism and the variations in the discharge are much smaller than in the rain.Practical calculations of the autocorrelation for the normalized flow series for groundwater fed rivers from aquifers with very high timeconstants, K_n, show that they have very typical correlograms. See fig. 3.11. The figure

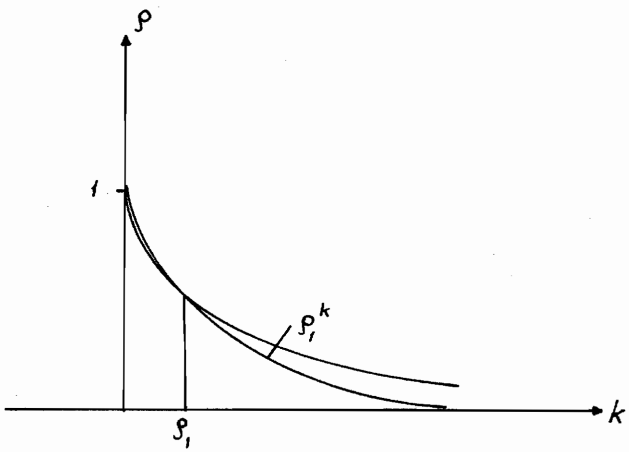

fig. 3.11

shows the autocorrelation for the normalized flow series where the seasonal variations have been extracted. A very commonly used stochastic model for the normalized flow series is the Markov model:

$$\varepsilon_t = \rho_1 \varepsilon_{t-1} + \sqrt{1-\rho_1^2}\,\eta_t \tag{3.92}$$

where η_t is white noise with some probability distribution.

The correllogram for the Markov process is also shown in fig. 3.11. It is given by:

$$\rho = \rho_1^{\,k} \qquad (3.93)$$

where ρ_1 is the first autocorrelation coefficient. The figure shows that the Markov process is fitted to the actual process by matching the first autocorrelation-coefficient. We see that the area under ρ_1^k is much smaller than the area under the actual curve. This is due to the high timeconstants for the river flow. The Markov process could of course also be fitted to the actual one by matching the area under the curves but then the first autocorre-lationcoefficients would not fit. The Markov model cannot therefore be used to describe the actual flow series, if it is important that both the first autocorrelation co-efficient and the area are conserved in the model. Then some other models must be used. It will not be discussed more here, but several models are available. See /54/, /71/, /51/ and /52/. In many cases it is more convenient to operate in the frequency domain than in the time domain. To show that we Fouriertransform our timedependent variables. The Fouriertransform of the groundwaterlevel and the infilt-ration is given by 3.9 and 3.10 respectively. Taking the Fouriertransform of 3.42 and inserting 3.9 and 3.10 we get:

$$H(x,y,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{1}{S}\sum_{n=1}^{\infty}\varphi_n(x,y)\int_0^{\infty}\int_{\Omega}\varphi_n(\xi,\eta)\left(R(\xi,\eta,t-\tau)-\bar{R}(\xi,\eta)\right)e^{-\frac{\tau}{K_n}}d\xi\, d\eta\, d\tau\right\}e^{-i\omega t}dt$$

$$= \frac{1}{S}\sum_{n=1}^{\infty}\varphi_n(x,y)\int_{\Omega}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(R(\xi,\eta,t-\tau)-\bar{R}(\xi,\eta)\right)e^{-i\omega(t-\tau)}d(t-\tau)\int_0^{\infty}e^{-\left(\frac{1}{K_n}-i\omega\right)\tau}d\tau\right\}\varphi_n(\xi,\eta)\,d\xi\, d\eta$$

$$H(x,y,\omega) = \frac{1}{S}\sum_{n=1}^{\infty}\frac{K_n}{1+i\omega K_n}\varphi_n(x,y)\int_{\Omega}F(\xi,\eta,\omega)\,\varphi_n(\xi,\eta)\,d\xi\, d\eta \qquad (3.94)$$

Now defining:

$$Q(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(q(t) - \int_{\Omega} \vec{R}(x,y)\,dx\,dy \right) e^{-i\omega t}\, dt \qquad (3.95)$$

and inserting q(t) from 3.70 we get:

$$Q(\omega) = \sum_{n=1}^{\infty} \frac{1}{1+i\omega K_n} \int_{\Omega} \varphi_n(x,y)\,dx\,dy \int_{\Omega} F(\xi,\eta,\omega)\,\varphi_n(\xi,\eta)\,d\xi\,d\eta \qquad (3.96)$$

We see from 3.94 and 3.96 that operations are much easier in the frequency domain than in the timedomain. If the groundwaterlevel and the boundary outflow is Fouriertransformed, then 3.94 and 3.96 can be used to estimate the storage coefficient, S, and the transmissivity, T. If the infiltration is constant over the area the equations become more simple :

$$\frac{1}{S}\sum_{n=1}^{\infty} \frac{K_n}{1+i\omega K_n}\,\varphi_n(x,y)\int_{\Omega} \varphi_n(\xi,\eta)\,d\xi\,d\eta = \frac{H(x,y,\omega)}{F(\omega)} \qquad (3.97)$$

and

$$\sum_{n=1}^{\infty} \frac{1}{1+i\omega K_n} \left(\int_{\Omega} \varphi_n(x,y)\,dx\,dy \right)^2 = \frac{Q(\omega)}{F(\omega)} \qquad (3.98)$$

Equations 3.97 and 3.98 can now be used for estimation purposes. The above equations can also be solved for the Fouriertransform of the infiltration, getting:

$$F(\omega) = \frac{H(x,y,\omega)}{\sum_{n=1}^{\infty} \frac{1}{S}\frac{K_n}{1+i\omega K_n}\,\varphi_n(x,y)\int_{\Omega}\varphi_n(\xi,\eta)\,d\xi\,d\eta} \qquad (3.99)$$

and

$$F(\omega) = \frac{Q(\omega)}{\sum_{n=1}^{\infty} \frac{1}{1+i\omega K_n} \left(\int_{\Omega} \varphi_n(x,y)\, dx\, dy \right)^2} \qquad (3.100)$$

Eq. 3.99 or 3.100 can be used to calculate the Fouriertransform of the infiltration, if the Fouriertransform of the groundwaterlevel or the boundary outflow is known. Then it can be transformed back to give the infiltration. In most cases it is much easier to measure the groundwaterlevel than the boundary outflow, therefore eq. 3.99 would be used more frequently for practical purposes. Then, of course, the aquifer parameters, S and T, must be determined by other means. The groundwaterlevel can be measured with much more accuracy than the infiltration. If the aquifer parameters can be determined with reasonable accuracy, then there is a reason to believe that the results obtained from 3.99, would be more accurate than those from direct measurements.

3.5 Variable aquifer coefficients.

In nature we do not have constant values of the storage coefficient and the transmissivity. We will, therefore, now consider a method of solution for eq. 2.22 and 2.31 with timeinvariant boundary conditions and variable S and T. We have the following equations:

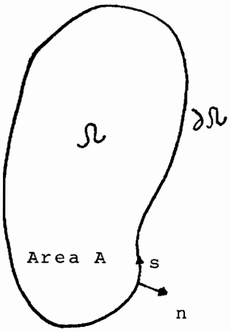

fig. 3.12

Differential equation:

$$\nabla \cdot (T \nabla h) = S \frac{\partial h}{\partial t} - R \quad in\ \Omega \tag{3.101}$$

Boundary conditions:

$$h + \alpha \frac{\partial h}{\partial n} = \beta \quad on\ \partial\Omega \tag{3.102}$$

Initial condition:

$$h(x, y, t_o) = f(x, y) \tag{3.103}$$

Let us, as before, divide the groundwaterlevel into a stationary and a transient part:

$$h(x, y, t) = h_1(x, y, t) + h_o(x, y)$$

Let the stationary level satisfy the following equation:

Differential equation:

$$\nabla \cdot (T \nabla h_o) = -\bar{R} \quad \text{in } \Omega \tag{3.104}$$

Boundary conditions:

$$h_o + \alpha \frac{\partial h_o}{\partial n} = \beta \quad \text{on } \partial\Omega \tag{3.105}$$

where $\bar{R}$ is the mean infiltration level as defined in 3.6. The transient part must then satisfy:

Differential equations:

$$\left(\frac{T_o}{S_o}\right)\cdot\left(\frac{S_o}{S}\right) \nabla \cdot \left(\frac{T}{T_o} \nabla h_1\right) = \frac{\partial h_1}{\partial t} - \frac{(R-\bar{R})}{S} \quad \text{in } \Omega \tag{3.106}$$

Boundary conditions:

$$h_1 + \alpha \frac{\partial h_1}{\partial n} = 0 \quad \text{on } \partial\Omega \tag{3.107}$$

Initial condition:

$$h_1(x,y,t_o) = f(x,y) - h_o(x,y) \tag{3.108}$$

T_o and S_o are reference values. We now define the eigen-problem:

$$\nabla \cdot \left(\frac{T}{T_o} \nabla \varphi_n\right) + \frac{\lambda_n}{A} \frac{S}{S_o} \varphi_n = 0 \quad \text{in } \Omega$$

$$\varphi_n + \alpha \frac{\partial \varphi_n}{\partial n} = 0 \quad \text{on } \partial\Omega \tag{3.109}$$

It is easy to prove that all the eigenvalues are positive. We will now proceed to find the orthogonal relationship. We have:

$$\nabla\cdot\left(\frac{T}{T_0}\nabla\varphi_n\right)\varphi_m - \nabla\cdot\left(\frac{T}{T_0}\nabla\varphi_m\right)\varphi_n + \frac{\lambda_n}{A}\frac{S}{S_0}\varphi_n\varphi_m - \frac{\lambda_m}{A}\frac{S}{S_0}\varphi_m\varphi_n = 0$$

Integrating over Ω we get:

$$\frac{(\lambda_m-\lambda_n)}{A}\int_\Omega\left(\frac{S}{S_0}\right)\varphi_n\varphi_m\,dx\,dy = \int_\Omega\left(\nabla\cdot\left(\frac{T}{T_0}\nabla\varphi_n\right)\varphi_m - \nabla\cdot\left(\frac{T}{T_0}\nabla\varphi_m\right)\varphi_n\right)dx\,dy$$

$$= \int_\Omega\left(\nabla\cdot\left(\frac{T}{T_0}\nabla\varphi_n\,\varphi_m\right) - \nabla\cdot\left(\frac{T}{T_0}\nabla\varphi_m\,\varphi_n\right)\right)dx\,dy$$

Now using Gauss theorem we get:

$$\frac{(\lambda_m-\lambda_n)}{A}\int_\Omega\left(\frac{S}{S_0}\right)\varphi_n\varphi_m\,dx\,dy = \int_{\partial\Omega}\frac{T}{T_0}\left(\varphi_m\frac{\partial\varphi_n}{\partial n} - \varphi_n\frac{\partial\varphi_m}{\partial n}\right)ds$$

Inserting the boundary conditions we finally have:

$$\frac{(\lambda_m-\lambda_n)}{A}\int_\Omega\left(\frac{S}{S_0}\right)\varphi_n\varphi_m\,dx\,dy = 0$$

We therefore have the orthogonal relationship:

$$\int_\Omega\left(\frac{S}{S_0}\right)\varphi_n\varphi_m\,dx\,dy = 0 \qquad (3.110)$$

We now define the inner product and the norm in our function space as:

$$\langle f, g\rangle = \int_\Omega\left(\frac{S}{S_0}\right)f\cdot g\,dx\,dy \qquad (3.111)$$

$$\|g\|^2 = \int_\Omega \left(\frac{S}{S_o}\right) g^2 \, dx\, dy \qquad (3.112)$$

Supposing the eigenfunctions have been normalized we have from these definitions and 3.110:

$$\langle \varphi_n, \varphi_m \rangle = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases} \qquad (3.113)$$

We now wish to expand h_1 in the series:

$$h_1(x,y,t) = \sum_{n=1}^{\infty} a_n(t)\, \varphi_n(x,y) \qquad (3.114)$$

Writing the source term in 3.106 as an eigenfunction expansion, we get:

$$\frac{\left(R(x,y,t) - \bar{R}(x,y)\right)}{S(x,y)} = \sum_{n=1}^{\infty} b_n(t)\, \varphi_n(x,y) \qquad (3.115)$$

By using the orthogonal relationship in 3.113, we find:

$$b_n(t) = \frac{1}{S_o} \int_\Omega \left(R(x,y,t) - \bar{R}(x,y)\right) \varphi_n(x,y)\, dx\, dy \qquad (3.116)$$

From 3.108 and 3.114 we get for the initial condition:

$$a_n(t_o) = \frac{1}{S_o}\int_{\Omega} S(x,y)\left(f(x,y) - h_o(x,y)\right)\varphi_n(x,y)\,dx\,dy \qquad (3.117)$$

Inserting 3.114 in the differential equation for h_1, 3.106, we find:

$$\sum_{n=1}^{\infty}\left\{\frac{T_o}{S_o}\,a_n(t)\,\frac{S_o}{S}\,\nabla\cdot\left(\frac{T}{T_o}\nabla\varphi_n\right) - \frac{da_n(t)}{dt}\,\varphi_n + b_n(t)\,\varphi_n\right\} = 0$$

using 3.109, we get:

$$\sum_{n=1}^{\infty}\left\{\frac{\lambda_n T_o}{A\,S_o}\,a_n(t) + \frac{da_n(t)}{dt} - b_n(t)\right\}\varphi_n = 0$$

Defining:

$$K_n = \frac{A\,S_o}{\lambda_n T_o} \qquad (3.118)$$

inserting and using the orthogonal relationship, we get:

$$\frac{da_n}{dt} + \frac{1}{K_n}\,a_n - b_n = 0 \qquad (3.119)$$

This is an ordinary differential equation with the initial condition given in 3.117.

Its solution is given by:

$$a_n(t) = \frac{1}{S_o}\int_{\Omega} S(x,y)\left(f(x,y) - h_o(x,y)\right)\varphi_n(x,y)\,dx\,dy\; e^{\frac{-(t-t_o)}{K_n}}$$

$$+\frac{1}{S_o}\int_{\Omega}\varphi_n(x,y)\int_{t_o}^{t}\left(R(x,y,\tau) - \bar{R}(x,y)\right)e^{\frac{-(t-\tau)}{K_n}}\,d\tau\,dx\,dy$$

(3.120)

We have used 3.116 and 3.117. We have finally for h_1:

$$h_1(x,y,t) = \frac{1}{S_0}\sum_{n=1}^{\infty} \varphi_n(x,y)\int_{\Omega} S(\xi,\eta)(f(\xi,\eta) - h_0(\xi,\eta))\,\varphi_n(\xi,\eta)\,d\xi\, d\eta\; e^{-\frac{(t-t_0)}{K_n}}$$

$$+\frac{1}{S_0}\sum_{n=1}^{\infty} \varphi_n(x,y)\int_{\Omega} \varphi_n(\xi,\eta)\left\{\int_{t_0}^{t} (R(\xi,\eta,\tau) - \bar{R}(\xi,\eta))\, e^{-\frac{(t-\tau)}{K_n}} d\tau\right\} d\xi\, d\eta \tag{3.121}$$

Now taking $t_0 = -\infty$ we get:

$$h_1(x,y,t) = \frac{1}{S_0}\sum_{n=1}^{\infty} \varphi_n(x,y)\int_{\Omega} \varphi_n(\xi,\eta)\left\{\int_0^{\infty} e^{-\frac{\tau}{K_n}} (R(\xi,\eta,t-\tau) - \bar{R}(\xi,\eta))\, d\tau\right\} d\xi\, d\eta \tag{3.122}$$

This equation is exactly the same as equation 3.42, but we must remember that the variable S and T coefficients are hidden in the eigenfunctions and eigenvalues. We now proceed to calculate the boundary outflow which is given by 3.69:

$$q(t) = -\int_{\partial\Omega} T \frac{\partial h}{\partial n}\, ds \tag{3.69}$$

By using Gauss' theorem we get:

$$q(t) = -\int_{\partial\Omega} T\frac{\partial h}{\partial n}\, ds = -\int_{\Omega} (T\nabla^2 h + \nabla T\cdot\nabla h)\, dx\, dy$$

$$= -\int_{\Omega} \nabla\cdot(T\nabla h)\, dx\, dy$$

Now inserting 3.104 and 3.122 we get:

$$q(t) = \int_{\Omega} \bar{R}(x,y)\,dx\,dy - \frac{1}{S_o}\sum_{n=1}^{\infty}\int_{\Omega} \nabla\cdot(T\nabla\varphi_n)\,dx\,dy \cdot I_n$$

$$I_n = \int_{\Omega} \varphi_n(\xi,\eta)\left\{\int_0^{\infty} e^{-\frac{\tau}{K_n}}\left(R(\xi,\eta,t-\tau) - \bar{R}(\xi,\eta)\right)d\tau\right\}d\xi\,d\eta$$

By making use of 3.109 we have:

$$q(t) = \int_{\Omega} \bar{R}(x,y)\,dx\,dy + \sum_{n=1}^{\infty}\frac{1}{K_n}\int_{\Omega}\frac{S}{S_o}\varphi_n\,dx\,dy \cdot I_n$$

By moving the first term inside the summation, we get:

$$q(t) = \sum_{n=1}^{\infty}\frac{1}{K_n}\int_{\Omega}\frac{S}{S_o}\varphi_n\,dx\,dy\int_{\Omega}\varphi_n(\xi,\eta)\left\{\int_0^{\infty} e^{-\frac{\tau}{K_n}} R(\xi,\eta,t-\tau)\,d\tau\right\}d\xi\,d\eta \tag{3.123}$$

By differentating the n'th term in 3.123 we find:

$$q_n + K_n\frac{dq_n}{dt} = \int_{\Omega}\frac{S}{S_o}\varphi_n\,dx\,dy\int_{\Omega}\varphi_n(\xi,\eta)\,R(\xi,\eta,t)\,d\xi\,d\eta$$

$$q(t) = \sum_{n=1}^{\infty} q_n(t) \tag{3.124}$$

We therefore have exactly the same linear reservoirs as before, but the inflow term to the n'th reservoir is given by:

$$Z_n = \int_\Omega \frac{S(x,y)}{S_o} \varphi_n(x,y)\,dx\,dy \int_\Omega \varphi_n(\xi,\eta)\, R(\xi,\eta,t)\, d\xi\, d\eta \qquad (3.125)$$

Now, checking the necessary continuity equation is fulfilled for the total inflow. We have:

$$\sum_{n=1}^{\infty} Z_n = \int_\Omega \frac{S(x,y)}{S_o} \left\{ \sum_{n=1}^{\infty} \varphi_n(x,y) \int_\Omega \frac{S(\xi,\eta)}{S_o} \varphi_n(\xi,\eta) \frac{R(\xi,\eta,t)}{S(\xi,\eta)/S_o}\, d\xi\, d\eta \right\} dx\,dy$$

$$= \int_\Omega R(x,y,t)\,dx\,dy$$

It is therefore satisfied as was to be expected.

The IUH was defined in 3.79 for constant values of S and T, now it is given by:

$$u(t) = \frac{1}{A} \sum_{n=1}^{\infty} \frac{1}{K_n} \int_\Omega \frac{S(x,y)}{S_o} \varphi_n(x,y)\,dx\,dy \int_\Omega \varphi_n(\xi,\eta)\,d\xi\,d\eta \; e^{-\frac{t}{K_n}} \qquad (3.126)$$

The area under the IUH function is:

$$\int_0^\infty u(t)\,dt = \frac{1}{A} \sum_{n=1}^{\infty} \frac{1}{K_n} \int_\Omega \frac{S}{S_o} \varphi_n\,dx\,dy \int_\Omega \varphi_n\,d\xi\,d\eta \int_0^\infty e^{-\frac{t}{K_n}}\,dt$$

$$= \frac{1}{A} \sum_{n=1}^{\infty} \int_\Omega \frac{S}{S_o} \varphi_n\,dx\,dy \int_\Omega \varphi_n\,d\xi\,d\eta$$

$$= \frac{1}{A} \int_\Omega \left\{ \sum_{n=1}^{\infty} \varphi_n(\xi,\eta) \int_\Omega \frac{S(x,y)}{S_o} \varphi_n(x,y)\,dx\,dy \right\} d\xi\,d\eta$$

Now using the definition of the inner product in 3.111, we have:

$$\int_0^{\infty} u(t)\,dt = \frac{1}{A}\int_{\Omega}\left\{\sum_{n=1}^{\infty} \varphi_n(\xi,\eta)\langle\varphi_n, 1\rangle\right\}d\xi\,d\eta$$

$$= \frac{1}{A}\int_{\Omega} 1\,d\xi\,d\eta = 1$$

which is the necessary result.
Performing Fourieranalysis on boundary outflow and the transient groundwater level we get almost the same equations as for constant values of S and T. Eq. 3.83, 3.84, 3.85, 3.86, 3.87, 3.88, 3.89, 3.90 and 3.91 remain almost unchanged, if we write S_o instead of S. But we must remember that the variation in the storage coefficient and the transmissivity is hidden in the eigenfunctions and eigenvalues. Our conclusions now become that the soil properties have influence on both the amplitude and the phase difference except in the high frequency case where we have constant phase difference equal to $\pi/2$.

Eq. 3.94, 3.96, 3.97, 3.98, 3.99 and 3.100 also remain nearly unchanged and can, therefore, be used for the same purpose, but the estimation procedure is much more difficult, as it now is two continuous functions S(x,y) and T(x,y), that must be estimated instead of two constant values. The backward transformation to the infiltration should not be more difficult if the storage coefficient function and the transmissivity function are known.

3.6 Examples.

We will now in the following look at some simple examples to demonstrate the principles we have been proving. Let us take Ω as a rectangle, and the source term be given in the form of constant pumping. If we write the pumping term as $Q\,\delta(x-\xi)\,\delta(y-\eta)$, where δ is Dirac's delta function, then the pumping can be treated as the infiltration before. We have for constant values of S and T:

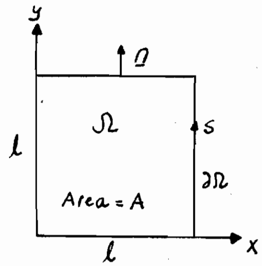

fig. 3.13

Differential equation:

$$\nabla^2 h = \frac{S}{T}\frac{\partial h}{\partial t} + \frac{Q}{T}\delta(x-\xi)\delta(y-\eta) \quad \text{in } \Omega$$

Boundary condition:

$$h = g = \text{constant} \quad \text{on } \partial\Omega$$

The solution to the stationary problem in 3.4 and 3.5 is obviously given by:

$$h_0 = g = \text{constant.}$$

For the transient term we have from 3.38:

$$h_1(x,y,t) = -\frac{1}{S}\sum_{n=1}^{\infty}\varphi_n\int_{\Omega}\varphi_n\left\{\int_0^t e^{-\frac{(t-\tau)}{K_n}}Q\,d\tau\right\}\delta(x-\xi)\,\delta(y-\eta)\,dx\,dy$$

Performing the integration, we get:

$$h_1(x,y,t) = -\frac{AQ}{T}\sum_{n=1}^{\infty}\frac{\varphi_n(x,y)}{\lambda_n}\,\varphi_n(\xi,\eta)\left(1-e^{-\frac{t}{K_n}}\right)$$

The solution to the eigenproblem in 3.34 is given by:

$$\lambda_{nm} = \pi^2(m^2+n^2)$$

$$\varphi_{nm} = \frac{2}{\ell} \sin\frac{m\pi}{\ell}x \, \sin\frac{n\pi}{\ell}y$$

Inserting these values in the equation for h_1 above, we get:

$$h_1(x,y,t) = -\frac{4Q}{\pi^2 T}\sum_{n,m}\frac{\sin\frac{m\pi x}{\ell}\sin\frac{n\pi y}{\ell}\sin\frac{m\pi\xi}{\ell}\sin\frac{n\pi\eta}{\ell}}{(m^2+n^2)}\left(1-e^{\frac{-\pi^2(n^2+m^2)T}{AS}t}\right)$$

Let us now insert numerical values.

Let us take $(x,y) = \left(\frac{\ell}{4},\frac{\ell}{2}\right), (\xi,\eta) = \left(\frac{\ell}{2},\frac{\ell}{2}\right),$

$A = 10^8\, m^2,\ \ell = 10^4\, m,\ T = 0{,}2\ m^2/sec$

$S = 0{,}06,\ Q = 1{,}0\ m^3/sec$

h (x,y,t) is plotted in fig. 3.14 together with the solution to 3.68,

$$h(x,y,t) = g + \frac{Q}{4\pi T}W(u)$$

$$W(u) = \int_u^\infty \frac{e^{-z}}{z}\,dz$$

$$u = \frac{r^2 S}{4Tt}$$

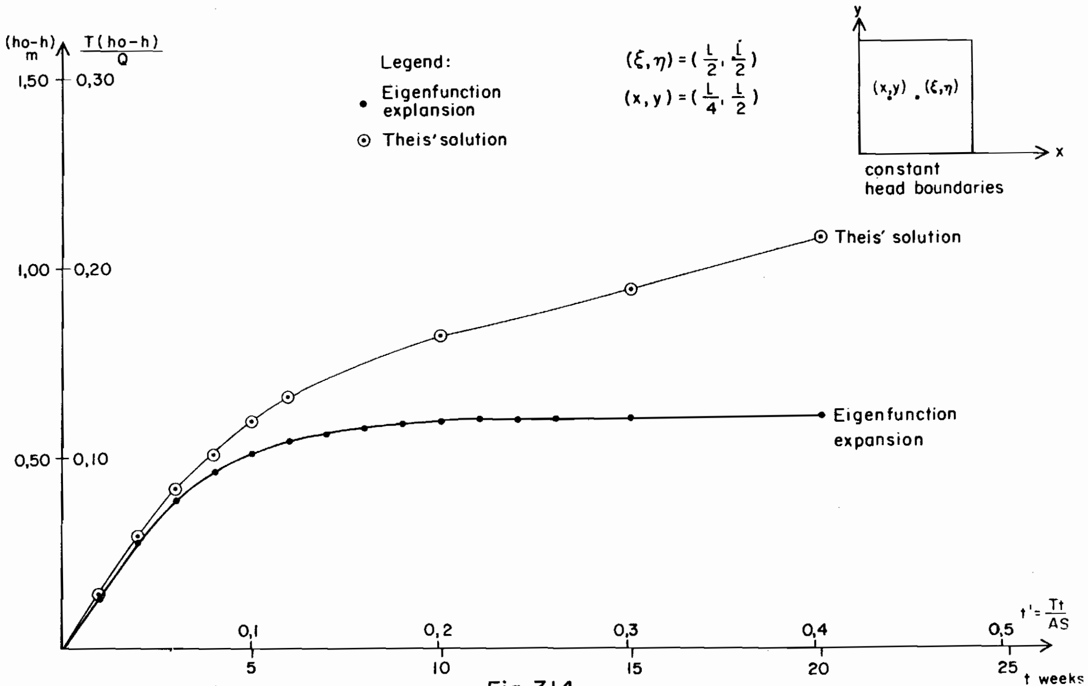

Fig. 3.14.

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This is Theis' solution, it is valid when our Ω extends to infinity on all four sides.

We see that Theis' solution only approximates the exact solution well in the starting period of the pumping.

In the case of just one boundary line Theis' solution could be modified by using an image well. Here, this cannot be done because of the constant head boundaries all around the point. We would get infinitely many image wells according to the infinite eigenfunction expansion. In nature we can have many kinds of boundary conditions, care must therefore be taken that Theis' equation is valid. The boundary inflow is given by eq. 3.78:

$$q(t) = -\sum_{n=1}^{\infty} \frac{1}{K_n} \int_{\Omega} \varphi_n \, dx\,dy \int_{\Omega} \varphi_n \left\{ \int_0^t e^{-\frac{(t-\tau)}{K_n}} Q \, d\tau \right\} \delta(x-\xi)\,\delta(y-\eta)\,dx\,dy$$

Performing the integration we get:

$$q(t) = -Q \sum_{n=1}^{\infty} \varphi_n(\xi,\eta) \int_{\Omega} \varphi_n \, dx\,dy \left(1 - e^{-\frac{t}{K_n}}\right)$$

Now using the definition of the inner product we get, when $t \rightarrow \infty$:

$$q(t) = -Q \sum_{n=1}^{\infty} \varphi_n(\xi,\eta) < \varphi_n, 1 > \; = -Q$$

That is to say $q(t) \rightarrow -Q$, when $t \rightarrow \infty$, as was to be expected.

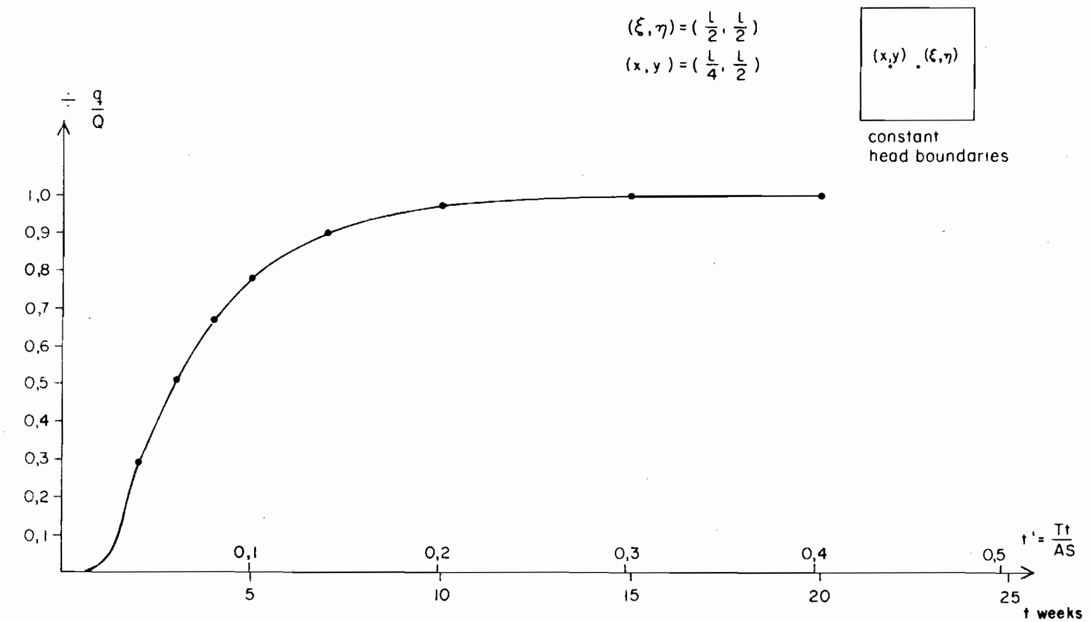
Inserting the eigenvalues and eigenvectors we have:

$$q(t) = -\frac{4Q}{\pi^2} \sum_{m,n} \frac{(1-(-1)^m)(1-(-1)^n)}{n\,m} \sin\frac{m\pi}{\ell}\xi \, \sin\frac{n\pi}{\ell}\eta \left(1 - e^{-\frac{\pi^2(n^2+m^2)T}{AS}t}\right)$$

which can be written as:

$$q(t) = -\frac{16Q}{\pi^2} \sum_{m=1,3\cdots} \sum_{n=1,3\cdots} \frac{\sin\frac{m\pi}{\ell}\xi \, \sin\frac{n\pi}{\ell}\eta}{n\,m} \left(1 - e^{-\frac{\pi^2(n^2+m^2)T}{AS}t}\right)$$

It is plotted in fig. 3.15.



Boundary outflow. Fig. 3.15

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The IUH-function is given by 3.79.

$$u(t) = \frac{64}{\pi^2} \frac{T}{AS} \sum_{m=1,3\cdots} \sum_{n=1,3\cdots} \frac{(m^2+n^2)}{m^2 n^2} e^{-\frac{\pi^2(m^2+n^2)T}{AS} t}$$

The IUH is plotted in fig. 3.16.

The area under the IUH-curve is given by:

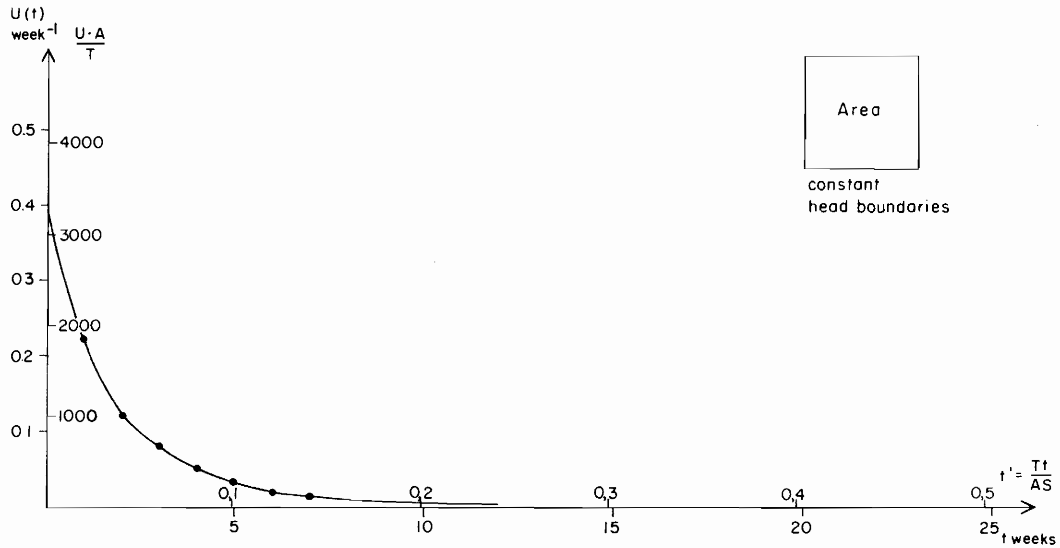
$$\int_0^\infty u(t)\,dt = \frac{64}{\pi^4} \sum_{m=1,3\cdots} \sum_{n=1,3\cdots} \frac{1}{m^2 n^2}$$

$$= \frac{64}{\pi^4} \left(\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \right)^2$$

$$= \frac{64}{\pi^4} \left(\frac{\pi^2}{8} \right)^2 = 1$$

Of course the same result as we obtained before in 3.81.

The expression for the IUH-function above is in a better agreement with the correlogram for a groundwater fed river given in fig. 3.11, than the correlogram for the Markov-process shown in the same figure. The above expression is an infinite expansion of exponentialfunctions, but the Markovprocess just corresponds to a single exponential function.



Instantaneous unit hydrograph. Fig. 3.16.

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4. THE GALERKIN FINITE ELEMENT METHOD.

4.1 The weak formulation.

Analytical solutions to the flow problem do not exist except in a very few simple cases. We will, therefore, use numerical methods to solve the following flow problem.

Differential equation:

$$\nabla \cdot (T \nabla h) = S\frac{\partial h}{\partial t} - R - \sum_i Q_i\, \delta(x-\xi_i)\,\delta(y-\eta_i) \quad \text{in } \Omega \qquad (4.1)$$

Boundary conditions:

$$h(x,y,t) = g(x,y,t) \quad \text{on } \partial\Omega_1$$

$$T\frac{\partial h}{\partial n} = -q(x,y,t) \quad \text{on } \partial\Omega_2 \qquad (4.2)$$

Initial condition:

$$h(x,y,t_0) = f(x,y) \quad \text{in } \Omega \qquad (4.3)$$

where Q_i is pumping from wells or recharge, and δ are Dirac's delta functions.

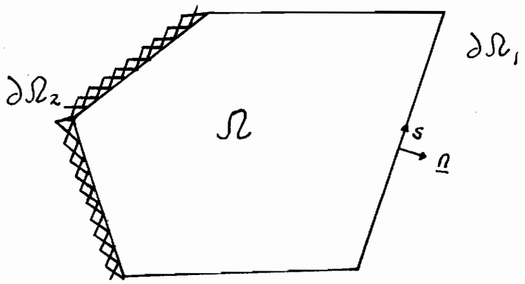

fig. 4.1

We now turn to the weak formulation of the problem. Let us define φ to be piecewise differentiable and vanish on $\partial\Omega_1$. We now multiply the differential-equation 4.1 by φ and integrate over the area:

$$\int_\Omega \nabla\cdot(T\nabla h)\varphi\,dx\,dy = \int_\Omega \left(S\frac{\partial h}{\partial t} - R - \sum_i Q_i\,\delta(x-\xi_i)\,\delta(y-\eta_i)\right)\varphi\,dx\,dy \tag{4.4}$$

For convenience we multiply 4.4 by some reference values T_o and S_o, getting:

$$T_o\int_\Omega \nabla\left(\left(\frac{T}{T_o}\right)\nabla h\right)\varphi\,dx\,dy = \int_\Omega \left(S_o\left(\frac{S}{S_o}\right)\frac{\partial h}{\partial t} - R - \sum_i Q_i\,\delta(x-\xi_i)\,\delta(y-\eta_i)\right)\varphi\,dx\,dy \tag{4.5}$$

Now using Gauss' theorem we get:

$$\begin{aligned} S_o\int_\Omega \left(\frac{S}{S_o}\right)\frac{\partial h}{\partial t}\varphi\,dx\,dy = &\int_{\partial\Omega_1} T\frac{\partial h}{\partial n}\varphi\,ds - \int_{\partial\Omega_2} q\varphi\,ds \\ &- T_o\int_\Omega\left(\frac{T}{T_o}\right)\nabla h\cdot\nabla\varphi\,dx\,dy + \int_\Omega R\varphi\,dx\,dy \\ &+ \sum_i Q_i\,\varphi(\xi_i,\eta_i) \end{aligned} \tag{4.6}$$

Remembering that φ was chosen to be zero on $\partial\Omega_1$, we get the weak formulation of the problem:

$$\begin{aligned} S_o\int_\Omega \left(\frac{S}{S_o}\right)\frac{\partial h}{\partial t}\varphi\,dx\,dy = &\int_\Omega R\varphi\,dx\,dy - \int_{\partial\Omega_2} q\varphi\,ds \\ &- T_o\int_\Omega\left(\frac{T}{T_o}\right)\nabla h\cdot\nabla\varphi\,dx\,dy \\ &+ \sum_i Q_i\,\varphi(\xi_i,\eta_i) \end{aligned} \tag{4.7}$$

This is the starting point for the finite element approximation. Given an N-dimensional subspace S^h of the space defined for the φ-functions, the Galerkin principle is to find a function $\varphi^P + h^h(x,y,t)$ with the following properties. φ^P takes care of the inhomogeneous constant head boundary conditions, and for $t > t_o$, h^h lies in S^h and satisfies:

$$S_o\int_\Omega \left(\frac{S}{S_o}\right)\frac{\partial h^h}{\partial t}\varphi^h\,dx\,dy + S_o\int_\Omega\left(\frac{S}{S_o}\right)\frac{\partial \varphi^P}{\partial t}\varphi^h\,dx\,dy$$

$$= \int_\Omega R\varphi^h\,dx\,dy - \int_{\partial\Omega_2} q\varphi^h\,ds - T_o\int_\Omega\left(\frac{T}{T_o}\right)\nabla h^h\cdot\nabla\varphi^h\,dx\,dy$$

(4.8)

$$-T_o\int_\Omega\left(\frac{T}{T_o}\right)\nabla\varphi^P\cdot\nabla\varphi^h\,dx\,dy + \sum_i Q_i\,\varphi^h(\xi_i,\eta_i)$$

We now choose a basis $\varphi_1, \varphi_2, \cdots\cdots\cdots, \varphi_N$ for the trial space S^h and expand the unknown solution as:

$$h^h(x,y,t) = \sum_{i=1}^{N} h_i(t)\,\varphi_i(x,y) \tag{4.9}$$

Inserting this in 4.8 we get:

$$\sum_{i=1}^{N}\left\{S_o\int_\Omega\left(\frac{S}{S_o}\right)\frac{dh_i}{dt}\varphi_i\varphi_j\,dx\,dy + T_o\int_\Omega\left(\frac{T}{T_o}\right)h_i\,\nabla\varphi_i\cdot\nabla\varphi_j\,dx\,dy\right\}$$

$$= \int_\Omega R\varphi_j\,dx\,dy - \int_{\partial\Omega_2} q\,\varphi_j\,ds + \sum_k Q_k\,\varphi_j(\xi_k,\eta_k)$$

$$- S_o\int_\Omega\left(\frac{S}{S_o}\right)\frac{\partial\varphi^P}{\partial t}\varphi_j\,dx\,dy - T_o\int_\Omega\left(\frac{T}{T_o}\right)\nabla\varphi^P\cdot\nabla\varphi_j\,dx\,dy \tag{4.10}$$

$j = 1, \cdots\cdots, N,.$

Since every φ^h is a combination of the basis functions φ_j , it is enough to apply the Galerkin principle only to the basis functions, as is done above. The result is a system of N ordinary differential equations for the N unknowns $h_1(t), \ldots\ldots, h_N(t)$, and the boundary conditions are already incorporated in these equations. The initial condition $h = f - \varphi^P$ is still to be accounted for, and here there are several possibilities. Mathematically, a natural choice of approximate initial condition $(f-\varphi^P)^h$ is the best least-squares approximation to $(f-\varphi^P)$ in S^h.

That is:

$$\langle (f-\varphi^P)^h, \varphi^h \rangle = \langle (f-\varphi^P), \varphi^h \rangle, \quad \forall \varphi^h$$

or in other words:

$$\sum_{i=1}^{N} h_{io} \int_{\Omega} \varphi_i \varphi_j \, dx\,dy = \int_{\Omega} (f-\varphi^P)\, \varphi_j \, dx\,dy \qquad (4.11)$$

$$J = 1, 2, \ldots, N$$

where

$$(f-\varphi^P)^h = \sum_{i=1}^{N} h_{io}\, \varphi_i(x,y) \qquad (4.12)$$

In practice, this means that the integrals in 4.11 on the right have to be computed, and the matrix on the left has to be inverted.

The least square method will not be used in the following and it will be shown later what is used instead.

4.2 Choise of basis function.

We now turn to the selection of a suitable approximation space, S^h, and basis functions for the space. In the following we will work with piecewise linear approximation spaces and linear roof functions as basis functions. We approximate the area and the boundary by a set of triangles (finite elements).

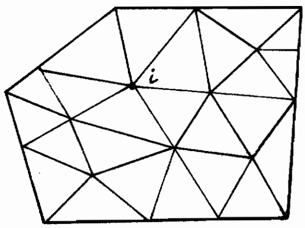

fig. 4.2

We now have that h^h is a linear function over each triangle and we take S and T to be constant over each triangle. For the basis functions we have:

$$\varphi_i(i) = 1$$
$$\varphi_i(j) = 0 \qquad (4.13)$$

That is to say basis function number i has the value one in point i and is zero elsewhere and is linear in between.

We now define the φ^p-function which was extended from the g boundary function over the area, to be piecewise linear. We define it to take the boundary values on the boundary and then decay lineary to zero inside Ω , in such way that it is equal to zero in each internal point.

To approximate the initial condition we will not use the least square approximation in 4.11, but use instead the method of collocation. That is:

$$(f-\varphi^{p})^{h} = (f(i) - \varphi^{p}(i)) = f(i) \qquad (4.14)$$

Where we have used the definition of the φ^{p}- function, f(i) is to be understood as the value of f in point number i.

If we place one of the points where the pumping and recharge take place we can write 4.10 in a more simplified form.

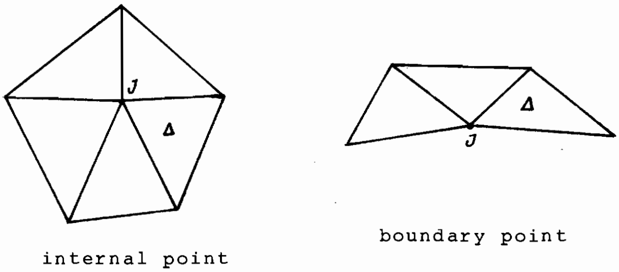

fig. 4.3

$$\sum_{i\in\Delta\in J}\left\{S_o \frac{dh_i}{dt}\int_{\Delta}\left(\frac{S}{S_o}\right)\varphi_i\varphi_j\,dxdy + T_o h_i \int_{\Delta}\left(\frac{T}{T_o}\right)\nabla\varphi_i\cdot\nabla\varphi_j\,dxdy\right\}$$

$$= \sum_{i\in\Delta\in J}\left\{\int_{\Delta} R\varphi_j\,dxdy - \int_{\Delta\cap\partial\Omega_2} q\varphi_j\,ds\right.$$

$$\left. - S_o\int_{\Delta}\left(\frac{S}{S_o}\right)\frac{\partial\varphi^{P}}{\partial t}\varphi_j\,dxdy - T_o\int_{\Delta}\left(\frac{T}{T_o}\right)\nabla\varphi^{P}\cdot\nabla\varphi_j\,dxdy\right\}$$

$$+ Q_j \qquad (4.15)$$

$$J = 1, 2, \cdots\cdots, N$$

The summation is to be understood to be taken over all triangles, Δ, including the point j.

We now define θ_k in a similar way to φ_j, except that k goes from one to N + M, where N is the number of internal points and M is equal to the number of points on $\partial\Omega_1$. In practice the infiltration is not given as a continuous function, but is just given pointwise. We will, therefore, define it as piecewise linear function. We then have:

$$R(x,y,t) = \sum_{i=1}^{(N+M)} R_i(t)\,\theta_i(x,y) \tag{4.16}$$

φ^P is given by:

$$\varphi^P(x,y,t) = \sum_{i=1}^{(N+M)} g_i(t)\,\theta_i(x,y) \tag{4.17}$$

where g_i is equal to g on the boundary $\partial\Omega_1$, and is equal to zero on $\partial\Omega_2$ and inside Ω.

Eq. 4.15 can then be rewritten as:

$$\sum_{i\in\Delta\in j}\left\{ S_o \frac{dh_i}{dt}\int_\Delta \left(\frac{S}{S_o}\right)\varphi_i\varphi_j\,dxdy + T_o h_i\int_\Delta \left(\frac{T}{T_o}\right)\nabla\varphi_i\cdot\nabla\varphi_j\,dxdy\right\}$$

$$= \sum_{i\in\Delta\in j}\left\{ R_i\int_\Delta \theta_i\varphi_j\,dxdy - \int_{\Delta\cap\partial\Omega_2} q\varphi_j\,ds\right.$$

$$\left. - S_o\frac{dg_i}{dt}\int_\Delta\left(\frac{S}{S_o}\right)\theta_i\varphi_j\,dxdy - T_o g_i\int_\Delta\left(\frac{T}{T_o}\right)\nabla\theta_i\cdot\nabla\varphi_j\,dxdy\right\}$$

$$+ Q_j$$

$$j = 1, 2 \cdots\cdots\cdots, N \tag{4.18}$$

We can now write 4.18 as a matrix equation:

$$S_o \underline{\underline{C}}_1 \frac{d\underline{h}}{dt} = -T_o \underline{\underline{B}}_1 \underline{h} + \underline{\underline{D}}\,\underline{R} + \underline{\mu}$$

$$-\left(S_o \underline{\underline{C}}_2 \frac{d\underline{g}}{dt} + T_o \underline{\underline{B}}_2 \underline{g} + \underline{\Omega} \right) \tag{4.19}$$

The matrices are defined in the following way:

$$c_1^{ij} = \int_\Omega \left(\frac{S}{S_o}\right) \varphi_i \varphi_j \, dx dy = \sum_{\Delta \in i} \int_\Delta \left(\frac{S}{S_o}\right) \varphi_i \varphi_j \, dx dy \tag{4.20}$$

$$b_1^{ij} = \int_\Omega \left(\frac{T}{T_o}\right) \nabla\varphi_i \cdot \nabla\varphi_j \, dx dy = \sum_{\Delta \in i} \int_\Delta \left(\frac{T}{T_o}\right) \nabla\varphi_i \cdot \nabla\varphi_j \, dx dy \tag{4.21}$$

$$d^{ik} = \int_\Omega \theta_k \varphi_i \, dx dy = \sum_{\Delta \in i} \int_\Delta \theta_k \varphi_i \, dx dy \tag{4.22}$$

$$c_2^{ik} = \int_\Omega \left(\frac{S}{S_o}\right) \theta_k \varphi_i \, dx dy = \sum_{\Delta \in i} \int_\Delta \left(\frac{S}{S_o}\right) \theta_k \varphi_i \, dx dy \tag{4.23}$$

$$b_2^{ik} = \int_\Omega \left(\frac{T}{T_o}\right) \nabla\theta_k \cdot \nabla\varphi_i \, dx dy = \sum_{\Delta \in i} \int_\Delta \left(\frac{T}{T_o}\right) \nabla\theta_k \cdot \nabla\varphi_i \, dx dy \tag{4.24}$$

$$\mu^i = Q_i \tag{4.25}$$

$$r^i = R(i) \tag{4.26}$$

$$g^k = \varphi^P(k) \tag{4.27}$$

$$\Omega^i = \sum_{\Delta \in i} \int_{\Delta \cap \partial \Omega_2} q \varphi_i \, ds \tag{4.28}$$

$$i = 1, 2, \cdots, N; \quad J = 1, 2, \cdots, N; \quad K = 1, 2, \cdots, (N+M)$$

We see that calculations of the matrix coefficients involves integration of the form $\int_\Delta \varphi_i \varphi_j \, dxdy$ and $\int_\Delta \nabla\varphi_i \cdot \nabla\varphi_j \, dxdy$ We will, therefore, proceed to calculate it. Let us for convenience place the local coordinate system as shown in fig. 4.4.

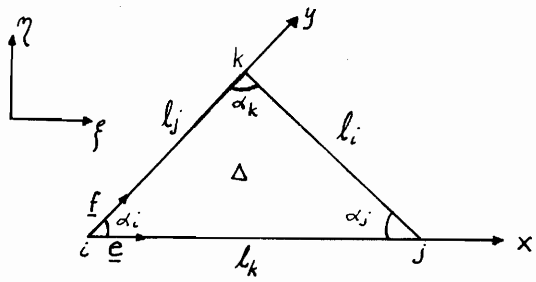

fig. 4.4

We have:

$$\varphi_i = \left(1 - \frac{x}{\ell_k} - \frac{y}{\ell_j}\right)$$

$$\varphi_j = \frac{x}{\ell_k}$$

$$\int_\Delta \varphi_i \varphi_j \, d\xi d\eta = \int_0^{\ell_j} \int_0^{(1-\frac{y}{\ell_j})\ell_k} \left(1 - \frac{x}{\ell_k} - \frac{y}{\ell_j}\right) \frac{x}{\ell_k} \sin\alpha_i \, dx dy \qquad (4.30)$$

$$= \frac{F}{12}$$

$$\int_\Delta \varphi_i \cdot \varphi_i \, d\xi d\eta = \int_0^{\ell_j} \int_0^{(1-\frac{y}{\ell_j})\ell_k} \left(1 - \frac{x}{\ell_k} - \frac{y}{\ell_j}\right)^2 \sin\alpha_i \, dx dy$$

$$= \frac{F}{6} \qquad (4.31)$$

where F is the area of the triangle.

We have :

$$\text{grad}\,\varphi_i \cdot \underline{e} = \frac{\partial \varphi_i}{\partial x} \; ; \; \text{grad}\,\varphi_i \cdot \underline{f} = \frac{\partial \varphi_i}{\partial y} \qquad (4.32)$$

where $\underline{e}$ and $\underline{f}$ are the unit vectors shown in fig. 4.4.
From 4.32 we get:

$$\text{grad}\,\varphi_i = \frac{1}{\sin^2\alpha_i} \left(\left(\frac{1}{\ell_j} \cos\alpha_i - \frac{1}{\ell_k}\right) \underline{e} + \left(\frac{1}{\ell_k} \cos\alpha_i - \frac{1}{\ell_j}\right) \underline{f} \right) \qquad (4.33)$$

In the same way we get:

$$\text{grad}\,\varphi_j = \frac{1}{l_k \sin^2\alpha_i}\left(\underline{e} - \cos\alpha_i\,\underline{f}\right) \tag{4.34}$$

We now have:

$$\int_\Delta \nabla\varphi_i \cdot \nabla\varphi_j \, d\xi\, d\eta = F\,\text{grad}\,\varphi_i \cdot \text{grad}\,\varphi_j$$

$$= \frac{1}{8F}\left(l_k^2 - l_i^2 - l_j^2\right) \tag{4.35}$$

$$\int_\Delta \nabla\varphi_i \cdot \nabla\varphi_i \, d\xi\, d\eta = F\,|\text{grad}\,\varphi_i|^2 = \frac{l_i^2}{4F} \tag{4.36}$$

From eq. 4.20 - eq. 4.24 we see that the matrices are formed from these element contributions by taking the summation over all triangles including point number i.

4.3 Solution of the differentialequation system.

From 4.19 and 4.14 we have the system of differential equations and initial condition. The $\underline{\underline{B}}$-matrices are dimensionless as can be seen from 4.35 and 4.36, but the $\underline{\underline{C}}$- and $\underline{\underline{D}}$ matrices have the dimension m^2, as can be seen from 4.30 and 4.31. We now consider the $\underline{\underline{C}}$- and $\underline{\underline{D}}$ matrices to be normalized with respect to the area A. We then get:

$$\underline{\underline{C}}_1 \frac{d\underline{h}}{dt} = -\frac{T_0}{AS_0}\underline{\underline{B}}_1 \underline{h} + \frac{1}{S_0}\underline{\underline{D}}\,\underline{R} + \frac{1}{AS_0}\underline{\mu} - \left(\underline{\underline{C}}_2 \frac{d\underline{g}}{dt} + \frac{T_0}{AS_0}\underline{\underline{B}}_2 \underline{g} + \frac{1}{AS_0}\underline{\Omega}\right)$$

$$\underline{h}(t_0) = \underline{f} \qquad (4.37)$$

We have now used the weak formulation to discretizise the differential equation 4.2 in the x and y coordinates, but the time is still continuous. A very commonly used method to solve the time problem is the Crank-Nicholson finite difference scheme which is centered at $(n+1/2)\Delta t$ and therefore achieves second-order accuracy in time.

$$\frac{\underline{\underline{C}}_1 + \frac{T_0 \Delta t}{AS_0}\underline{\underline{B}}_1}{2}\,\underline{h}^{n+1} = \frac{\underline{\underline{C}}_1 - \frac{T_0 \Delta t}{AS_0}\underline{\underline{B}}_1}{2}\,\underline{h}^{n} + \underline{\underline{D}}\,\frac{(\underline{R}^{n+1} + \underline{R}^{n})\Delta t}{2S_0}$$

$$+ \frac{(\underline{\mu}^{n+1} + \underline{\mu}^{n})\Delta t}{2AS_0} - \Big(\underline{\underline{C}}_2 (\underline{g}^{n+1} - \underline{g}^{n})$$

$$+ \frac{T_0 \Delta t}{AS_0}\underline{\underline{B}}_2 \frac{(\underline{g}^{n+1} + \underline{g}^{n})}{2} + \frac{\Delta t}{AS_0}\frac{(\underline{\Omega}^{n+1} + \underline{\Omega}^{n})}{2}\Big)$$

$$\underline{h}^{0} = \underline{f} \qquad (4.38)$$

As the matrices $\underline{\underline{C}}_1$ and $\underline{\underline{B}}_1$ are symmetric, positive definit, band matrices, the matrix on the left can be factored by Cholesky's square root method into LL^T, where L is a lower triangular matrix, and then $\underline{h}^{n+1}$ could be computed at each time step by two substitutions. Each time the input function changes the two substitutions have to be done again for each time step. The Cholesky's factorization would of course remain unchanged. In the following we will not use this finite difference scheme, but use instead mode superposition. For that purpose it seems practical to split 4.37 into two problems, a stationary part and a transient part. We have for the stationary part:

$$\underline{\underline{B}}_1 \underline{h}_o = \frac{A}{T_o} \underline{\underline{D}} \bar{\underline{R}} - \underline{\underline{B}}_2 \bar{\underline{q}} - \frac{1}{T_o} \bar{\underline{\Omega}} \tag{4.39}$$

where $\bar{\underline{R}}, \bar{\underline{q}}, \bar{\underline{\Omega}}$ mean long time average values. Long time average values of $\underline{\mu}$, pumping or recharge, are in most cases equal to zero.

For the transient part we get:

$$\underline{\underline{C}}_1 \frac{d\underline{h}_1}{dt} = -\frac{T_o}{AS_o} \underline{\underline{B}}_1 \underline{h}_1 + \frac{1}{S_o} \underline{\underline{D}} (\underline{R} - \bar{\underline{R}}) + \frac{1}{AS_o} \underline{\mu}$$

$$- \left(\underline{\underline{C}}_2 \frac{d(\underline{q} - \bar{\underline{q}})}{dt} + \frac{T_o}{AS_o} \underline{\underline{B}}_2 (\underline{q} - \bar{\underline{q}}) + \frac{1}{AS_o} (\underline{\Omega} - \bar{\underline{\Omega}}) \right)$$

$$\underline{h}_1(t_o) = \underline{f} - \underline{h}_o \tag{4.40}$$

where

$$\underline{h} = \underline{h}_o + \underline{h}_1 \tag{4.41}$$

Eq. 4.39 can be solved simply by Cholesky's factorization. Solution to 4.40 is given by:

$$\underline{h}_1(t) = e^{-(t-t_0)\underline{\underline{C}}_1^{-1}\underline{\underline{B}}_1 \frac{T_0}{AS_0}} (\underline{f} - h_0)$$

$$+ \frac{1}{S_0}\int_{t_0}^{t} e^{-(t-\tau)\underline{\underline{C}}_1^{-1}\underline{\underline{B}}_1 \frac{T_0}{AS_0}} \underline{\underline{C}}_1^{-1}\underline{\underline{D}}\,(\underline{R}-\bar{\underline{R}})\,d\tau$$

$$+ \frac{1}{AS_0}\int_{t_0}^{t} e^{-(t-\tau)\underline{\underline{C}}_1^{-1}\underline{\underline{B}}_1 \frac{T_0}{AS_0}} \underline{\underline{C}}_1^{-1}\underline{\mu}\,d\tau \tag{4.42}$$

$$-\int_{t_0}^{t} e^{-(t-\tau)\underline{\underline{C}}_1^{-1}\underline{\underline{B}}_1 \frac{T_0}{AS_0}} \underline{\underline{C}}_1^{-1}\left(\underline{\underline{C}}_2 \frac{d(\underline{g}-\bar{\underline{g}})}{dt} + \frac{T_0}{AS_0}\underline{\underline{B}}_2(\underline{g}-\bar{\underline{g}}) + \frac{1}{AS_0}(\underline{\Omega}-\bar{\underline{\Omega}})\right)$$

We now define the eigenvalue problem:

$$\underline{\underline{B}}_1\,\underline{\varphi} = \lambda\,\underline{\underline{C}}_1\,\underline{\varphi} \tag{4.43}$$

The eigenfunctions are orthonormalized in the following sense:

$$\underline{\underline{\varphi}}^T\,\underline{\underline{C}}_1\,\underline{\underline{\varphi}} = \underline{\underline{E}} \tag{4.44}$$

where $\underline{\underline{E}}$ is the identitymatrix.

We can now write the following as an eigenvector series expansion. By making use of 4.44, we get:

$$(\underline{f}-\underline{h}_o) = \sum_{n=1}^{N} \underline{\varphi}_n^T \underline{\underline{C}}_1 (\underline{f}-\underline{h}_o)\underline{\varphi}_n \tag{4.45}$$

$$\underline{\underline{C}}_1^{-1}\underline{\underline{D}}\,(\underline{R}-\bar{\underline{R}}) = \sum_{n=1}^{N} \underline{\varphi}_n^T \underline{\underline{D}}\,(R-\bar{\underline{R}})\,\underline{\varphi}_n \tag{4.46}$$

$$\underline{\underline{C}}_1^{-1}\underline{\mu} = \sum_{n=1}^{N} \underline{\varphi}_n^T \underline{\mu}\,\underline{\varphi}_n \tag{4.47}$$

$$\underline{\underline{C}}_1^{-1}\left(\underline{\underline{C}}_2 \frac{d(\underline{g}-\bar{\underline{g}})}{dt} + \frac{T_o}{AS_o}\underline{\underline{B}}_2(\underline{g}-\bar{\underline{g}}) + \frac{1}{AS_o}(\underline{\Omega}-\bar{\underline{\Omega}})\right)$$
$$= \sum_{n=1}^{N} \underline{\varphi}_n^T\left(\underline{\underline{C}}_2 \frac{d(\underline{g}-\bar{\underline{g}})}{dt} + \frac{T_o}{AS_o}\underline{\underline{B}}_2(\underline{g}-\bar{\underline{g}}) + \frac{1}{AS_o}(\Omega-\bar{\underline{\Omega}})\right)\varphi_n \tag{4.48}$$

$\underline{\varphi}_n^T$ means $\underline{\varphi}_n$ transposed.

We now introduce the timeconstants for the linear reservoirs.

$$K_n = \frac{A\,S_o}{\lambda_n T_o} \tag{4.49}$$

By inserting all these equations into 4.42, we get:

$$\underline{h}_1(t) = \sum_{n=1}^{N} \underline{\varphi}_n \Big\{ e^{-\frac{(t-t_0)}{K_n}} \underline{\varphi}_n^T \underline{\underline{C}}_1 (\underline{f} - h_0)$$

$$+ \frac{1}{S_0} \int_{t_0}^{t} e^{-\frac{(t-\tau)}{K_n}} \underline{\varphi}_n^T \underline{\underline{D}} (\underline{R} - \underline{\bar{R}})\, d\tau \qquad (4.50)$$

$$+ \frac{1}{AS_0} \int_{t_0}^{t} e^{-\frac{(t-\tau)}{K_n}} \underline{\varphi}_n^T \underline{\mu}\, d\tau$$

$$- \int_{t_0}^{t} e^{-\frac{(t-\tau)}{K_n}} \varphi_n^T \Big(\underline{\underline{C}}_2 \frac{d(\underline{g} - \underline{\bar{g}})}{dt} + \frac{T_0}{AS_0} \underline{\underline{B}}_2 (\underline{g} - \underline{\bar{g}}) + \frac{1}{AS_0} (\underline{\Omega} - \underline{\bar{\Omega}}) \Big) d\tau \Big\}$$

Now using partial integration on $\frac{d(\underline{g} - \underline{\bar{g}})}{dt}$ we get:

$$h_1(t) = \sum_{n=1}^{N} \varphi_n \Big\{ e^{-\frac{(t-t_0)}{K_n}} \underline{\varphi}_n^T \underline{\underline{C}}_1 (\underline{f} - \underline{h}_0)$$

$$+ \frac{1}{S_0} \int_{t_0}^{t} e^{-\frac{(t-\tau)}{K_n}} \underline{\varphi}_n^T \underline{\underline{D}} (\underline{R} - \underline{\bar{R}})\, d\tau \qquad (4.51)$$

$$+ \frac{1}{AS_0} \int_{t_0}^{t} e^{-\frac{(t-\tau)}{K_n}} \underline{\varphi}_n^T \underline{\mu}\, d\tau$$

$$- \underline{\varphi}_n^T \underline{\underline{C}}_2 \Big((\underline{g}(t) - \underline{\bar{g}}) - (\underline{g}(t_0) - \underline{\bar{g}}) e^{-\frac{(t-t_0)}{K_n}} \Big)$$

$$- \int_{t_0}^{t} e^{-\frac{(t-\tau)}{K_n}} \underline{\varphi}_n^T \Big(\frac{1}{K_n} \big(\frac{1}{\lambda_n} \underline{\underline{B}}_2 - \underline{\underline{C}}_2 \big) (\underline{g} - \underline{\bar{g}}) + \frac{1}{AS_0} (\underline{\Omega} - \underline{\bar{\Omega}}) \Big) d\tau \Big\}$$

If we now take $t_o = -\infty$ and assume that the boundary conditions are independent of time we get:

$$\underline{h}_1(t) = \sum_{n=1}^{N} \underline{\varphi}_n \left\{ \frac{1}{S_o} \int_0^{\infty} \underline{\varphi}_n^T \left(\underline{\underline{D}} \left(\underline{R}(t-\tau) - \bar{\underline{R}} \right) + \frac{1}{A} \mu(t-\tau) \right) e^{-\frac{\tau}{K_n}} d\tau \right\} \tag{4.52}$$

This is a simple convolution integral. We see that if the input function changes we just have to perform the convolution calculations. In practice the infiltration series and the pumping or/and recharge series are given as discrete time series. In that case eq. 4.52 can be made much simpler. Let us look at the convolution integral of some discrete time series, I(t).

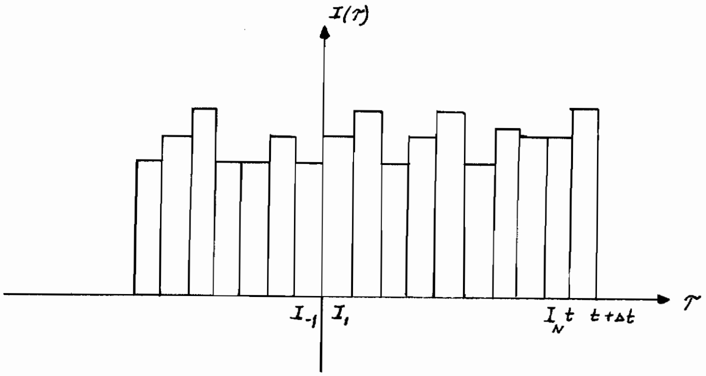

fig. 4.5

We have:

$$CONV(t) = \int_0^{\infty} I(t-\tau) e^{-\tau/K} d\tau$$

$$= \sum_{i=0}^{\infty} \int_{i\Delta t}^{(i+1)\Delta t} I(t-\tau) e^{-\tau/K} d\tau$$

$$\simeq \sum_{i=0}^{\infty} I_{N-i} \int_{i\Delta t}^{(i+1)\Delta t} e^{-\tau/K}\, d\tau$$

$$= K(1-e^{-\Delta t/K}) \sum_{i=0}^{\infty} I_{N-i}\, e^{-i\Delta t/K}$$

$$= K(1-e^{-\Delta t/K}) \left\{ \sum_{i=0}^{N-1} I_{N-i}\, e^{-i\Delta t/K} + \sum_{i=N}^{\infty} I_{N-i}\, e^{-i\Delta t/K} \right\}$$

$$= CONV_N + K(1-e^{-\Delta t/K}) \sum_{i=N}^{\infty} I_{N-i}\, e^{-i\Delta t/K}$$

$$= CONV_N + \mathcal{E}_N$$

$$\mathcal{E}_N \leq maxI \cdot K \cdot e^{-N\Delta t/K} \tag{4.53}$$

We see that $\mathcal{E}_N \to 0$ when $N \to \infty$
Therefore, if $N\Delta t/K >> 1$ we approximate:

$$CONV(t) \simeq CONV_N = K(1-e^{-\Delta t/K}) \sum_{i=0}^{N-1} I_{N-i}\, e^{-i\Delta t/K} \tag{4.54}$$

We now proceed to calculate:

$$CONV(t+\Delta t) = \int_{-\infty}^{t+\Delta t} I(\tau)\, e^{-(t+\Delta t-\tau)/K}\, d\tau$$

$$= \int_{-\infty}^{t} I(\tau)\, e^{-(t-\tau)/K}\, e^{-\Delta t/K}\, d\tau$$

$$+ \int_t^{t+\Delta t} I(\tau) e^{-(t-\tau)/K} e^{-\Delta t/K} d\tau$$

$$= e^{-\Delta t/K} CONV(t) + K(1-e^{-\Delta t/K}) I(t+\Delta t)$$

That is:

$$CONV_{N+1} = e^{-\Delta t/K} CONV_N + K(1-e^{-\Delta t/K}) I_{N+1} \quad (4.55)$$

Eq. 4.52 can now be written as:

$$\underline{h}_{1,n}^{(k)} = \frac{K_n}{S_o} \underline{\varphi}_n (1-e^{-\Delta t/K_n}) \sum_{i=0}^{k-1} \underline{\varphi}_n^T (\underline{\underline{D}} (\underline{R}_{k-i} - \underline{\bar{R}}) + \frac{1}{A} \underline{\mu}_{k-i}) e^{-i\Delta t/K_n}$$

$$\underline{h}_{1,n}^{(k+1)} = e^{-\Delta t/K_n} \underline{h}_{1,n}^{(k)} + \frac{K_n}{S_o} \underline{\varphi}_n (1-e^{-\Delta t/K_n}) (\underline{\varphi}_n^T (\underline{\underline{D}} (\underline{R}_{k+1} - \underline{\bar{R}}) + \frac{1}{A} \underline{\mu}_{k+1}))$$

(4.56)

$$\underline{h}_1^{(k)} = \sum_{n=1}^{N} \underline{h}_{1,n}^{(k)}$$

$$\underline{h}^{(k)} = \underline{h}_o + \underline{h}_1^{(k)}$$

From 4.56 we see that in practical calculations it could be enough to use just the few first eigenfunctions in the expansion. We thus introduce some truncation error. We now see the reason for why we splitted eq. 4.37 into two problems. Eq. 4.39 for the stationary problem can be solved without any truncation error. Acceptable truncation error in the transient term could give too large errors in the stationary part, if it had been solved by the eigenfunction expansion too.

If the input function is not very rich in the higher harmonics, that is to say does not fluctuate very much over the area, one can be satisfied with a very limited number of eigenfunctions.

Stability and convergence problems will not be treated here, but the reader is referred to standard books on the subject. See f.ex. /78/ and /26/.

4.4 Solution of the eigenproblem.

According to eq. 4.43 we must solve the generalized eigenvalue problem:

$$\underline{\underline{B}}\,\underline{\varphi} = \lambda\,\underline{\underline{C}}\,\underline{\varphi} \tag{4.57}$$

where we have dropped the subscript 1 on the matrices.

We use the subspace iteration solution. See /78/ and /2/. We want to determine the p lowest eigenvalues and corresponding eigenvectors. The iteration can be interpreted as a repeated application of the Ritz method, /21/, in which the computed eigenvectors from one step are used as the trial basis vectors for the next iteration until convergence to the required p eigenvalues and eigenvectors is obtained.

The solution is carried out by iterating simultaneously with q linearly independent vectors, where q>p. In the k'th iteration the vectors span the q-dimensional subspace $\mathcal{E}_k$ and 'best' eigenvalue and eigenvector approximations are calculated, i.e. when the vectors span the p-dimensional least dominant subspace, the required eigenvalues and eigenvectors are obtained.

Let $\underline{\underline{V}}_o$ be the starting vectors of the iteration, then the k'th iteration is described as follows:
Solve for vectors $\underline{\underline{\bar{V}}}_k$ which span $\mathcal{E}_k$.

$$\underline{\underline{B}}\,\underline{\underline{\bar{V}}}_k = \underline{\underline{C}}\,\underline{\underline{V}}_{k-1} \tag{4.58}$$

Calculate the projections of $\underline{\underline{B}}$ and $\underline{\underline{C}}$ onto $\mathcal{E}_k$:

$$\underline{\underline{B}}_k = \underline{\underline{\bar{V}}}_k^T\,\underline{\underline{B}}\,\underline{\underline{\bar{V}}}_k \tag{4.59}$$

$$\underline{\underline{C}}_k = \underline{\underline{\bar{V}}}_k^T\,\underline{\underline{C}}\,\underline{\underline{\bar{V}}}_k \tag{4.60}$$

Solve the eigenvalue problem:

$$\underline{\underline{B}}_k \, \underline{\underline{Q}}_k = \lambda_k \, \underline{\underline{C}}_k \, \underline{\underline{Q}}_k \tag{4.61}$$

Calculate the k'th improved approximation to the eigenvectors:

$$\underline{\underline{V}}_k = \underline{\underline{\bar{V}}}_k \, \underline{\underline{Q}}_k \tag{4.62}$$

Provided that the starting subspace is not orthogonal to any of the required eigenvectors, the iteration converges to the desired result, i.e. $\lambda_k \rightarrow \lambda$ and $V_k \rightarrow \varphi$ as $k \rightarrow \infty$

The number of vectors q used in the iteration is taken greater than the desired number of eigenvectors in order to accelerate the convergence of the process. Bathe and others, /2/, use q = min (2p, p + 8), which has proven to be effective in general applications. The number of iterations required to achieve satisfactory convergence depends, of course, on the quality of the starting vectors V_o. In many cases a good estimate of the required eigensystem is known beforehand, also in case the number of eigenvalues and vectors required is increased, the already calculated eigenvectors can be specified as part of the starting iteration vectors.

As convergence criterium we have used

$$\frac{|\lambda_p(k+1) - \lambda_p(k)|}{\lambda_p(k)} < \varepsilon \tag{4.63}$$

As the $\underline{\underline{B}}$-matrix is positive definite eq. 4.58 can be solved by the Cholesky's method. The Cholesky's factorization need only be solved once and for all and not repeated in each iteration step. The small eigenvalue problem 4.61 is solved by Cholesky factorizing the $\underline{\underline{C}}_k$ matrix into $\underline{\underline{L}}_k \underline{\underline{L}}_k^T$, where $\underline{\underline{L}}_k$ is the lower triangular matrix, and then solving the eigenvalue problem:

$$(\underline{\underline{L}}_k^{-1} \underline{\underline{B}}_k \underline{\underline{L}}_k^{-T})(\underline{\underline{L}}_k^T \underline{\underline{Q}}_k) = \lambda_k (\underline{\underline{L}}_k^T Q_k) \tag{4.64}$$

The matrix $(\underline{\underline{L}}_k^{-1} \underline{\underline{B}}_k \underline{\underline{L}}_k^{-T})$ is then transformed to a tridiagonal form by Householder's transformation. The eigenvalues are then found by QR iteration. The eigenvector corresponding to a given eigenvalue is then found by applying inverse iteration to the tridiagonal matrix using the given eigenvalue as a shift. The eigenvectors are then transformed back to the original matrix $\underline{\underline{L}}_k^{-1} \underline{\underline{B}}_k \underline{\underline{L}}_k^{-T}$. Finally we perform the transformation $\underline{\underline{L}}_k^{-T}$ on the eigenvectors in order to find the eigenvectors of 4.61.

The eigenvectors are orthonormalized according to 4.44. If we normalize the eigenvectors in 4.64 in the following way:

$$\underline{\underline{Q}}_k^T \underline{\underline{L}}_k \underline{\underline{L}}_k^T \underline{\underline{Q}}_k = \underline{\underline{Q}}_k^T \underline{\underline{C}}_k \underline{\underline{Q}}_k = \underline{\underline{E}} \tag{4.65}$$

we get from 4.60 and 4.62:

$$\underline{\underline{Q}}_k^T \bar{\underline{\underline{V}}}_k^T \underline{\underline{C}} \bar{\underline{\underline{V}}}_k \underline{\underline{Q}}_k = \underline{\underline{V}}_k^T \underline{\underline{C}} \underline{\underline{V}}_k = \underline{\underline{E}} \tag{4.66}$$

That is to say if we normalize the eigenvectors of the small eigenproblem in 4.61 according to 4.65 then the eigenvectors of the original problem will be normalized according to 4.44 in each iteration step.

In practical calculations we would of course use the band structure of the matrices B and C to save memory space in a digital computer.

4.5 Boundary outflow.

In many cases it is interesting to know the flow through the boundary. We will now proceed to calculate it.

Let $\partial\Omega_S$ be any part of the boundary. The flow through $\partial\Omega_S$ is given by:

$$q_S = -\int_{S_1}^{S_2} T\,\frac{\partial h}{\partial n}\,ds = -\int_{S_1}^{S_2} T\,\mathrm{grad}h\cdot\underline{n}\;ds \qquad (4.67)$$

Remembering that T and gradh are constants inside each triangle, we get:

$$q_S = -\sum_{\Delta\cap\partial\Omega_S} T_k\,(\mathrm{grad}h\cdot\bar{n})_k\,l_k = \sum_{\Delta\cap\partial\Omega_S} q_{l_k} \qquad (4.68)$$

We now must determine q_{l_k}. We have:

$$q_{l_k} = -T_k\,(\mathrm{grad}h\cdot\underline{n})_k\,l_k \qquad (4.69)$$

We take l_k to be a part of the boundary $\partial\Omega_S$. See fig.4.6.

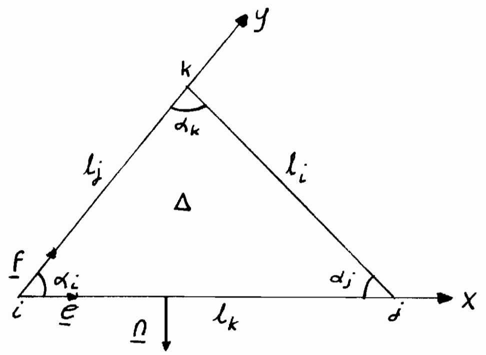

fig. 4.6

We have:

$$h = ax + by + c$$

where:

$$a = \frac{h_j - h_i}{l_k} \tag{4.70}$$

$$b = \frac{h_k - h_i}{l_j}$$

$$c = h_i$$

gradh is given by:

$$\text{grad}h = \frac{1}{\sin^2\alpha_i}\Big((a - b\cos\alpha_i)\underline{e} + (b - a\cos\alpha_i)\underline{f} \Big) \tag{4.71}$$

We now get from 4.69 and 4.71:

$$q_{l_k} = T_k l_k \frac{l_j l_k}{2F_k} (b - a\cos\alpha_i) \tag{4.72}$$

Now using 4.70 we get:

$$q_{l_k} = \frac{T_k}{2F_k}\Big(l_k^2 h_k + \frac{1}{2}(l_j^2 - l_i^2 - l_k^2) h_i + \frac{1}{2}(l_i^2 - l_j^2 - l_k^2) h_j \Big) \tag{4.73}$$

where F_k is the area of the triangle.

4.6 Model calibration.

Calibrating the model consists of finding the S and T values in the model in order to fit the model to data. Current methods of calibrating groundwaterflow models are either indirect or direct. The indirect approach is essentially a trial and error procedure that seeks to improve an existing estimate of the parameters until the model response is 'sufficiently close' to that of the real system. Direct methods consist either of using some iterative methods to improve the estimate of the parameters or to solve the inverse problem. Let us begin with to use subjective information. Kisiel, Duckstein and Lovell have discussed the use of such information, see /50/. To start with we use geological information to tell us where the same values of S and T can be used, and thus determining the number of unknown constants. Now the calibration procedure can start. It can contain some physical constraints on the unknown parameters, where these constraints again are determined from subjective information. The most commonly used procedure is the method of trial and error. Let us define:

H_t : measured value of groundwater level

h_t : theoretical value of groundwater level

We now must construct some norms to measure the difference between H_t and h_t. Using the following norm would of course satisfy all our requirements.

$$\| H_t - h_t \| = \sup_{t,x,y} | H_t - h_t | \tag{4.74}$$

It demands a great deal of calculation and in many cases we would be satisfied with a less demanding norm. We define:

$$\| H_t - h_t \|^2 = \frac{1}{AT} \int_0^T \int_\Omega (H_t - h_t)^2 \, dx\, dy\, dt \qquad (4.75)$$

and

$$\| H_t - h_t \| = \sup_t \left(\frac{1}{A} \int_\Omega (H_t - h_t)^2 \, dx\, dy \right)^{1/2} \qquad (4.76)$$

where T is the total time length.

If we are not interested in knowing the groundwater level very well in each point but just on the average over the area, this norm would be very useful. If we were f.ex. just interested in calculating the boundary outflow, this norm would be quite satisfactory.

As we see, it involves integration over the area and therefore demands that we know H_t in enough number of points in order to justify interpolation of the H_t-values in the area. If the number of observation points in the area does not justify interpolation we are forced to use some 'degenerate' form of norms. The degenerate form of the norm in 4.74 is defined as:

$$\| H_t - h_t \| = \sup_{t \in T^1, (x,y) \in \Omega^1} | H_t - h_t | \qquad (4.77)$$

where:

T^1 : the set of times we have measured H_t

Ω^1 : the set of points we have measured H_t in

The 'degenerate' form of the norms in 4.75 and 4.76 are defined as:

$$\| H_t - h_t \|^2 = \sum_{t \in T'} \sum_{(x,y) \in \Omega'} (H_t - h_t)^2 / N_x N_t \tag{4.78}$$

$$\| H_t - h_t \| = \sup_{t \in T'} \left(\sum_{(x,y) \in \Omega'} (H_t - h_t)^2 / N_x \right)^{1/2} \tag{4.79}$$

Where N_t and N_x are the number of time intervals and observation points respectively.

The two 'degenerate' norms:

$$\| H_t - h_t \| = \sum_{t \in T'} \sum_{(x,y) \in \Omega'} | H_t - h_t | / N_x N_t \tag{4.80}$$

and

$$\| H_t - h_t \| = \sup_{t \in T'} \sum_{(x,y) \in \Omega'} | H_t - h_t | / N_x \tag{4.81}$$

are defined from the norms respectively:

$$\| H_t - h_t \| = \frac{1}{AT} \int_0^T \int_\Omega | H_t - h_t | \, dx \, dy \, dt \tag{4.82}$$

and

$$\| H_t - h_t \| = \sup_{t \in T'} \left(\frac{1}{A} \int_\Omega | H_t - h_t | \, dx \, dy \right) \tag{4.83}$$

We can now make an initial guess of the unknown's and then proceed to calculate h_t. We can then calculate $||H_t-h_t||$, and if we are not satisfied with the result we can correct our initial guess until we find $\min_{S,T} ||H_t - h_t||$ which is satisfactory.

4.7 The inverse problem.

In the following we will give a brief description of the inverse problem. We will not try to solve the problem, but show recent research and ideas for further development.

From eq. 4.39 and 4.40 we see that it would be very practical to split our problem into a stationary part and a transient part in the estimation procedure. The stationary level would then be used to estimate the unknown transmissivity values or/and the mean infiltration and then the transient level to estimate the storage coefficients or/and the variations in the infiltration.

The discrete matrix equation 4.39 is equivalent to the differential equation:

$$\nabla \cdot (T \cdot \nabla h_o) = -\bar{R} \qquad (4.84)$$

with the appropriate boundary conditions. Eq. 4.84 is a second order partial differential equation when h_o is considered unknown and T is given, but a first order partial differential equation when T is unknown and h_o is given. The latter case is the formulation of the inverse problem. It can be shown that it has an unique solution, if the boundary conditions for T are given on a line Γ cutting every streamline of the flow in Ω or when a value of T is known on every streamline of the region Ω. See fig. 4.7.

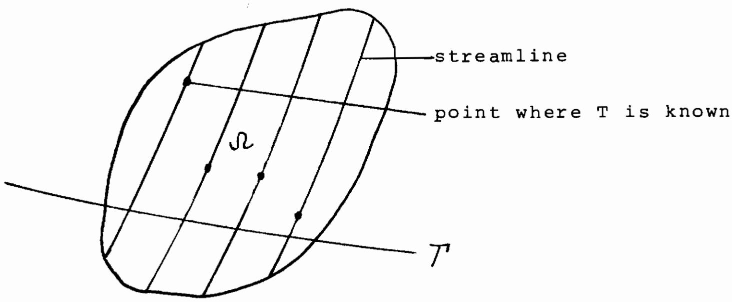

fig. 4.7

In practice it is very difficult to find such boundary values for T, therefore a different approach to the inverse problem is needed. Instead of stating the absolute boundary value of T, it would be much easier to formulate a kind of periodic boundary conditions. That is we use geological information to tell us where the same value of T can be assigned. We see later that in case that $\bar{R} \neq 0$, these conditions are sufficient, but in case that $\bar{R} = 0$ we just get the relative transmissivity values and then we need to know it in one subarea to calculate the absolute values.

Let us write down eq. 4.39 the discrete equation corresponding to eq. 4.84:

$$\underline{\underline{B}}_1 \underline{h}_0 + \underline{\underline{B}}_2 \underline{q} = \underline{b} \tag{4.85}$$

where we have defined $\underline{b}$ as:

$$\underline{b} = \frac{A}{T_0} \underline{\underline{D}} \underline{R} - \frac{1}{T_0} \underline{\Omega} \tag{4.85b}$$

As the matrices $\underline{\underline{B}}_1$ and $\underline{\underline{B}}_2$ are linear functions of T eq. 4.85 can also be written as:

$$\left(\underline{\underline{B}}_1 \underline{h}_0 + \underline{\underline{B}}_2 \bar{\underline{q}} \right) = \underline{\underline{A}}_1 \underline{T} = \underline{b} \tag{4.86}$$

where $\underline{\underline{A}}_1$ is a (N, K) matrix and $\underline{T}$ is a (K,1) vector. We assume that we have used the periodic boundary conditions for T to make K < N. Eq. 4.86 is the discrete formulation of the inverse problem by using periodic boundary conditions for T. Let us first suppose that the transmissivity is unknown, but the groundwater level and the $\underline{b}$-vector (the infiltration and the boundary inflow) are known, but may contain some error. If we assume that the error in $\underline{b}$ is much greater than the error in $\underline{h}_0$ we get a linear estimation procedure. Neuman, /58/, uses this approach and linear programming to solve:

$$\min_{T} \left| \underline{\underline{W}} \left(\underline{\underline{A}}_1 \underline{T} - \underline{b} \right) \right| \tag{4.87}$$

with several subjective constraints on the T-values. According to Neuman, $\underline{\underline{W}}$ is some weighting matrix. We choose to formulate it as a linear least square procedure:

$$\min_{T} \| \Delta \underline{b} \|_{\underline{\underline{W}}_b} = \min_{T} \| \underline{\underline{W}}_b^{1/2} (\underline{\underline{A}}_1 T - \underline{b}) \| \tag{4.88}$$

$$\Delta \underline{b} = \underline{\underline{A}}_1 \underline{T} - \underline{b} \tag{4.89}$$

where $\| \Delta b \|_{\underline{\underline{W}}_b}$ means the norm $\Delta \underline{b}^T \underline{\underline{W}}_b \, \Delta \underline{b}$.
$\underline{\underline{W}}_b$ is some weighting matrix. It can be shown that if the errors in $\underline{b}$ are normally distributed and the weighting matrix is chosen as :

$$\underline{\underline{W}}_b = \underline{\underline{COV(\underline{b})}}^{-1} \tag{4.90}$$

$\underline{\underline{COV(\underline{b})}}$ is the covariance matrix of $\underline{b}$.

We then get a maximum likelihood estimate for $\underline{T}$. $\underline{\underline{W}}_b^{1/2}$ is defined in the following way:

$$\left(\underline{\underline{W}}_b^{1/2} \right)^T \left(\underline{\underline{W}}_b^{1/2} \right) = \underline{\underline{W}}_b \tag{4.91}$$

The solution to this minimization problem is given by:

$$\hat{\underline{T}} = \left(\underline{\underline{A}}_1^T \underline{\underline{W}}_b \underline{\underline{A}}_1 \right)^{-1} \left(\underline{\underline{A}}_1^T \underline{\underline{W}}_b \underline{b} \right) \tag{4.92}$$

where $\hat{\underline{T}}$ is used to denote the estimate for T.

If, on the other hand, the error in $\underline{h}_o$ were much greater, we would get a nonlinear estimation procedure. If we use the least square method as before, we get:

$$\min_{T} \| \underline{\Delta h_o} \|_{\underline{\underline{W}}_{h_o}} \tag{4.93}$$

$$\underline{\Delta h_o} = \underline{\underline{B}}_1^{-1} \underline{b} - \underline{\underline{B}}_1^{-1} \underline{\underline{B}}_2 \underline{q} - \underline{h}_o \tag{4.94}$$

This is a nonlinear least square problem with the weighting matrix defined as:

$$\underline{\underline{W}}_{h_o} = \underline{\underline{COV(\underline{h}_o)}}^{-1} \tag{4.95}$$

It could be solved f.ex. by the Gauss-Newton iteration method. We have:

$$\underline{\underline{W}}_{h_o}^{1/2} \underline{\underline{J}}^{(i)} (\underline{T}^{(i+1)} - \underline{T}^{(i)}) = \underline{\underline{W}}_{h_o}^{1/2} \Delta \underline{h}_o^{(i)} \tag{4.96}$$

where $\underline{\underline{J}}$ is defined as:

$$\underline{\underline{J}}^{(i)} = \frac{\partial \underline{\Delta h_o}}{\partial \underline{T}} = \begin{bmatrix} \frac{\partial \underline{\Delta h_{o1}}}{\partial T_1} & \cdots & \frac{\partial \underline{\Delta h_{o1}}}{\partial T_k} \\ \vdots & & \vdots \\ \frac{\partial \underline{\Delta h_{oN}}}{\partial T_1} & \cdots & \frac{\partial \underline{\Delta h_{oN}}}{\partial T_k} \end{bmatrix} \tag{4.97}$$

As a startvector $\underline{T}_o$ we could use the solution to the linear least square problem in eq. 4.92. If the errors in $\underline{h}_o$ are normally distributed and the weighting matrix is chosen to be the inverse of the covariance matrix of $\underline{h}_o$, we would get maximum likelihood estimate of T. If some of the eigenvalues of $\underline{\underline{J}}^T \underline{\underline{W}}_{h_o} \underline{\underline{J}}$ are very small, then our iteration procedure might have difficulties in converging. Then it could be very wise to use the method of ridge regression, see /43/. It consists of solving instead of 4.96 the following problem:

$$\begin{bmatrix} \underline{\underline{W}}_{h_o}^{1/2} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{W}}_T^{1/2} \end{bmatrix} \begin{bmatrix} \underline{\underline{J}}^{(i)} \\ \lambda^{(i)} \underline{\underline{I}} \end{bmatrix} \underline{\Delta T} = \begin{bmatrix} \underline{\underline{W}}_{h_o}^{1/2} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{W}}_T^{1/2} \end{bmatrix} \begin{bmatrix} \underline{\Delta h}_o^{(i)} \\ \underline{0} \end{bmatrix} \tag{4.98}$$

where $\underline{\underline{I}}$ is the identity matrix and $\underline{\underline{W}}_T$ is a weighting matrix to keep the relative size of the eigenvalues of $\underline{J}^T\underline{J}$ equal. λ_i is a parameter having large values in the first iterations but must be made smaller and smaller and is equal to zero in the last few iterations.

We have now considered two cases, first where the error in $\underline{b}$ dominated, and then where the error in $\underline{h}_o$ dominated. If the errors in $\underline{b}$ and $\underline{h}_o$ have the same order of magnitude, we have to consider some weighted average between the two procedures. We will not go further into that problem here, but just point out the need for further research in that area.

We will at last give examples of two cases. Let $\underline{b} \equiv \underline{0}$, then of course the error in $\underline{b}$ is equal to zero, and we must use 4.93 as our procedure. It is easy to see that we are just able to determine the relative values of $\underline{T}$. In order to determine their absolute value, we must know T in at least one subarea. This is very natural, because if we put $\bar{R} = 0$ in the differential equation 4.84, we can multiply through with any reference value, T_o, without changing the result. The solution h_o is just a function of the relative transmissivity values, but not of their absolute values.

If the $\underline{b}$-vector were unknown then we would have to use 4.93 and the minimization procedure would have to be taken with respect to $\underline{b}$ also. That is:

$$\min_{\underline{T},\underline{b}} \| \Delta \underline{h}_o \|_{\underline{\underline{W}}_{h_o}} \tag{4.99}$$

We must remember that the $\underline{\underline{J}}$-matrix in 4.97 must be determined now with respect to the $\underline{b}$'s too. This procedure gives also just the relative values. In order to determine the absolute values we have to know the $\underline{T}$'s or the $\underline{b}$'s at least in one subarea.

In connection with the minimization procedures a sensitivity analysis plays an important role. We will finally show that singular value decomposition gives us the important tool, when solving a linear least square problem. The same arguments hold for nonlinear least square problems, if we consider the neighbourhood of the minimum value, because there our problem is linear. Let us take 4.88 as an example. We can write $\underline{\underline{W}}_b^{1/2} \underline{\underline{A}}_1$ as:

$$\underline{\underline{W}}_b^{1/2} \underline{\underline{A}}_1 = \underline{\underline{U}}\, \underline{\underline{\Lambda}}\, \underline{\underline{V}}^T \tag{4.100}$$

where $\underline{\underline{V}}$ is a (K,K) orthogonal matrix, $\underline{\underline{U}}$ is a (N,K) orthogonal matrix and $\underline{\underline{\Lambda}}$ is a (K,K) diagonal matrix with non-negative elements (the singular values of the matrix $\underline{\underline{W}}_b^{1/2}\underline{\underline{A}}_1$).

If none of the elements of $\underline{\underline{\Lambda}}$ is equal to zero we get for the estimate of $\underline{T}$.

$$\hat{\underline{T}} = \underline{\underline{V}}\, \underline{\underline{\Lambda}}^{-1} \underline{\underline{U}}^T \underline{\underline{W}}_b^{1/2} \underline{b} \tag{4.101}$$

and we have for $\underline{\underline{V}}$ and $\underline{\underline{\Lambda}}$:

$$\left(\underline{\underline{A}}_1^T \underline{\underline{W}}_b \underline{\underline{A}}_1\right) \underline{\underline{V}} = \underline{\underline{V}}\, \underline{\underline{\Lambda}}^2 \tag{4.102}$$

and

$$\left(\underline{\underline{W}}_b^{1/2} \underline{\underline{A}}_1 \underline{\underline{A}}_1^T (\underline{\underline{W}}_b^{1/2})^T \right) \underline{\underline{U}} = \underline{\underline{U}} \underline{\underline{\Lambda}}^2 \qquad (4.103)$$

That is to say $\underline{\underline{V}}$ is the eigenvector matrix and $\underline{\underline{\Lambda}}^2$ are the eigenvalues of the matrix $\underline{\underline{A}}_1^T \underline{\underline{W}}_b \underline{\underline{A}}_1$, and $\underline{\underline{U}}$ is the eigenvector matrix of the matrix $\underline{\underline{W}}_b^{1/2} \underline{\underline{A}}_1 \underline{\underline{A}}_1^T (W_b^{1/2})^T$.

From 4.101 we get:

$$\underline{\underline{V}}^T \underline{\hat{T}} = \underline{\underline{\Lambda}}^{-1} \underline{\underline{U}}^T \underline{\underline{W}}_b^{1/2} \underline{b} \qquad (4.104)$$

From eq. 4.104 we see that the linear combinations of $\hat{T}$ values, which are not very sensitive to variations in the $\underline{b}$-vector, correspond to the largest eigenvalues of $\underline{\underline{A}}_1^T \underline{\underline{W}}_b \underline{\underline{A}}_1$, and the linear combinations of $\hat{T}$-values, which correspond to the smallest eigenvalues, are most sensitive to variation in the $\underline{b}$-vector. Eq. 4.104 also shows, which linear combination of $\underline{b}$-values affect most certain linear combinations of $\hat{T}$-values.

We have just discussed the estimation of the transmissivity values or/and the mean of the infiltration and the boundary inflow. There remains to estimate the storage coefficient or/and the variations with time in the infiltration and boundary inflow. For that purpose we must use eq. 4.40, the transient part of our equations. This problem is much more complex and will not be discussed here, but research in that area is a natural continuation of this thesis.

In many cases we just have a short time series for the groundwater level. Then it might be impossible to estimate long time average values of the groundwater level. This makes it impossible to split our problem into two problems. In this case we would have to solve the estimation problem simultaneously for the storage coefficients, the transmissivity or/and the infiltration and the boundary inflow. To solve the two last mentioned problems, the discrete time equation, 4.38, seems most suitable for that purpose. But further development of estimation methods to solve these problems will not be given in this thesis.

For many practical purposes great simplifications seem to be justified, such as assuming homogeneous aquifer, constant infiltration over the area and constant boundary inflow with time. In that case much simpler estimation procedures can be derived, such as those given by 3.99, 3.100, 3.101 and 3.102.

Constraints on the estimated parameters can be reinforced in all the estimation procedures. Physically we know f.ex. that $\underline{T} > 0$. However, it would be wise to solve the problem first without these constraints and look at the results. If f.ex. some of the $\underline{T}$-values become negative, it might be possible that the weighting matrix had been chosen incorrectly. If not, the problem could be solved again with the constraints. If we had solved the problem with the constraints in the first case, we might not have discovered the contradiction in our assumptions.

4.8 Examples.

Let us look at the same example as in chapter 3.6. We take Ω as a rectangle and the source term to be given in the form of constant pumping. To begin with we take S and T to be constant values over the area.

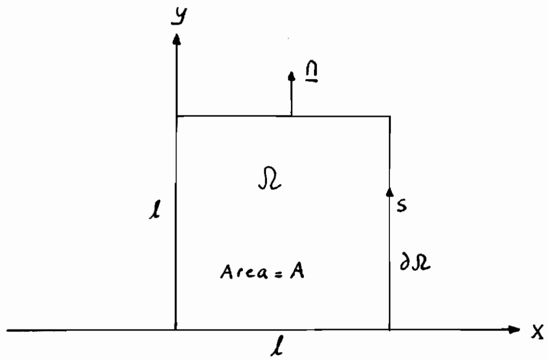

fig. 4.8

From 4.56 we have the approximate solution :

$$\underline{h}^{(k)} = \underline{h}_o - \frac{Q}{T} \sum_{n=1}^{N} \frac{\underline{\varphi}_n}{\lambda_n} \underline{\varphi}_n^T \underline{I} \left(1 - \left(e^{-\Delta t/k_n}\right)^k\right) \qquad (4.105)$$

where we have defined:

$$\underline{I} = \begin{Bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{Bmatrix} \qquad (4.106)$$

with zero everywhere, except in the point where the pumping takes place.

We now insert the same numerical values as before. That is to say: $(x,y) = (\frac{\ell}{4}, \frac{\ell}{2})$, $(\xi, \eta) = (\frac{\ell}{2}, \frac{\ell}{2})$, $A = 10^8\ m^2$, $\ell = 10^4\ m$, $T = 0{,}2\ m^2/s$, $S = 0{,}06$, $Q = 1{,}0\ m^3/s$, $\Delta t = 2$ weeks

Eq. 4.105 is plotted in fig. 4.9 for 9, 25 and 49 internal points together with the theoretical solution and Theis' solution. There are two scales on the axis. On the vertical axis we have the actual drawdown $s = (h-h_o)$ and the dimensionless drawdown $s^1 = \frac{TS}{Q}$. On the horizontal axis we have the actual time in weeks and the dimensionless time $t^1 = \frac{Tt}{AS}$. Then the diagram can be used for different values of Q, T, S, and A by using the dimensionless scale on the axis.

If just the stationary part of the solution is of interest and not the whole time variation, then the eigenfunctions and the convolution integrals need not to be evaluated. From eq. 4.37 we would get:

$$\underline{\underline{B}}_1\, \underline{h}_s = \frac{A}{T_o}\, \underline{\underline{D}}\, \underline{R} + \frac{1}{T_o}\, \underline{\mu} - \underline{\underline{B}}_2\, \underline{q} \tag{4.107}$$

which can be solved simply by Cholesky's factorization. If it is also of interest to know how long time it takes to reach the stationary level, then the eigenvalues must be evaluated, but there is no need to calculate the time variations. From eq. 4.105 and the theoretical solution given by:

$$h_1 = -\frac{AQ}{T} \sum_{n=1}^{\infty} \frac{\varphi_n(x,y)}{\lambda_n}\, \varphi_n(\xi,\eta)\left(1 - e^{-t/k_n}\right)$$

we have:

$$|h_1| > |h_s|\left(1 - e^{-t/k_1}\right) \tag{4.108}$$

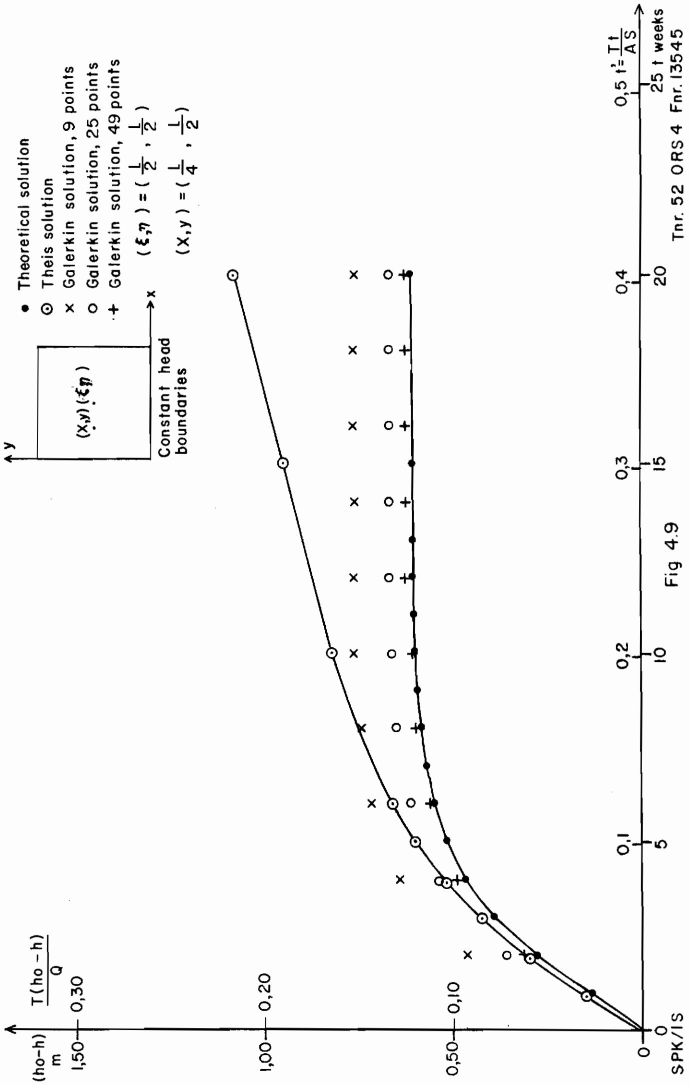
Theoretical solution
Theis solution
Galerkin solution, 9 points
Galerkin solution, 25 points
Galerkin solution, 49 points
(ξ,η) = (L/2, L/2)
(X,y) = (L/4, L/2)
(X,y)(ξ,η)
Constant head boundaries
(ho-h) m
T(ho-h)/Q
1,50 0,30
1,00 0,20
0,50 0,10
0
0 SPK/IS
0,1 5
0,2 10
0,3 15
0,4 20
0,5 t'= Tt/AS
25 t weeks
Tnr. 52 ORS 4 Fnr. 13545

Fig 4.9

where h_s means the stationary level, and K_1 is the largest timeconstant. See table 4.3. The time for the groundwater level to have reached $(1-\alpha)|h_s|$ is given by:

$$(1-\alpha)|h_s| > |h_s|(1-e^{-t_s/K_1})$$

That is to say:

$$t_s < -K_1 \ln \alpha \tag{4.109}$$

Now taking $\alpha = 0.01$ we get:

$t_s < -1{,}20 \ln (0.01) \simeq 11{,}05$ weeks corresponding to the dimensionless time:

$$t_s^1 < 0{,}22$$

As most of the flow goes through linear reservoir number one, see fig. 3.8, $t_s \simeq -K_1 \ln\alpha$ should not be a very bad approximation.

The timeconstants for 9,25 and 49 points respectively are given in tables 4.1, 4.2 and 4.3.

We will finally give a simple example of the solution to the inverse problem. Let us use the same area as before with 49 internal points, see fig. 4.10.

TIMECONSTANTS FOR LINEAR RESERVOIRS IN UNITS OF TWO WEEKS

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1.09	.42	.34	.22	.16	.15	.13	.09								

Table 4.1

TIMECONSTANTS FOR LINEAR RESERVOIRS IN UNITS OF TWO WEEKS.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1.20	.46	.43	.25	.20	.19	.15	.12	.10	.10	.09	.08	.07	.06	.06	.06
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
.05	.05	.04	.04	.04	.03	.03	.03	.03							

Table 4.2

TIMECONSTANTS FOR LINEAR RESERVOIRS IN UNITS OF TWO WEEKS.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1.20	.47	.45	.27	.21	.21	.16	.14	.11	.11	.11	.09	.08	.07	.07	.06
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
.06	.05	.05	.05	.04	.04	.04	.04	.04	.03	.03	.03	.03	.03	.03	.03
33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
.03	.02	.02	.02	.02	.02	.02	.02	.02	.02	.02	.01	.01	.01	.01	.01
49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
.01															

Table 4.3

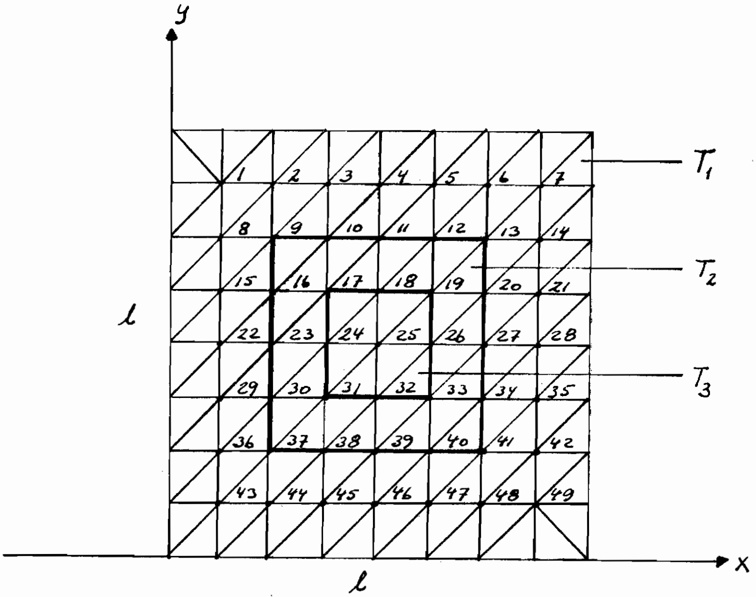

fig. 4.10

The triangular mesh is shown in fig. 4.10 together with the number of points. We divide the area into three sub-areas. One close to the boundary with $T_1 = 0,2\ m^2/s$, one at the center of the area with $T = 0,2\ m^2/s$ and they are separated by a more impermeable area with $T_2 = 0,02\ m^2/s$. We take the input term to be in the form of constant infiltration, $\overline{R} = 1000$ mm/year. We assume as before, that we have constant head boundary, with g = 200 m all around the boundary. Eq. 4.39 now gives us the groundwater level. The solution is given in table 4.4 and 4.5. Table 4.4 gives the part of the solution due to the boundary conditions, and is of course equal to 200 m everywhere. Table 4.5 gives the part of the solution due to the infiltration. The total groundwater level is given by the sum of these two terms. We will now use the calculated groundwater level as an input to the inverse problem, but we change the solution slightly in order to introduce

STATIONARY LEVEL ,m.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00
33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00	200.00
49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
200.00															

Table 4.4

INFILTRATION TERM ,m.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
.28	.43	.53	.56	.52	.43	.27	.43	.67	.87	.93	.87	.66	.43	.53	.87
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
2.71	2.82	2.71	.87	.52	.56	.93	2.82	2.89	2.82	.93	.56	.52	.87	2.71	2.82
33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
2.71	.87	.53	.43	.66	.87	.93	.87	.67	.43	.27	.43	.52	.56	.53	.43
49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
.28															

Table 4.5

some error into the inverse problem. In this case we have $\underline{\Delta b} \equiv 0$ and we should use 4.93. As this is just an illustrative example we choose to use eq. 4.88 instead of 4.93, thus imagining that the error is in the infiltration but not in the groundwater level. The minimization procedure involved in eq. 4.88 is much simpler than in eq. 4.93 and thus gives better explanation of the principles. We now suppose that we have used geological information to tell us that there are three different transmissivity values in the area. Let us denote them by T_1, T_2, T_3, see fig. 4.10.

We choose the weighting matrix in 4.88 to be the identity-matrix. The solution to 4.88 is given by:

$$T_1 = 0{,}20 \;\; m^2/s$$

$$T_2 = 0{,}02 \;\; m^2/s$$

$$T_3 = 0{,}19 \; m^2/s$$

We now perform the singular value decomposition. The solution is given in fig. 4.11. We must have eq. 4.104 in mind to understand its meaning. The linear combination that is best determined from the data is the one corresponding to the largest eigenvalue. From the parameter eigenvector, V, in fig. 4.11, we see that it is T_2. That is very reasonable, because in that subarea we have the largest gradients. The data eigenvector, U, gives us information about the degree of sensitivity of the corresponding linear combination of T-values to changes in the infiltration in each point. We see that the linear combination that is worst determined from the data is T_3. That is also very reasonable, because there we have the smallest gradients. From the corresponding data eigenvector

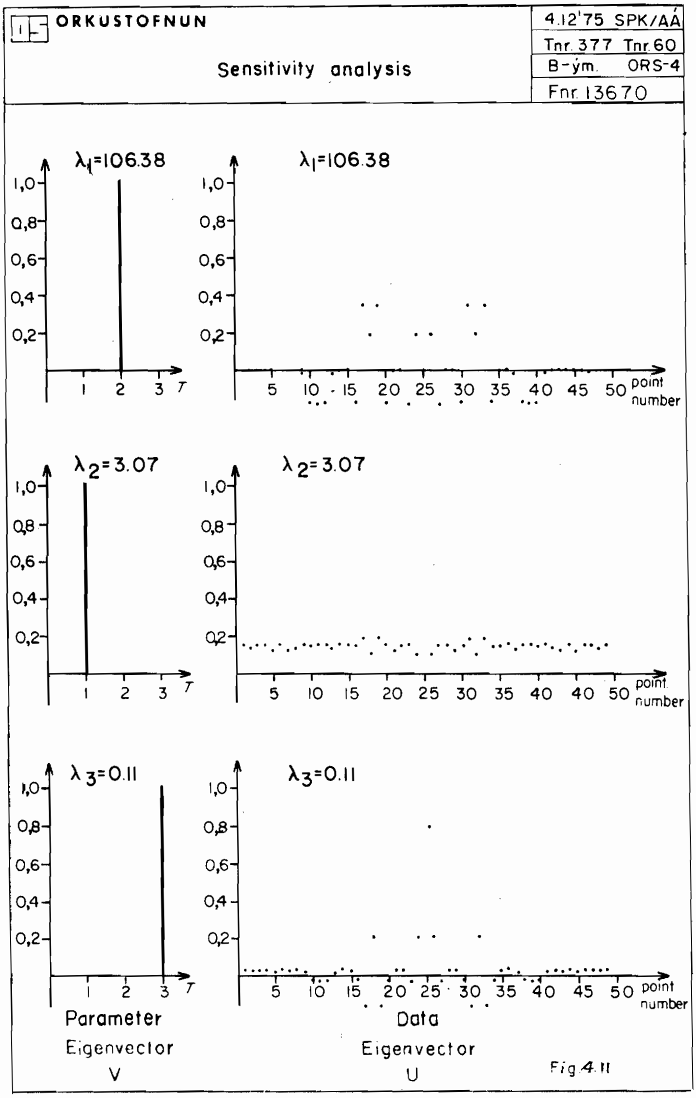
ORKUSTOFNUN
Sensitivity analysis
4.12'75 SPK/AÁ
Tnr. 377 Tnr. 60
B-ým. ORS-4
Fnr. 13670
λ1=106.38
λ1=106.38
λ2=3.07
λ2=3.07
λ3=0.11
λ3=0.11
1,0
0,8
0,6
0,4
0,2
T
point number
Parameter Eigenvector V
Data Eigenvector U

Fig. 4.11

we see how important it is to know the infiltration in point 25 very accurately. Because of the small eigenvalue, T_3 is very sensitive to changes in the infiltration.

Let us look at another example. Exactly the same as before except for that the boundary conditions are given by, g = 300 m, for y = 0, and g = 200 for y = 1 and linear between. The solution is given in Table 4.6 and Table 4.7. The infiltration term is of course the same as before. We use this solution to solve the inverse problem as before. Its solution is given by:

$$T_1 = 0,20\ m^2/s$$
$$T_2 = 0,02\ m^2/s$$
$$T_3 = 0,20\ m^2/s$$

The singular value decomposition is given in fig. 4.12. We see that all the linear combinations are better determined now, because the eigenvalues are all larger in this case. That is very natural because the gradients are larger in this case. We also see that T_1 and T_3 now become equally well determined.

As mentioned before this is just an introduction to the inverse problem. It was not the meaning to solve all the aspects of the inverse problem, but just to introduce where the problems are and what should be studied in more detail.

STATIONARY LEVEL ,m.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
211.21	209.37	207.18	206.45	207.18	209.37	211.21	222.97	219.09	212.90	211.44	212.90	219.09	222.97	236.59	235.91
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
244.06	245.28	244.06	235.91	236.59	250.00	250.00	250.00	250.00	250.00	250.00	250.00	263.40	264.08	255.93	254.71
33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
255.93	264.08	263.40	277.02	280.90	287.09	288.55	287.09	280.90	277.02	288.78	290.62	292.81	293.54	292.81	290.62
49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
288.78															

Table 4.6

INFILTRATION TERM ,m.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
.28	.43	.53	.56	.52	.43	.27	.43	.67	.87	.93	.87	.66	.43	.53	.87
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
2.71	2.82	2.71	.87	.52	.56	.93	2.82	2.89	2.82	.93	.56	.52	.87	2.71	2.82
33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
2.71	.87	.53	.43	.66	.87	.93	.87	.67	.43	.27	.43	.52	.56	.53	.43
49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
.28															

Table 4.7

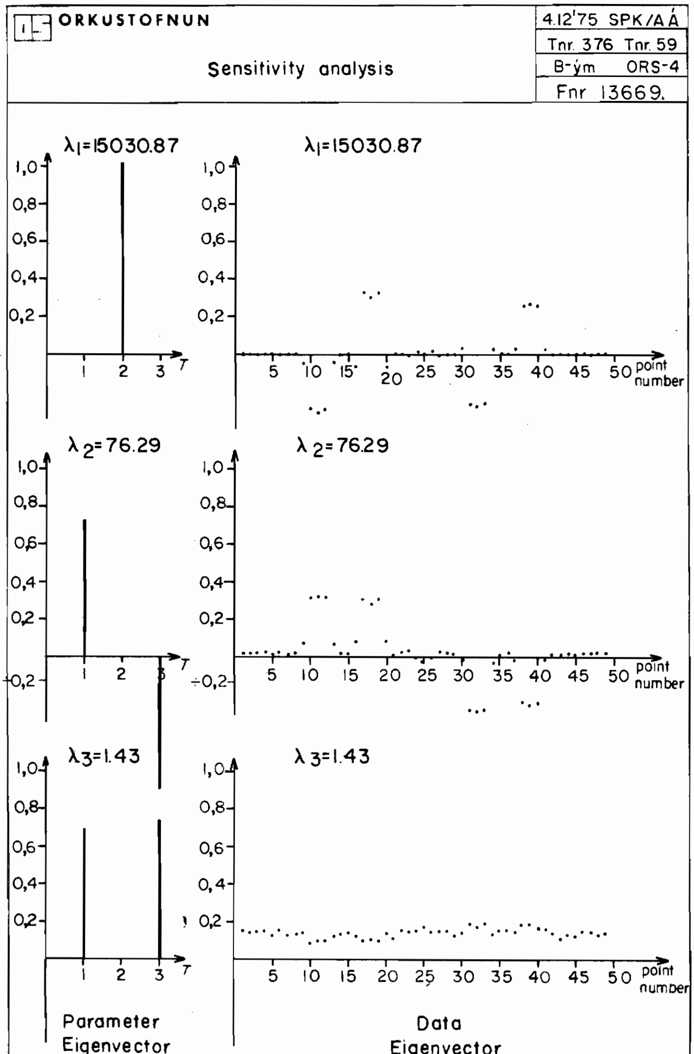
ORKUSTOFNUN
Sensitivity analysis
4.12'75 SPK/AÁ
Tnr. 376 Tnr. 59
B-ým ORS-4
Fnr 13669.
λ1=15030.87
λ1=15030.87
λ2=76.29
λ2=76.29
λ3=1.43
λ3=1.43
point number
T
Parameter Eigenvector V
Data Eigenvector U

Fig. 4.12.

5. A PRACTICAL EXAMPLE.

In the following we will give a small practical example to demonstrate the methods and principles we have been considering. It must not be looked upon as a complete solution to a practical case, but just as an example how the various methods can be used to solve such problems.

In many practical problems the fine mathematical methods cannot be used in full detail, because of great lack of data. Then geological and hydrological insight and 'engineering methods' become extremely valuable. Then an unique solution cannot be given, but the solution is rather of the form of various possibilities. The final solution should also contain information about, where more data is most valuable in order to improve the results.

In the practical example we are going to consider we are very much in the lack of data. We cannot, therefore, give concrete answers, but just intend to demonstrate how the methods we have been developing can be used combined with engineering methods' in order to solve actual problems.

Fig. 5.1 shows a location map of the Sigalda research area in the southern part of Iceland. The research area is 44.3 km^2 and is a very porous lava field separated from nearby lava fields by more impervious pillow lava formations. Fig. 5.2 shows a geological map of the area. We have a weather station at Hæll and four years of observation at Búrfell. See fig. 5.1. When determining the infiltration on the research area we choose to use the long observation series at Hæll multiplied with the ratio between the respective mean value of the four years of observation at Hæll and Búrfell. It is very unlikely that the seasonal variations in the rain are the same at Hæll and Sigalda,

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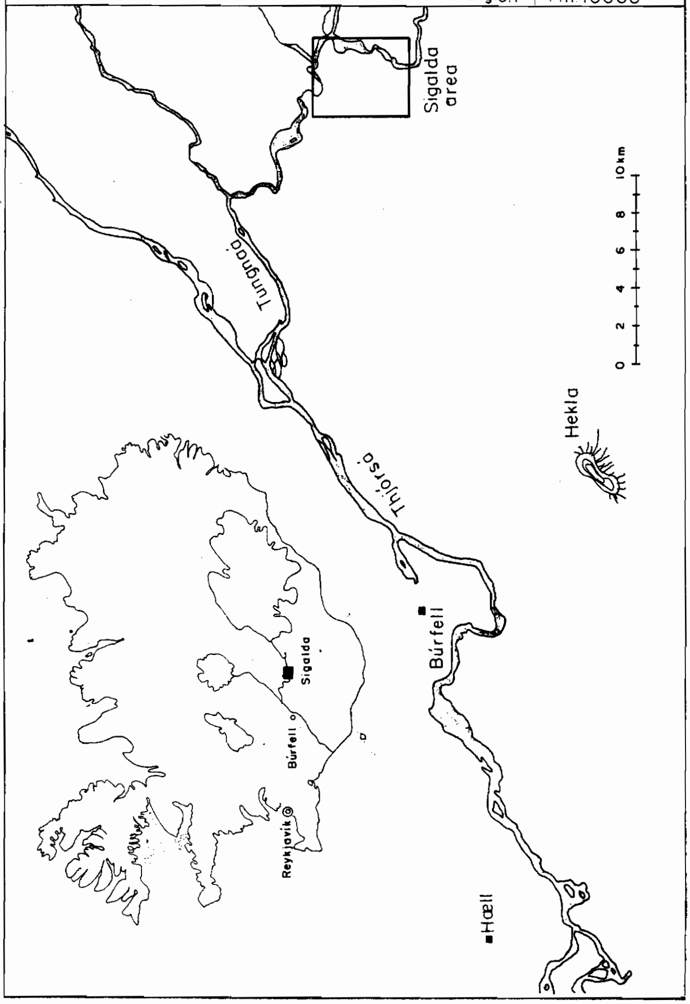
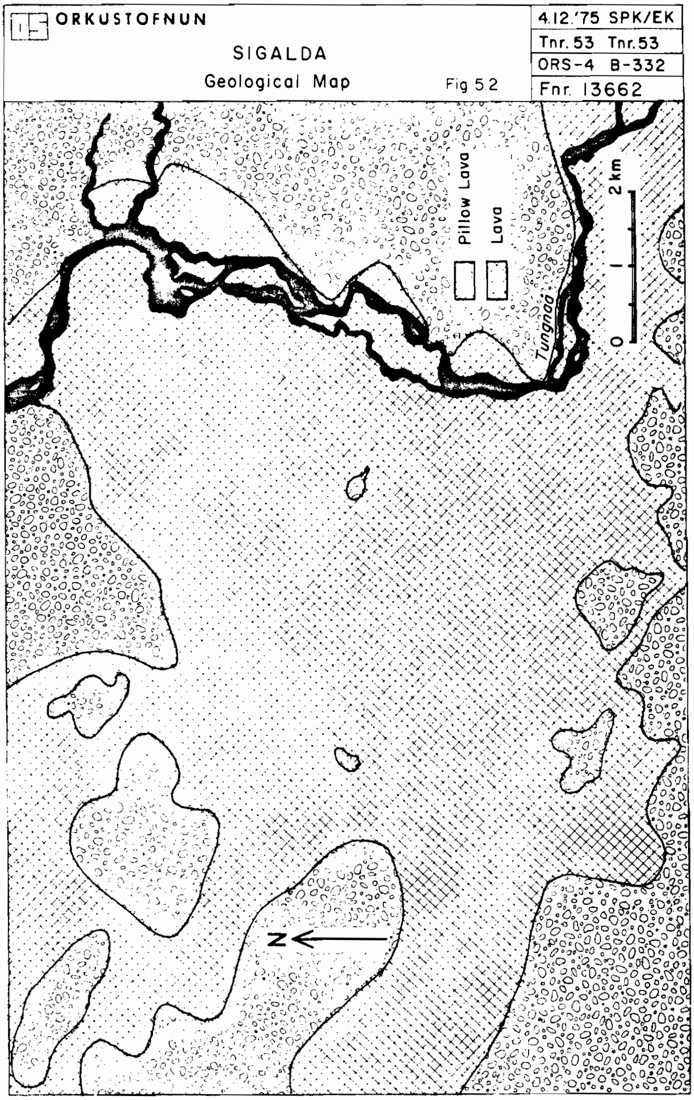
Sigalda
Location Map

Fig.5.1

4.12'75 SPK ÁÁ	
Tnr. 435 Tnr. 55	
B-332 ORS-4	
Fnr. 13665	




ORKUSTOFNUN
SIGALDA
Geological Map
Fig 5.2
4.12.'75 SPK/EK
Tnr. 53 Tnr. 53
ORS-4 B-332
Fnr. 13662
Pillow Lava
Lava
Tungnaa
0
1
2km
N

but there is nothing else to do with the exisiting data, but we see later that an error in the infiltration plays an unimportant role compared with the error in boundary inflow. We also see, that if we had long observation series from the Sigalda area, the error in the infiltration would probably be just as big. This is due to the great uncertainty in the infiltrationcoefficient, which is defined as the ratio between the infiltration and the rain. In the summer it is practically unity, in the winter it is equal to zero and in the springtime it goes from zero to one. In the spring when the snow begins to melt, it flows first as surface runoff before it goes as infiltration into the groundwater reservoir. The infiltrationcoefficient is also a function of place. We can have points in the research area where all the rain and the snowmelt flows directly away as surface runoff. The reverse situation can also be the case and everything inbetween.

In view of the great uncertainty in the infiltration, we use a heuristic method to calculate it. As said before we use the observation series at Hæll multiplied with the ratio between the respective mean value of the four years of observation at Hæll and Búrfell. The infiltration coefficient we use is shown in fig. 5.3. The rain that falls in the two weeks periods 1-7 and 20-26 is kept in a snowmagazine and is then distributed uniformly in period 8-11. We take the infiltration to be constant over the area. We use a two week period as a timestep and 24 years of observation at station Hæll is used. According to eq. 4.53 and the largest timeconstant in table 5.20 this is quite satisfactory. The calculation of the infiltration according to the above mentioned methods gives $\overline{R}$ = 666 mm/year.

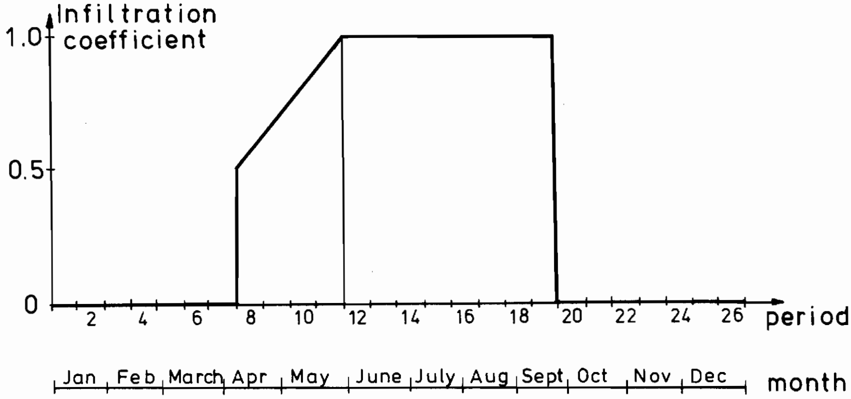


Fig. 5.3

In fig. 5.4 the boundary of the research area and the existing groundwater level gauging stations are shown. In fig. 5.5 the triangular mesh is shown together with the gauging stations and the boundary conditions. We take the pillowlava formation to be impermeable except for the great formation in the south, which must be taken into account due to its great areal extent. Fig. 5.6, 5.7 and 5.8 show the groundwater variation in E-28, E-30 and TK-3. The variations are not very large. We therefore assume in the following that we have constant head boundaries in the northeast corner.
The boundary values are given in table 5.1.

Triangunar mesh point	Groundwater elevation,m
61	477,0
62	475,0
63	473,5
64	472,0
65	475,0

Table 5.1

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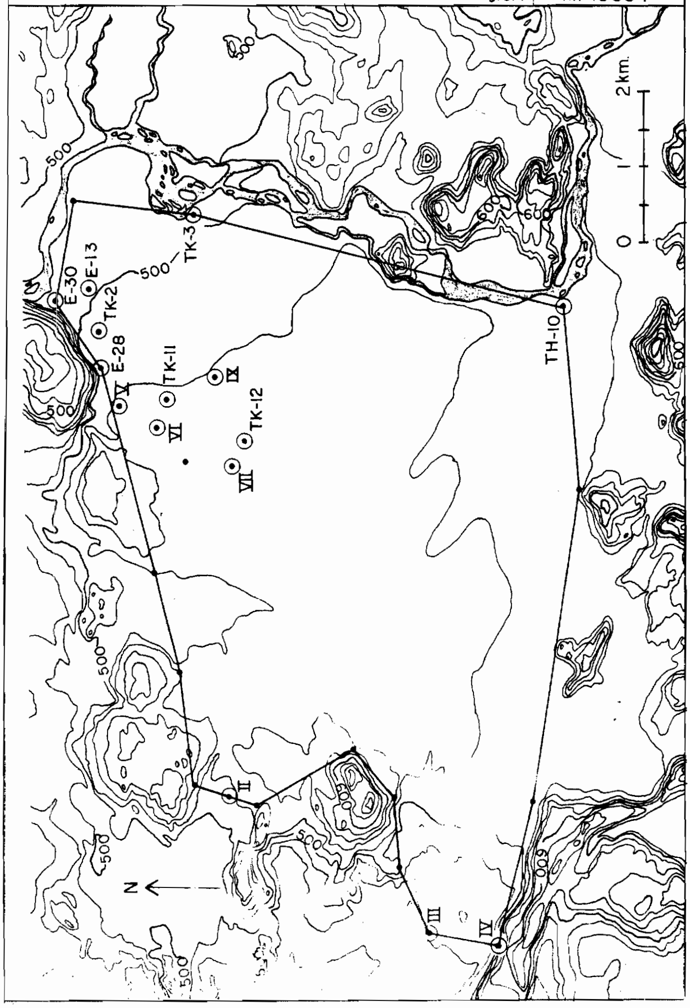
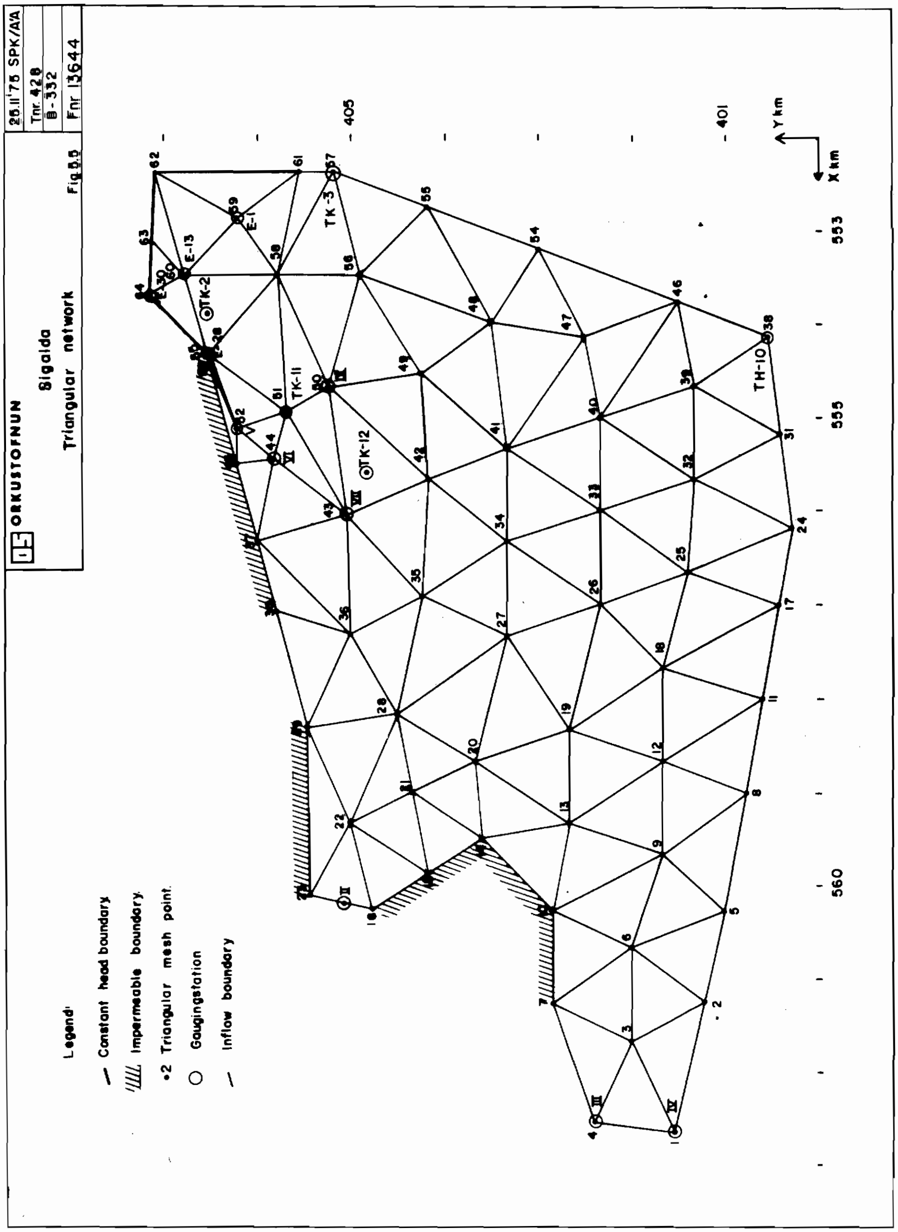
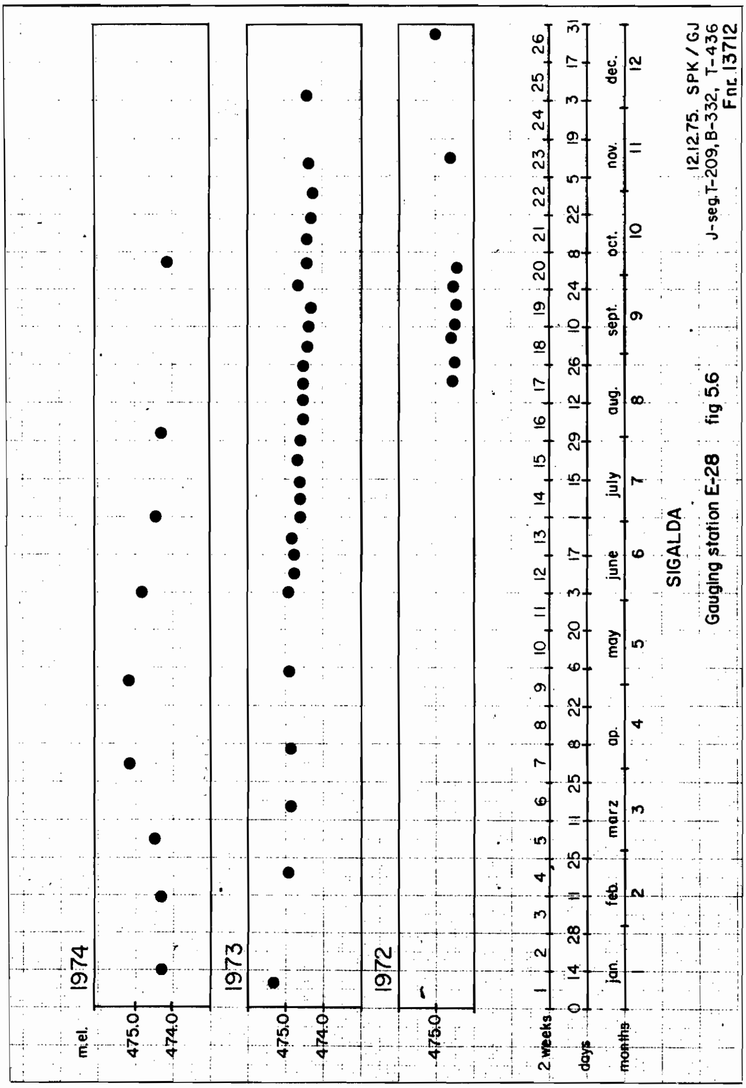
Sigalda
Research area

Fig. 5.4

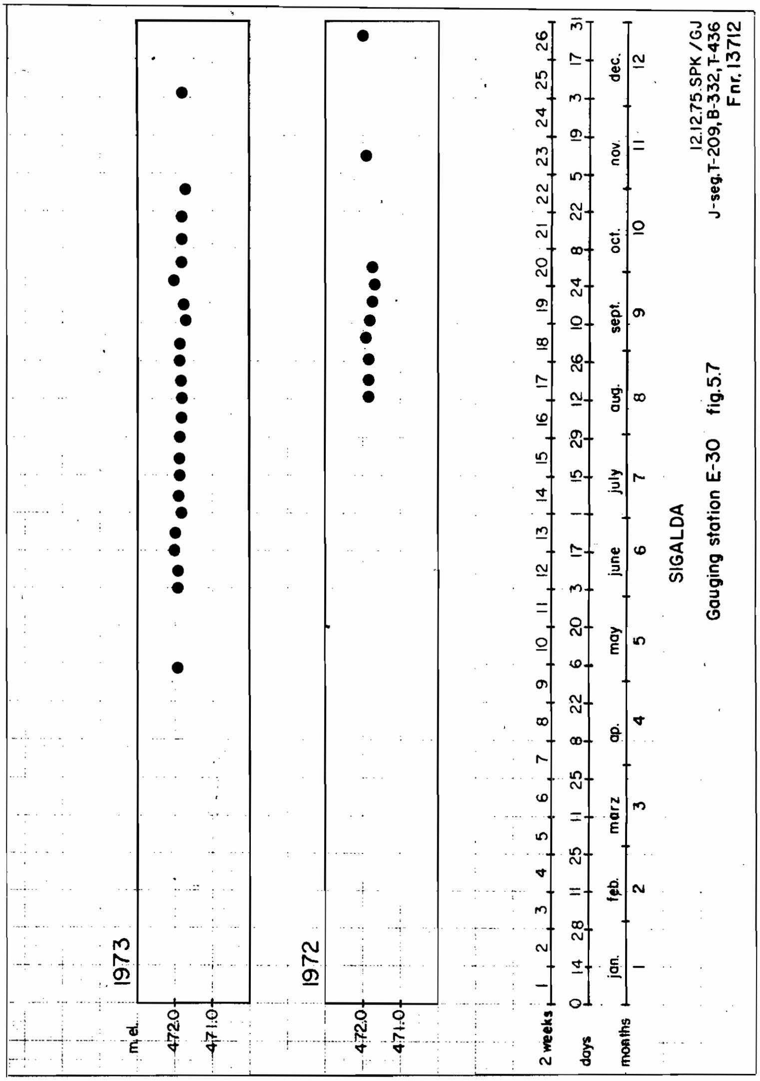
4.12.'75 SPK/AÁ	
Tnr. 434	Tnr. 54
B-332	ORS-4
Fnr. 13664	



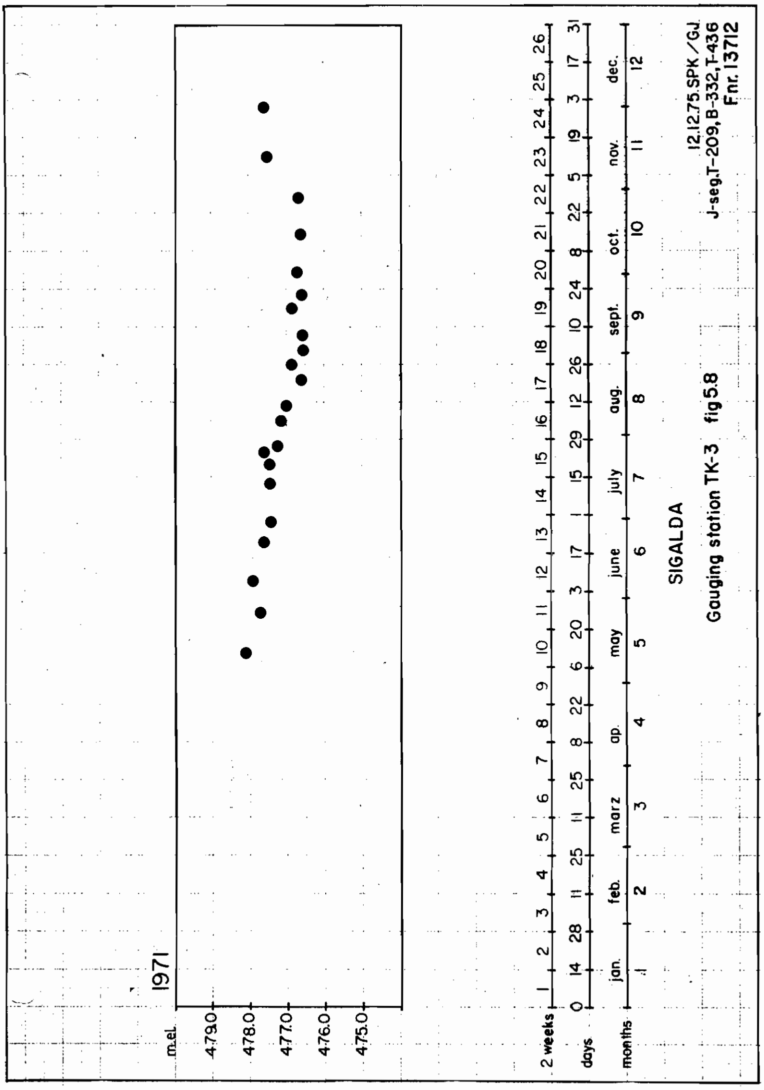

ORKUSTOFNUN
Sigalda
Triangular network
Fig. 5.5
25.11'75 SPK/AA
Tnr. 428
B-332
Fnr 13644
Legend:
Constant head boundary
Impermeable boundary
•2 Triangular mesh point
Gaugingstation
Inflow boundary
TK-2
TK-3
TK-10
TK-11
TK-12
E-1
E-13
X km
Y km
553
555
560
405
401



SIGALDA
Gauging station E-28
fig 5.6
12.12.75. SPK / GJ
J-seg.T-209, B-332, T-436
Fnr. 13712
1974
1973
1972
m.el.
475.0
474.0
2 weeks
days
months
jan.
feb.
marz
ap.
may
june
july
aug.
sept.
oct.
nov.
dec.



SIGALDA
Gauging station E-30 fig.5.7


1971
m.el.
479.0
478.0
477.0
476.0
475.0
2 weeks
days
months
jan.
feb.
marz
ap.
may
june
july
aug.
sept.
oct.
nov.
dec.
SIGALDA
Gauging station TK-3 fig 5.8
12.12.75.SPK/GJ
J-seg.T-209, B-332, T-436
F.nr. 13712

The value of the groundwater in points 62 and 63 is interpolated linearly. We take the groundwater level in point 61 as constant, but allow it to vary in point 67, that is to say TK-3, but choose point 61 to be very near point 57, in order to make the discontinuity in the boundary conditions as small as possible. Point 61 is estimated from the values in point 57. Point 53 is also chosen to be very near point 65, because of the discontinuity in the boundary conditions.

The boundary outflow/inflow is taken to be constant in time and evenly distributed along the boundary curves, from 57 to 38, from 38 to 24, from 24 to 1, from 1 to 4, from 4 to 7, from 16 to 23 and from 29 to 30. That is to say we have seven unknown boundary inflow/outflow values. Nothing is known about their absolute values, except that NEA's hydrologists have estimated the boundary inflow between point 24 and 38 to be approximately 10 m^3/s. It is also estimated that the inflow terms have the same order of magnitude as the outflow terms. This means practically that the northeast corner of our research area is not very active in the global groundwater flow. It is also estimated that the inflow/outflow through the boundary between point 38 and 57 is very small. We now use eq. 4.39 and the mean groundwater level in seven points near the boundary to calculate the seven unknown inflow terms and to estimate the transmissivity value. The method of trial and error is used to find the transmissivity value, that gives the inflow terms, which is in accordance with the small information we mentioned above. We will use the following notations in the calculations:

q_1 = boundary inflow between points 38 and 57

q_2 = - " - " - " 24 " 38

q_3 = - " - " - " 1 " 24

q_4 = boundary outflow between points 1 " 4

q_5 = - " - " - " 4 " 7

q_6 = - " - " - " 16 " 23

q_7 = - " - " - " 29 " 30

The mean level of the points we use to calculate the seven unknown q's is given in table 5.2.

Triangular mesh	Mean level
1	457,5
4	453,0
7	458,0
16/23	445,0
38	503,3
43	474,0
57	477,1

Table 5.2

16/23 is the point between the points 16 and 23, that is gauging station II. Point 7 is estimated.

We take the aquifer to be homogeneous and isotropic and we estimate the transmissivity value, as said before, by the method of trial and error. It gives $T = 0,3\ m^2/s$. It is not very well determined, because of the small information we have about the boundary inflow terms. The method will now be demonstrated in case of the estimated transmissivity value. To begin with we define inflow influence level as the water level in each point, when one of the q's is equal to one and the others are zero. The

INFLUENCELEVEL FOR q_1.

INFLOW TERM ,m.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2.37	2.37	2.37	2.37	2.38	2.37	2.37	2.46	2.39	2.36	2.46	2.42	2.35	2.29	2.25	2.24
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
2.45	2.50	2.37	2.28	2.24	2.22	2.24	2.73	2.62	2.47	2.30	2.19	2.16	1.94	3.03	2.82
33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
2.57	2.31	2.14	2.06	1.73	3.59	3.17	2.77	2.36	2.05	1.82	1.36	1.40	3.56	2.95	2.45
49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
1.94	1.38	1.18	.96	.33	3.09	2.31	1.64	.71	.62	.16	.08				

Table 5.3

INFLUENCELEVEL FOR q_2.

INFLOW TERM ,m.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
3.72	3.73	3.72	3.72	3.75	3.73	3.72	3.95	3.76	3.69	3.95	3.84	3.68	3.53	3.42	3.40
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
3.91	4.06	3.70	3.49	3.40	3.35	3.40	5.15	4.44	3.89	3.49	3.28	3.22	2.78	5.93	4.71
33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
3.94	3.39	3.12	3.02	2.40	5.71	4.78	3.88	3.20	2.79	2.52	1.81	1.87	4.43	3.57	2.70
49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
2.31	1.69	1.51	1.27	.44	2.99	1.85	1.60	.49	.66	.17	.09				

Table 5.4

INFLUENCELEVEL FOR q_3.

INFLOW TERM ,m.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
6.87	6.64	6.51	6.63	6.06	6.11	6.25	6.13	5.78	5.71	5.90	5.68	5.29	4.80	4.48	4.41
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
6.28	5.09	4.91	4.55	4.38	4.29	4.40	4.39	4.49	4.37	4.12	4.08	3.99	3.23	3.90	3.98
33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
3.84	3.61	3.58	3.56	2.68	3.63	3.61	3.39	3.11	2.91	2.76	1.94	2.02	3.38	3.04	2.46
49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
2.25	1.72	1.58	1.36	.47	2.64	1.72	1.52	.47	.64	.17	.08				

Table 5.5

INFLUENCELEVEL FOR q_4.

INFLOW TERM ,m.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
-13.28	-9.61	-10.50	-12.83	-7.23	-8.28	-9.27	-4.82	-6.71	-7.25	-5.o9	-5.75	-6.02	-5.35	-4.88	-4.78
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
-5.10	-4.78	-5.12	-4.89	-4.72	-4.62	-4.77	-4.04	-4.25	-4.26	-4.21	-4.32	-4.23	-3.35	-3.69	-3.78
33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
-3.73	-3.61	-3.67	-3.70	-2.75	-3.47	-3.46	-3.29	-3.07	-2.92	-2.81	-1.97	-2.05	-3.25	-2.95	-2.42
49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
-2.23	-1.72	-1.60	-1.38	-.47	-2.57	-1.69	-1.51	-.46	-.64	-.17	-.o8				

Table 5.6

INFLUENCELEVEL FOR q_5.

INFLOW TERM ,m.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
-10.60	-9.18	-9.91	-11.50	-7.17	-8.23	-9.98	-4.82	-6.67	-7.32	-5.09	-5.74	-6.03	-5.36	-4.89	-4.78
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
-5.10	-4.77	-5.12	-4.89	-4.73	-4.63	-4.77	-4.04	-4.25	-4.26	-4.21	-4.33	-4.23	-3.35	-3.69	-3.78
33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
-3.72	-3.61	-3.67	-3.71	-2.75	-3.47	-3.46	-3.29	-3.07	-2.92	-2.81	-1.97	-2.05	-3.25	-2.95	-2.42
49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
-2.23	-1.72	-1.60	-1.38	-.47	-2.57	-1.69	-1.50	-.46	-.64	-.17	-.08				

Table 5.7

INFLUENCELEVEL FOR q_6.

INFLOW TERM ,m.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
-4.77	-4.76	-4.77	-4.77	-4.71	-4.76	-4.78	-4.24	-4.66	-4.86	-4.26	-4.47	-4.87	-5.50	-6.96	-9.02
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
-4.27	-4.10	-4.55	-5.13	-5.96	-7.08	-9.27	-3.64	-3.78	-3.89	-4.11	-4.94	-5.12	-3.66	-3.38	-3.45
33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
-3.46	-3.51	-3.81	-4.09	-2.92	-3.20	-3.20	-3.09	-2.96	-2.91	-2.92	-2.04	-2.13	-3.03	-2.79	-2.33
49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
-2.19	-1.73	-1.63	-1.42	-.49	-2.45	-1.64	-1.47	-.45	-.63	-.16	-.08				

Table 5.8

INFLUENCELEVEL FOR q_7.

INFLOW TERM ,m.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
-3.79	-3.79	-3.79	-3.79	-3.76	-3.79	-3.79	-3.54	-3.74	-3.83	-3.54	-3.65	-3.84	-4.07	-4.33	-4.41
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
-3.55	-3.46	-3.73	-4.06	-4.33	-4.59	-4.38	-3.17	-3.27	-3.38	-3.61	-4.35	-5.24	-4.65	-2.99	-3.05
33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
-3.09	-3.21	-3.67	-4.29	-3.41	-2.86	-2.86	-2.79	-2.75	-2.82	-3.03	-2.18	-2.31	-2.73	-2.54	-2.17
49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
-2.10	-1.74	-1.69	-1.52	-.52	-2.27	-1.55	-1.40	-.43	-.62	-.16	-.08				

Table 5.9

STATIONARY LEVEL ,m.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
475.50	475.50	475.50	475.50	475.50	475.50	475.50	475.51	475.50	475.50	475.51	475.50	475.50	475.49	475.49	475.49
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
475.50	475.51	475.50	475.49	475.49	475.48	475.49	475.52	475.52	475.51	475.50	475.48	475.48	475.45	475.53	475.53
33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
475.52	475.50	475.48	475.47	475.42	475.54	475.54	475.54	475.53	475.49	475.44	475.36	475.37	475.55	475.56	475.59
49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
475.54	475.42	475.34	475.26	475.09	475.60	475.75	475.70	476.54	475.37	475.44	473.62				

Table 5.10

INFILTRATION TERM ,m.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
3.77	3.72	3.73	3.76	3.61	3.66	3.70	3.42	3.56	3.58	3.41	3.46	3.46	3.35	3.28	3.28
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
3.46	3.29	3.32	3.25	3.22	3.22	3.29	3.11	3.15	3.11	3.03	3.07	3.05	2.57	2.96	2.98
33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
2.90	2.78	2.78	2.78	2.20	2.85	2.83	2.69	2.51	2.36	2.25	1.64	1.70	2.71	2.48	2.08
49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
1.90	1.48	1.35	1.15	.40	2.21	1.50	1.34	.42	.59	.16	.09				

Table 5.11

influencelevel for the seven q's is given in table 5.3 to table 5.9. The stationary level is given in table 5.10. It is defined as the waterlevel created by the constant head boundaries with zero infiltration and zero boundary inflow. The infiltration level is given in table 5.11. It is defined as the waterlevel due to infiltration with zero at the constant head boundaries and zero boundary inflow. The sum of the stationary level and the infiltration level subtracted from the measured mean level in table 5.2 gives the desired inflow term. By using the inflow influence level we can set up seven equations with seven unknowns. We get:

$$\begin{bmatrix} 2,37 & 3,72 & 6,87 & -13,28 & -10,60 & -4,77 & -3,79 \\ 2,37 & 3,72 & 6,63 & -12,83 & -11,50 & -4,77 & -3,79 \\ 2,37 & 3,72 & 6,25 & -9,27 & -9,98 & -4,78 & -3,79 \\ 2,24 & 3,40 & 4,40 & -4,78 & -4,78 & -9,14 & -4,40 \\ 3,59 & 5,71 & 3,63 & -3,47 & -3,47 & -3,20 & -2,86 \\ 1,82 & 2,52 & 2,76 & -2,81 & -2,81 & -2,92 & -3,03 \\ 0,71 & 0,49 & 0,47 & -0,46 & -0,46 & -0,45 & -0,43 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \end{bmatrix} = \begin{bmatrix} -21,77 \\ -26,26 \\ -21,20 \\ -33,78 \\ 24,91 \\ -3,69 \\ 0,14 \end{bmatrix}$$

The solution is given by:

Inflow	Outflow
$q_2 = 9,7\ m^3/s$	$q_1 = 1,2\ m^3/s$
$q_3 = 3,7$ "	$q_4 = 0,1$ "
	$q_5 = 4,1$ "
	$q_6 = 5,1$ "
	$q_7 = 3,1$ "
$= 13,4\ m^3/s$	$= 13,6\ m^3/s$

We see that the solution satisfies the necessary conditions mentioned before. We can now calculate the inflow term in each point; it is given in table 5.12. The total mean level is the sum of the stationary level, the infiltration level and the inflow level. It is given in table 5.13. The calculated groundwater elevation is shown in fig. 5.9. The groundwater level is not known in enough number of points to draw piezometric surface, but we can use the observations as a check on our calculations, because we have just used seven of the gauging stations out of fourteen. See fig. 5.4 and fig. 5.5. The comparison is shown in table 5.14. The greatest difference is in station E-13, where there is one meter's difference in the observed and calculated elevation. In the following we will show, how the possible source of error in this point can be detected. From table 5.11 we see that the infiltration term in E-13, point 60, is 0,09 m, and from table 5.12 we get the inflow term 0,01 m. The error we have in E-13 can thus hardly be explained by the error in the infiltration or the inflow; therefore the stationary level must be the main source of error. Table 5.15 to table 5.19 show the influence level for the five points on the constant head boundary. The constant head influence level is defined as the water level in each point, when one of g_i's is equal to one and the others are zero. g is used as before to denote the water level at the constant head boundary, and g_i is the waterlevel in point number i at the constant head boundary. If we assume that the water level in other points than point 60 is approximately correct, the error must be in the water level in point 63 or 64, which gives the influenceordinate 0,42 and 0,32 respectively in point 60 and much smaller in other points. The groundwater level in point 64 is most probably better known than in point 63, as we have the gauging station E-30 in point 64, and the level in point 63 is interpolated linearly between the measured level in point 64 and the estimated level in point 62. Our conclusion is therefore, if we want the calculated level in point 60 to fit better the observed level, then better determination of the groundwaterlevel in point 63 is needed.

INFLOW TERM ,m.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
-21.93	-16.62	-20.21	-26.46	-9.80	-14.56	-21.47	5.11	-8.32	-13.20	3.13	-2.80	-9.58	-13.96	-22.50	-33.37
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
4.08	3.54	-5.09	-11.35	-17.28	-24.14	-34.48	17.56	9.26	2.76	-3.50	-12.60	-16.82	-10.62	26.32	14.04
33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
6.40	.41	-5.30	-9.83	-5.92	24.64	16.20	8.65	3.18	-.51	-3.67	-2.41	-2.76	13.56	7.78	3.56
49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
1.29	-.45	-1.22	-1.64	- .54	4.63	1.26	.99	.09	.13	.03	.01				

Table 5.12

MEAN-LEVEL ,m.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
457.33	462.60	459.02	452.80	469.30	464.59	457.73	484.04	470.73	465.87	482.06	476.17	469.38	464.88	456.28	445.40
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
483.05	482.35	473.73	467.40	461.43	454.56	444.30	496.20	487.94	481.39	475.03	465.96	461.71	467.40	504.82	492.56
33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
484.83	478.71	472.96	468.42	471.70	503.04	494.58	486.90	481.22	477.34	474.03	474.59	474.32	491.84	485.84	481.24
49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
478.75	476.45	475.47	474.77	474.95	482.45	478.53	478.03	477.06	476.11	475.65	473.73				

Table 5.13

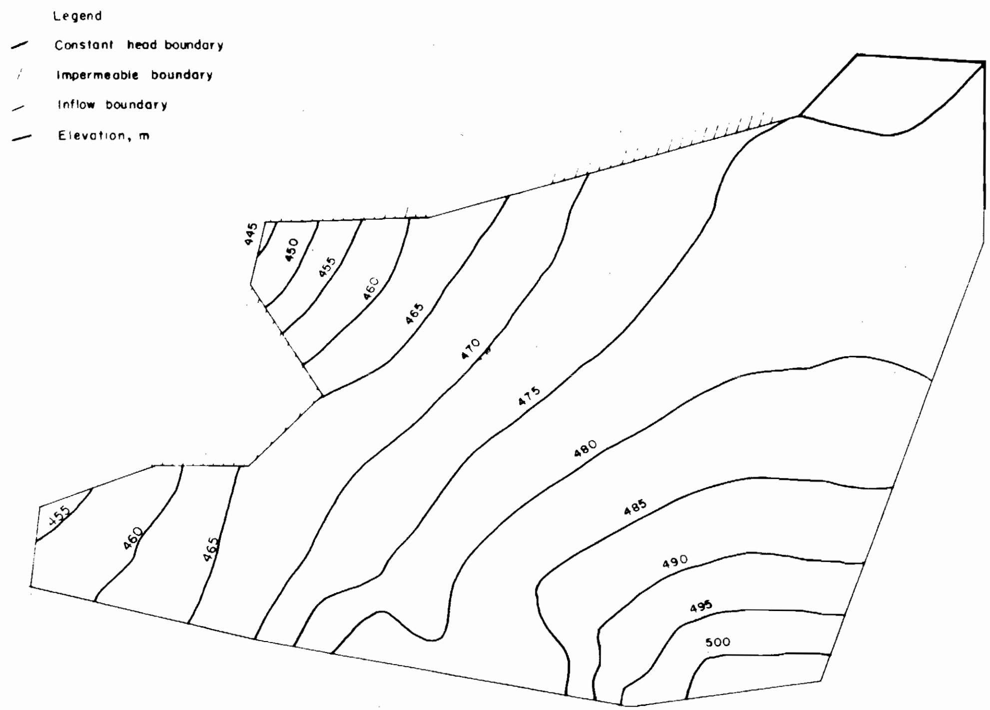
ORKUSTOFNUN

Sigalda

Groundwater elevation

Fig 59

26 11'75 SPK AÁ
Tnr 429
B-332
Fnr 13645



Point number	Gauging station	Measured mean elevation	Calculated mean elevation
1	IV	457,5	457,5
4	III	453,0	453,0
16/23	II	445,0	445,0
38	TH-10	503,0	503,0
43	VII	474,0	474,0
44	VI	475,0	474,5
43/50/42	TK-12	< 476,0	475,5
50	IX	476,0	476,5
51	TK-11	< 476,0	475,5
52	V	475,0	475,0
57	TK-3	477,0	477,0
59	E-1	476,0	476,0
65/58/60	TK-2	474,5	474,5
60	E-13	475,0	474,0

Table 5.14

16/23 means on the side between points 16 and 23. 43/50/42 means inside the triangle formed by the three points 43, 50 and 42.

From table 5.11 and table 5.12 we see that the inflow term is much larger than the infiltration term in most of the mesh points. We therefore conclude that the error in the infiltration is dominated by the error in the inflow. If we want to make our results more reliable, then the boundary inflow must first be determined better, before better determination of the infiltration is needed. Therefore more observation drillholes near the boundary is the first step to make the results more accurate.

INFLUENCELEVEL FOR g_{61}.

STATIONARY LEVEL,m.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
.34	.34	.34	.34	.34	.34	.34	.34	.34	.34	.34	.34	.34	.33	.33	.33
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
.34	.34	.34	.33	.33	.33	.33	.35	.35	.34	.34	.33	.33	.31	.36	.35
33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
.35	.34	.33	.32	.29	.36	.36	.36	.35	.34	.31	.26	.26	.37	.38	.39
49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
.37	.31	.24	.18	.06	.39	.47	.45	.81	.37	.43	.11				

Table 5.15

INFLUENCELEVEL FOR g_{62}.

STATIONARY LEVEL ,m.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03
33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.04	.04
49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
.04	.04	.03	.02	.00	.04	.04	.04	.01	.07	.24	-.04				

Table 5.16

INFLUENCELEVEL FOR g_{63}.

STATIONARY LEVEL ,m.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
.04	.04	.04	.04	.04	.04	.04	.04	.04	.04	.04	.04	.04	.04	.04	.04
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
.04	.04	.04	.04	.04	.04	.04	.04	.04	.04	.04	.04	.04	.04	.04	.04
33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
.04	.04	.04	.04	.04	.04	.04	.04	.04	.04	.04	.04	.04	.04	.05	.05
49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
.05	.05	.03	.02	.01	.05	.05	.05	.02	.09	.11	.42				

Table 5.17

INFLUENCELEVEL FOR g_{64}.

STATIONARY LEVEL ,m.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03
33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03	.03
49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
.03	.03	.03	.02	.00	.03	.03	.04	.01	.07	.08	.32				

Table 5.18

INFLUENCELEVEL FOR g_{65}.

STATIONARY LEVEL ,m.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
.53	.53	.53	.53	.53	.53	.53	.52	.53	.53	.52	.53	.53	.53	.54	.54
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
.52	.52	.53	.53	.54	.54	.54	.51	.52	.52	.53	.54	.54	.56	.51	.51
33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
.51	.52	.54	.55	.58	.50	.50	.50	.51	.53	.57	.63	.62	.49	.49	.47
49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
.49	.55	.64	.73	.90	.47	.39	.39	.13	.37	.12	.18				

Table 5.19

Mr. Sigurjón Rist*, at NEA, has measured the boundary outflow through the side between points 65 and 64. The river Tungnaá flows there in a canyon and the groundwater flows there out of the sidewalls. He has measured the outflow to be between 2,5 m^3/s and 3,5 m^3/s. Calculated outflow between the points 65 and 62 is 2,2 m^3/s, which is approximately the same as the measured one, but spread over longer part of the boundary. That could be explained by local flowways, which cannot be accounted for because of the assumption of isotropic aquifer. The error in the boundary outflow could also be due to the same reason as the error in point 60. If we increase the constant head value in point 63 from 473,5 m to 478 m, then point 60 becomes correct and the main part of the boundary outflow goes through the side between the points 64 and 65. As said before, we need better determination of the constant head values along the side from points 64 to 63 and from 63 to 62, if we want more reliable results. We have also investigated if it would be better to take the boundary from point 64 to 62 as impermeable. This gives too low value in point 60 as before and the boundary outflow between points 65 and 64 just becomes 1,0 m^3/s. The calculated inflow to the canyon is just from the lava aquifer, the rest of the inflow then comes from the pillow lava formations outside our model area. See fig. 5.2 and fig. 5.4. But the main part should come from the lava aquifer, as the calculations show.

We have now just solved the stationary part of our problem according to eq. 4.39 and the transient part according to eq. 4.56 remains to be solved. Because of great uncertainty in the seasonal variation of the infiltration, and uncertainty in the mean level, because of great uncertainty in the boundary inflow and last but not least because we take the boundary inflow to be constant in time, it would not be surprising that we would get very inaccurate results.

* Personal communication.

We have three gauging stations where we could get acceptable time series. They are stations VII, IV and TK-3, see fig. 5.4 and fig. 5.5. Because of the great uncertainties mentioned above, we will just demonstrate in the following how the various methods can be used, but the results are of very little practical value and better observations are needed in order to improve the reliability of the results. We use the method of trial and error to estimate the storage coefficient. The calculated amplitude in the three gauging stations is matched with the measured one, in order to estimate the storage coefficient. The estimated storage coefficient becomes 0,25, which is rather high and is probably due to the great uncertainties mentioned above. The timeconstants for the 28 largest linear reservoirs are shown in table 5.20. The largest timeconstant is 29,4 two week periods, or just over a year. This is in good agreement with the fact that the autocorrelationcoefficient between the yearly means in the river Tungnaá, which is partly groundwater fed from this lava aquifer, has a significant value between 0,3 and 0,4. The observed waterlevel in these three gauging stations together with the calculated one is shown in fig. 5.10 to fig. 5.13. We see that the calculated level is always above the measured level in the autumn. This could be caused by heavier rainfall in the autumn at station Hæll, which we use, than in the Sigalda area. The main reason is also due to less boundary inflow in the autumn. This can be seen by comparing stations VII and IV. Station VII is in the middle of the area but station IV is at the boundary and the difference between the calculated level and the observed one in station IV in the autumn is much greater than in station VII.

Station TK-3 is shown in fig. 5.13. We see that the variations are greater in the observed level than in the calculated one. This is very natural because station TK-3 is very near point 61, which we chose to be constant in time.

From fig. 5.12 we see that the yearly mean level in station IV increases by approximately 2 m between 1971 and 1972. We have no immediate explanation. One possible explanation is that the snowmelt in the spring 1972 has flown extremely much from the pillow lava formations down to station IV. See fig. 5.4. Fig. 5.14 shows the situation if input has been added in the neighbourhood of station IV, due to the snowmelt from the pillow lava. The necessary input is 0,25 m^3/s in the 8,9,10,11'th two week periods, but that corresponds approximately to a snowmagazine covering 2 km^2 in that area.

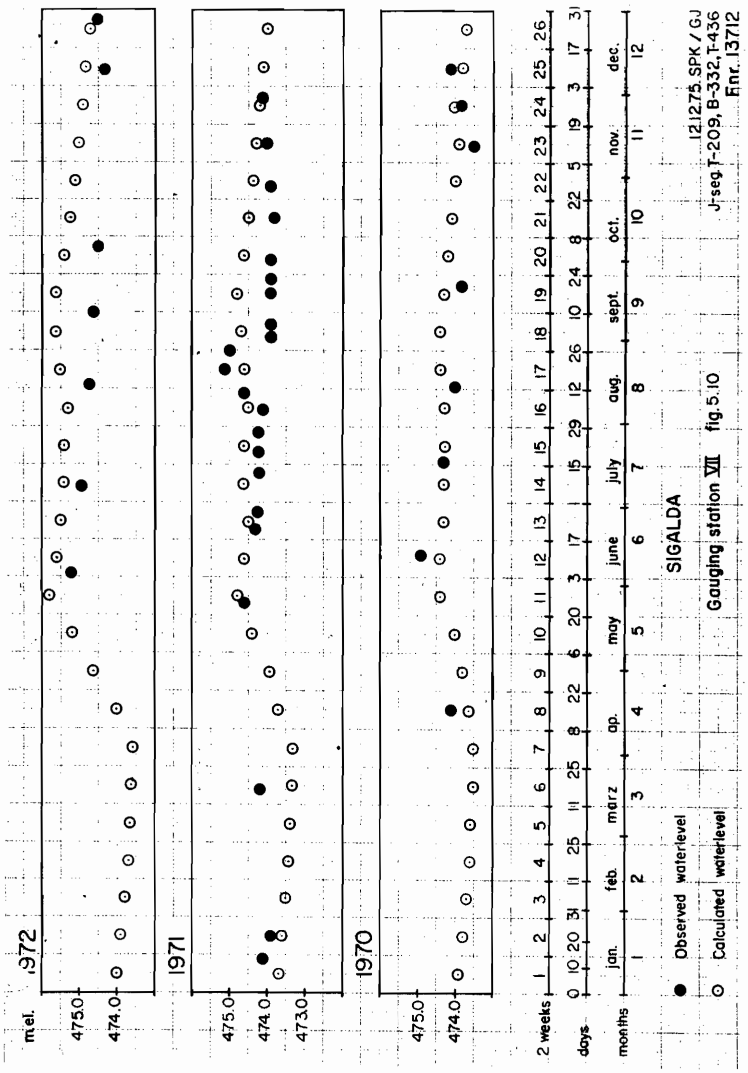
As mentioned before the uncertainty in both the boundary inflow and the seasonal variation of the infiltration make the practical use of the seasonal variation of the groundwater level very limited. That is obvious from the figures 5.10-5.14. It can also be seen from tables 5.21 and 5.22. They give the mathematical norms given in Chapter 4.6. As can be seen from f.ex. the standard deviation of estimate the time series calculations are not more valuable than just the mean level. The norms are calculated with the extra input 0,25 m^3/s in the neighbourhood of station IV.

Finally we will summarize what observation is needed to improve the results. If we want more reliable mean level in the area then first of all the boundary inflow/outflow must be better determined by locating more observation drillholes at the boundary. Then better determination of the infiltration is needed, which can be achieved by locating weather observation stations in the area and by field observations in order to improve the infiltration coefficient. If we want more reliable results for the mean level in the northeast corner of the model area and the boundary outflow to the canyon, then better determination of the boundary condition along the side between the points 64 and 62 is needed.

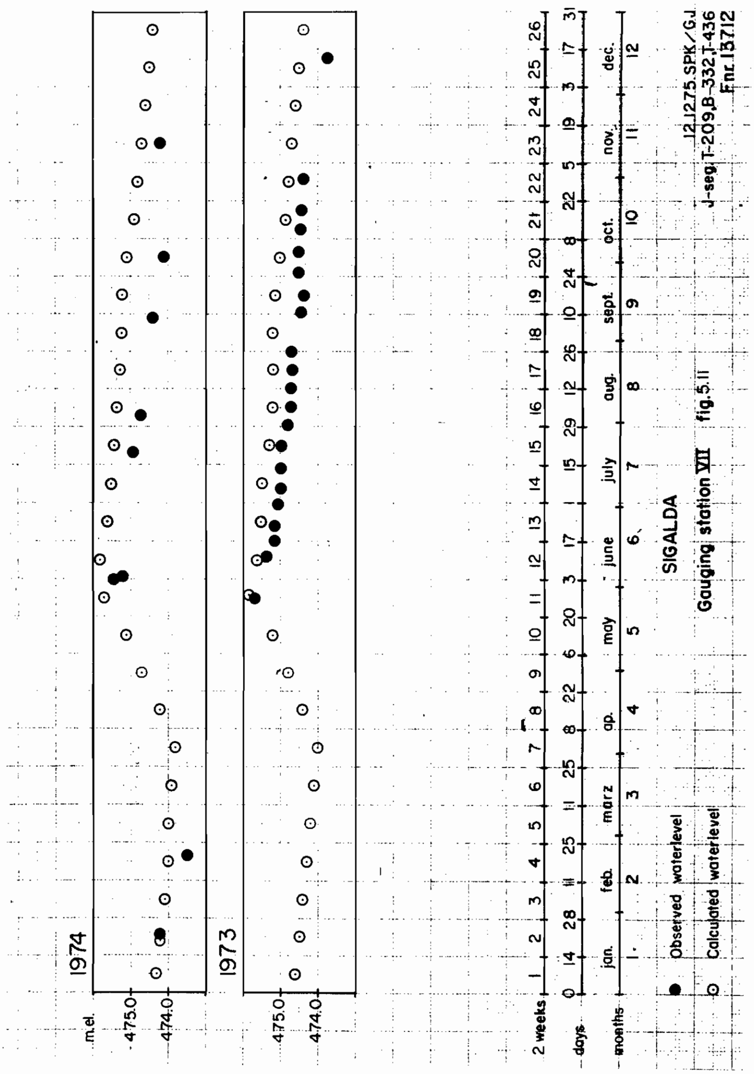
TIMECONSTANTS FOR LINEAR RESERVOIRS IN UNITS OF TWO WEEKS.

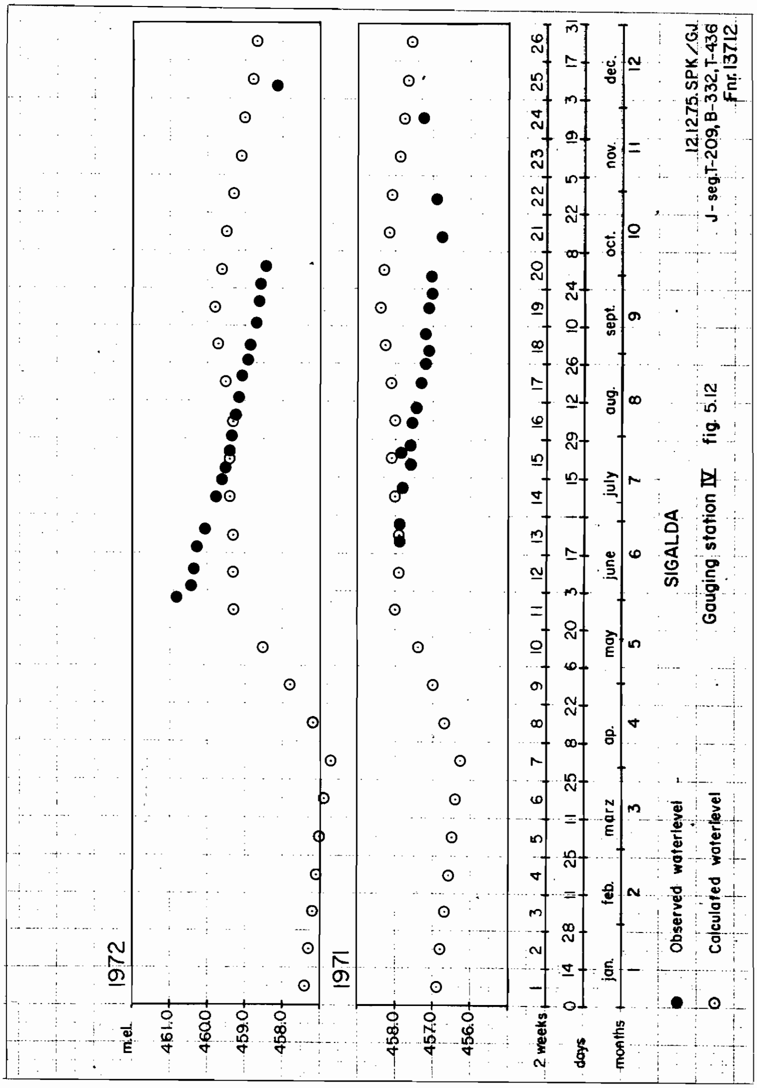
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
29.36	3.45	1.90	1.48	.89	.66	.50	.42	.37	.30	.26	.24	.20	.18	.16	.15
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
.14	.13	.12	.10	.10	.09	.08	.08	.07	.07	.06	.05				

Table 5.20

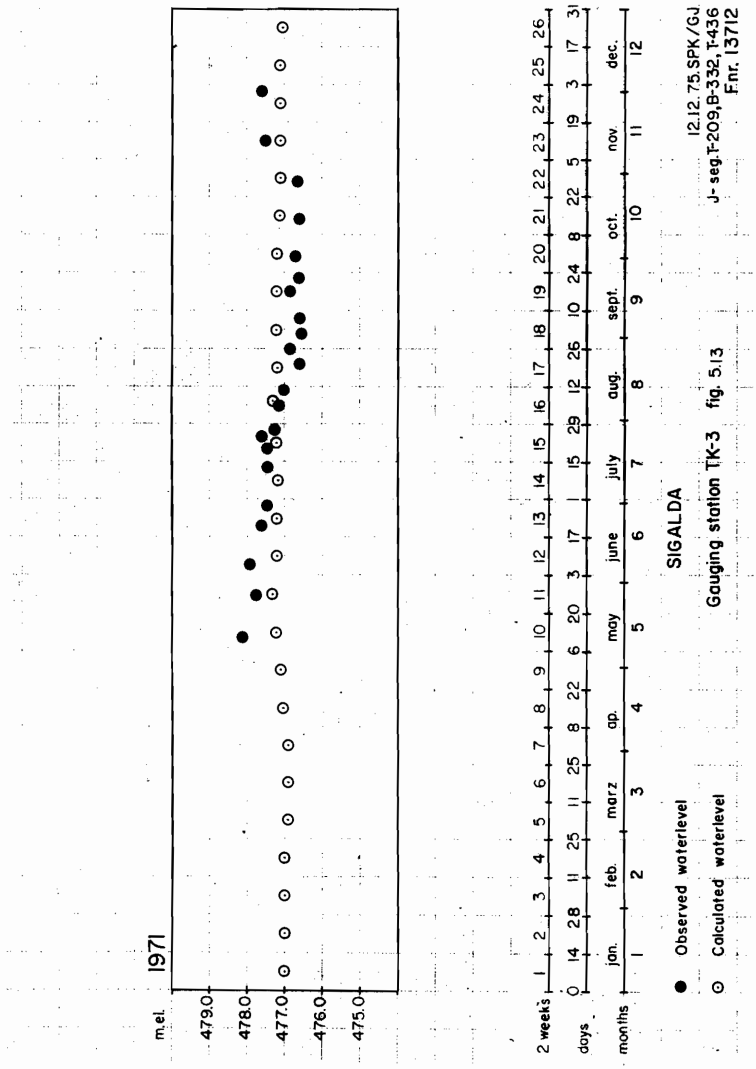


SIGALDA
Gauging station VII fig.5.10

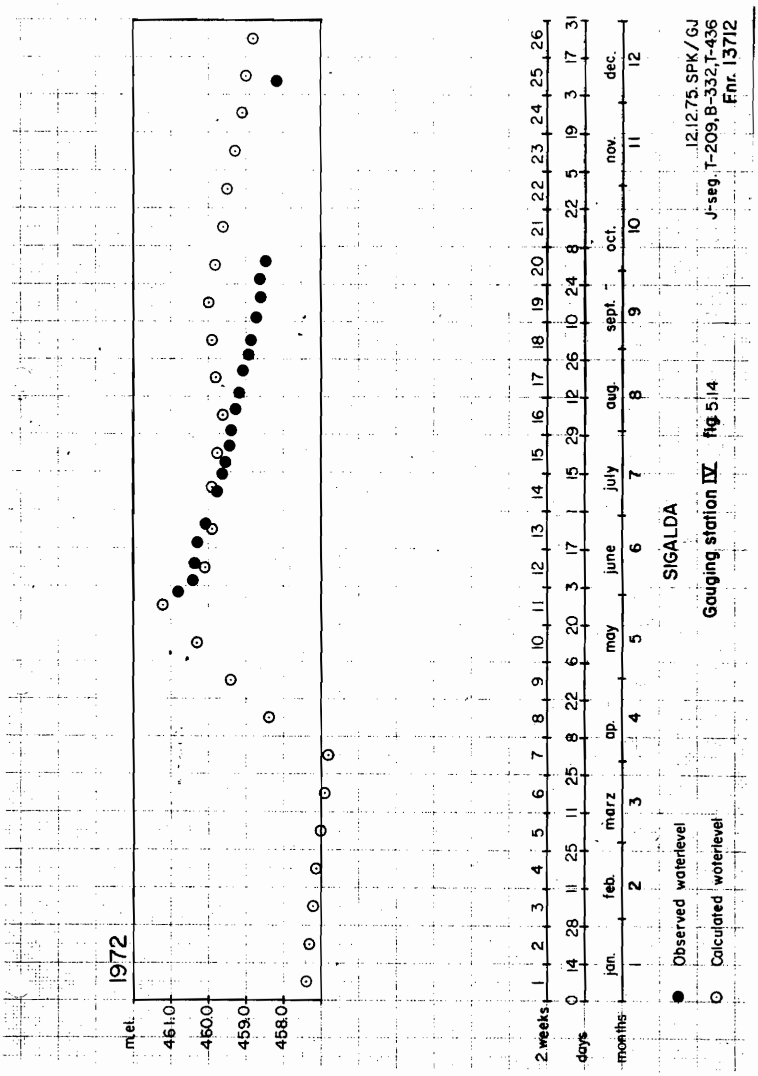

1974
1973
m.el.
475.0
474.0
2 weeks
days
months
jan.
feb.
marz
ap.
may
june
july
aug.
sept.
oct.
nov.
dec.
Observed waterlevel
Calculated waterlevel
SIGALDA
Gauging station VII
fig.5.11
12.1275.SPK/G.J
J-seg.T-209,B-332,T-436
Fnr.13712



1972
1971
m.el.
461.0
460.0
459.0
458.0
457.0
456.0
2 weeks
days
months
jan.
feb.
marz
ap.
may
june
july
aug.
sept.
oct.
nov.
dec.
Observed waterlevel
Calculated waterlevel
SIGALDA
Gauging station IV
fig. 5.12
12.12.75. SPK/GJ
J-seg.T-209, B-332, T-436
Fnr. 13712



SIGALDA
Gauging station TK-3 fig. 5.13


1972
m.el.
461.0
460.0
459.0
458.0
2 weeks
days
months
jan.
feb.
marz
ap.
may
june
july
aug.
sept.
oct.
nov.
dec.
Observed waterlevel
Calculated waterlevel
SIGALDA
Gauging station IV
fig 5.14
12.12.75.SPK/GJ
J-seg. T-209, B-332, T-436
Fnr. 13712

THEORETICAL MEAN LEVEL ,m.

NR	1	43	57
	459.01	474.73	477.17

MEASURED MEAN LEVEL ,m.

NR.	1	43	57
	458.27	474.37	477.25

STANDARD DEV. MEASURED ,m.

NR.	1	43	57
	1.10	.46	.51

STANDARD DEV. THEORET. ,m.

NR.	1	43	57
	.91	.63	.05

STANDARD DEV. ESTIMATE ,m.

NR.	1	43	57
	.89	.56	.51

ABSOLUTE DEV. MEASURED ,m.

NR.		43	57
	.91	.37	.46

ABSOLUTE DEV. THEORET. ,m.

NR.	1	43	57
	.82	.53	.04

ABSOLUTE DEV. ESTIMATE ,m.

NR.	1	43	57
	.77	.51	.47

MAXIMUM DEV. MEASURED ,m.

NR.	1	43	57
	2.42	1.32	.83

MAXIMUM DEV. THEORET. ,m.

NR.	1	43	57
	2.18	1.36	.13

MAXIMUM DEV. ESTIMATE ,m.

NR.	1	43	57
	1.37	1.16	.87

Table 5.21

AREAL MEAN VALUES, m.

STANDARD DEV. MEASURED	.74
STANDARD DEV. THEORET.	.64
STANDARD DEV. ESTIMATE	.67
ABSOLUTE DEV. MEASURED	.58
ABSOLUTE DEV. THEORET.	.46
ABSOLUTE DEV. ESTIMATE	.58
MAXIMUM DEV. MEASURED	2.42
MAXIMUM DEV. THEORET.	2.18
MAXIMUM DEV. ESTIMATE	1.37

Table 5.22

If more reliable timeseries for the groundwater level are needed, then first of all the seasonal variations of the boundary inflow/outflow must be known, and then the variations in the infiltration must be better determined in order to improve the results further.

In order to estimate the transmissivity coefficient and the storage coefficient for an inhomogeneous aquifer, then observation drillholes must be located in the area, according to geological information on homogeneous sub-areas.

6. COMPUTER PROGRAMS.

The following contains a small description of the computer programs developed to solve the various problems in this thesis. It is not the meaning to give a detailed description of the programs, but just to mention what is available and how the main procedures are interconnected. A detailed description together with program listings are available at the National Energy Authority in Reykjavík. The main program structure is shown in fig. 6.1.

SPKOS1: This main procedure generates the finite element matrices defined in chapter 4.2 together with the boundary outflow matrix in chapter 4.5. The matrices are written on an auxiliary file, DISK 1.

SPKOS2: This main procedure reads the file DISK1 and we get a printed output of the matrices $\underline{\underline{B}}_1$ and $\underline{\underline{C}}_1$ in bandform.

SPKOS3: This main procedure calculates the eigenvalues and eigenvectors according to eq. 4.43 and writes them on an auxiliary file DISK3.

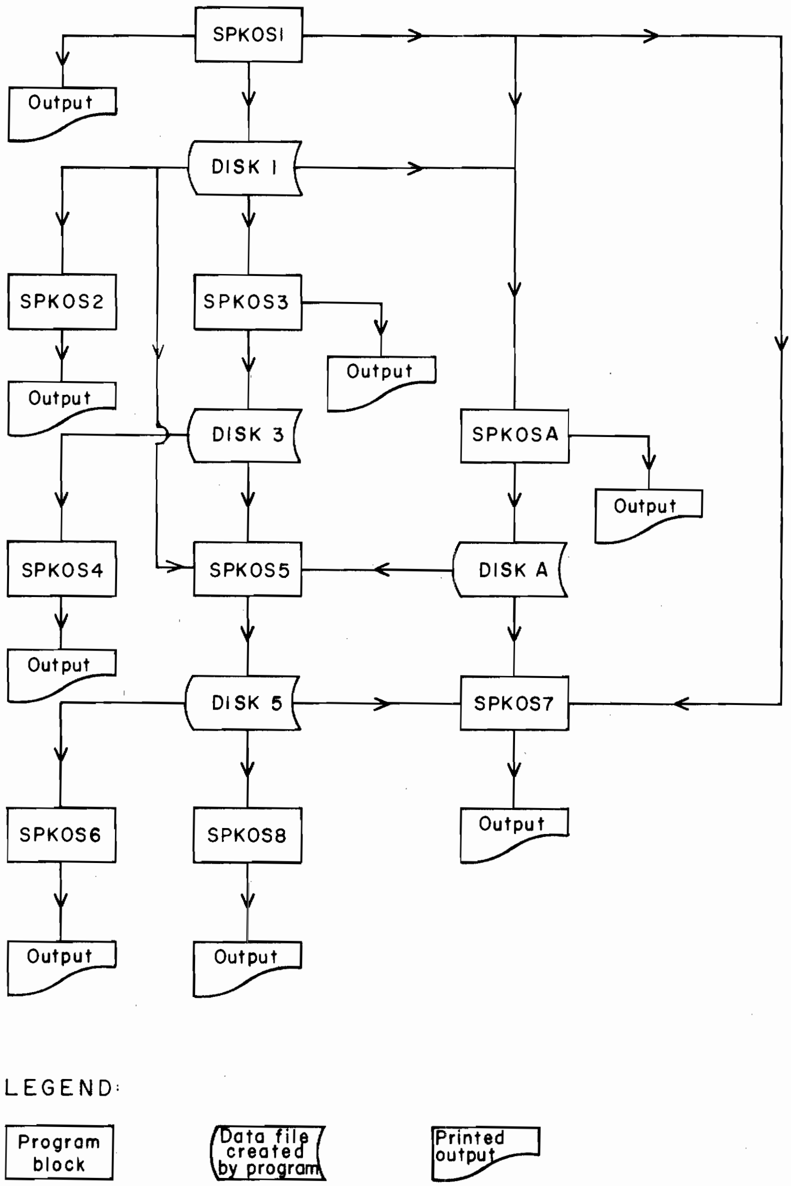
SPKOS4: This main procedure reads the file DISK 3 and prints out the eigenvalues and eigenvectors.

SPKOSA: This main procedure reads the file DISK 1 and calculates the mean level according to eq. 4.39 and writes it on an auxiliary file DISK A.

SPKOS5: This main procedure reads the files DISK 1, DISK 3 and DISK A and calculates the transient groundwater level according to eq. 4.56 and writes it on an auxiliary file DISK 5.

Fig. 6.1.

Main program structure



SPKOS6: This main procedure reads the file DISK 5 and prints out the timeseries for the groundwater level.

SPKOS7: This main procedure reads either the files DISK 1 and DISK A and calculates the mean boundary outflow, or it reads the files DISK 1 and DISK 5 and calculates time-series for the boundary outflow. Printout of the boundary outflow is also given.

SPKOS8: This main procedure reads the file DISK 5 and the observed water level from a card deck. The observed level and the calculated level are compared by computing the mathematical norms in chapter 4.6.

SPKOS0: This main procedure is not shown in fig. 6.1 because it is not connected with the other procedures. It solves the linear inverse problem according to eq. 4.88 and performs the singular value decomposition according to eq. 4.100 to eq. 4.104.

7. CONCLUSIONS.

The method presented in this thesis enables us to treat groundwater reservoir problems in a basic network of triangles that contain all the available field information (location of wells, springs, observation stations and hydrologically homogeneous areas with equal S and T values) and transform this information into a set of constant matrices that do not change with time even though the infiltration, the pumping or other time dependent variables change. The solution of the eigenvalue problem for the matrices involved gives us the geometrical and physical properties of the aquifer and need only be solved once and for all for the same aquifer. If the input function changes we can use the already calculated properties to give us the groundwater level. The eigenfunction expansion shows that the groundwater reservoir can be looked upon as a system of infinitly many parallel linear reservoirs with a definit timeconstant attached to each reservoir. If some hydrological action is forced upon the reservoir from the environment, then the calculation of the largest timeconstant gives you the upper timelimit for the groundwater level to become stationary again. Calculation of the whole timeseries is in that case not necessary.

All this makes it possible to test any specific utilization program prior to it is put in action, and thus reveal the properties of our reservoir and the consequences of the planned utilization before it is possibly too late. In such an investigation the infiltration series and the wellpumping as well, do not have to be measured quantities. They may be planned or estimated values (artificial recharge or planned pumping), or stochastic infiltration series obtained from stochastic models of hydrological data. The method set forth here, is in short, a method to calculate the response of a given groundwater reservoir against any kind of hydrological action forced upon it from the environment.

Finally, it is worth mentioning again that in the field of parameter estimation more research is needed.

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LIST OF SYMBOLS

Symbol	Explanation	Dimension
A	area	m^2
$\underline{\underline{B}}_1$	gives the internal flow between the triangular elements, caused by the variation with time of the groundwater level at each intersection point. It also includes the variation in the transmissivity (T) within the reservoir (Ω).	
$\underline{\underline{B}}_2$	gives the internal flow caused by the known variation of the groundwater level at the boundary. It contains the variation of T.	
$\underline{\underline{C}}_1$	gives the variation of water storage caused by the variation of the groundwater level. It contains the variation of the storage coefficient S.	
$\underline{\underline{C}}_2$	gives the variation of water storage caused by the known variation of the groundwater level at the boundary. It contains the variation in S too.	
$\underline{\underline{D}}$	gives the effect of the infiltration at each point on the groundwater level in every point.	
D	depth of aquifer	m
$\mathcal{D}$	aquifer diffusivity	m^2/sec
$\underline{f}$	gives the initial water level	m
F	fouriertransform of infiltration	
$\underline{g}$	known groundwater level at the boundary	m
g	acceleration of gravity	m/sec^2
$\underline{h}$	total groundwater level	m
$\underline{h}_o$	mean groundwater level	m

Symbol	Explanation	Dimension
$\underline{h}_1$	transient groundwater level	m
H	fouriertransform of groundwater level	
I	hydraulic gradient	
K	Darcy's coefficient	m/sec
K_i	timeconstant for linear reservoirs	sec
$\underline{n}$	normal to boundary curve	
q	boundary runoff	m^3/sec
$\underline{R}$	Infiltration	m/sec
s	parameter along the boundary	m
S	Storage coefficient	
S_o	reference storage coefficient	
t	time	sec
T	transmissivity	m^2/sec
T_o	reference transmissivity	m^2/sec
V ,U,W	velocity components	m/sec
x_1, y_1, z_1	dimensionless coordinates	
x_2, y_2, z_2	dimensionless coordinates	
x, y, z	dimensionless coordinates	
λ_i	eigenvalues	
ϕ_i	eigenfunctions	
ξ, η	dimensionless coordinates	
δ	Dirac's delta function	$1/m^2$
ρ	density of water	kg/m^3
γ	specific gravity	$kg/m^2/sec^2$
$\underline{\mu}$	point input	m^3/s

Symbol	Explanation	Dimension
$\underline{\Omega}$	gives the effect of the known boundary inflow/outflow on the groundwater level in each point	m^3/s
Ω	model area	
$\partial\Omega$	boundary of model area	
Ψ	Green's function	
Γ	heat kernel	

other symbols are defined as they appear. We use the notation $\underline{\underline{A}}$ to denote the matrix A and $\underline{a}$ to denote the vector a. $\underline{\underline{A}}^T$ means the transpose of the matrix A, and $\underline{\underline{A}}^{-1}$ means the inverse of the matrix A.